Chapter 1 Solutions

Solution 1.1-1

(a) $E = \int_0^2 (1)^2 dt + \int_2^3 (-1)^2 dt = 3$

(b) $E = \int_0^2 (-1)^2 dt + \int_2^3 (1)^2 dt = 3$

(c) $E = \int_0^2 (2)^2 dt + \int_2^3 (-2)^2 dt = 12$

(d) $E = \int_3^5 (1)^2 dt + \int_5^6 (-1)^2 dt = 3$

Comments: Changing the sign of a signal does not change its energy. Doubling a signal quadruples its energy. Shifting a signal does not change its energy. Multiplying a signal by a constant K increases its energy by a factor K^2.

Solution 1.1-2

$$E_x = \int_0^1 t^2 dt = \frac{1}{3} \left. t^3 \right|_0^1 = \frac{1}{3}, \qquad E_{x_1} = \int_{-1}^0 (-t)^2 dt = \frac{1}{3} \left. t^3 \right|_{-1}^0 = \frac{1}{3},$$

$$E_{x_2} = \int_0^1 (-t)^2 dt = \frac{1}{3} \left. t^3 \right|_0^1 = \frac{1}{3}, \qquad E_{x_3} = \int_1^2 (t-1)^2 dt = \int_0^1 x^2 dx = \frac{1}{3},$$

$$E_{x_4} = \int_0^1 (2t)^2 dt = \frac{4}{3} \left. t^3 \right|_{-1}^0 = \frac{4}{3}$$

Solution 1.1-3

(a)

$$E_x = \int_0^2 (1)^2 dt = 2, \qquad E_y = \int_0^1 (1)^2 dt + \int_1^2 (-1)^2 dt = 2,$$

$$E_{x+y} = \int_0^1 (2)^2 dt = 4, \qquad E_{x-y} = \int_1^2 (2)^2 dt = 4$$

Therefore $E_{x\pm y} = E_x + E_y$.

(b)

$$E_x = \int_0^{2\pi} \sin^2 t \, dt = \frac{1}{2} \int_0^{2\pi} (1) dt - \frac{1}{2} \int_0^{2\pi} \cos(2t) dt = \pi + 0 = \pi$$

$$E_y = \int_0^{2\pi} (1)^2 \, dt = 2\pi$$

$$E_{x+y} = \int_0^{2\pi} (\sin t + 1)^2 \, dt = \int_0^{2\pi} \sin^2(t) dt + 2 \int_0^{2\pi} \sin(t) dt + \int_0^{2\pi} (1)^2 dt = \pi + 0 + 2\pi = 3\pi$$

In both cases (a) and (b), $E_{x+y} = E_x + E_y$. Similarly we can show that for both cases $E_{x-y} = E_x + E_y$.

(c) As seen in part (a),

$$E_x = \int_0^\pi \sin^2 t \, dt = \pi/2$$

Furthermore,

$$E_y = \int_0^\pi (1)^2 \, dt = \pi$$

Thus,

$$E_{x+y} = \int_0^\pi (\sin t + 1)^2 \, dt = \int_0^\pi \sin^2(t) dt + 2 \int_0^\pi \sin(t) dt + \int_0^\pi (1)^2 dt = \frac{\pi}{2} + 2(2) + \pi = \frac{3\pi}{2} + 4$$

Additionally,

$$E_{x-y} = \int_0^\pi (\sin t - 1)^2 \, dt = \pi/2 - 4 + \pi = \frac{3\pi}{2} - 4$$

In this case, $E_{x+y} \neq E_{x-y} \neq E_x + E_y$. Hence, we cannot generalize the conclusions observed in parts (a) and (b).

Solution 1.1-4

$P_x = \frac{1}{4} \int_{-2}^2 (t^3)^2 dt = 64/7$

(a) $P_{-x} = \frac{1}{4} \int_{-2}^2 (-t^3)^2 dt = 64/7$

(b) $P_{2x} = \frac{1}{4} \int_{-2}^2 (2t^3)^2 dt = 4(64/7) = 256/7$

(c) $P_{cx} = \frac{1}{4} \int_{-2}^2 (ct^3)^2 dt = 64c^2/7$

Comments: Changing the sign of a signal does not affect its power. Multiplying a signal by a constant c increases the power by a factor c^2.

Solution 1.1-5

In the original design, the 3-second duration 10-volt square pulse has energy

$$E_{\text{orig}} = \int_0^3 (10)^2 \, dt = 300.$$

In the soft-start design, as shown in Fig. S1.1-5, the waveform follows a stair-step shape at the start, where each step increases by 1 volt from the previous and has a duration of 20 ms.

Let us call the initial 9 steps in this waveform $x_1(t)$, with duration $9(20) = 180$ ms, and the final step $x_2(t)$, with duration $T - 0.18$ s. The complete waveform $x(t) = x_1(t) + x_2(t)$ has total duration T seconds and, since $x_1(t)$ does not overlap with $x_2(t)$, energy $E_x = E_{x_1} + E_{x_2}$.

The nine steps before reaching the final 10 volt level have a combined energy of

$$E_{x_1} = \sum_{i=1}^9 i^2 (0.02) = 0.02 \left[\frac{9(9+1)(2(9)+1)}{6} \right] = 5.7$$

The final step has energy
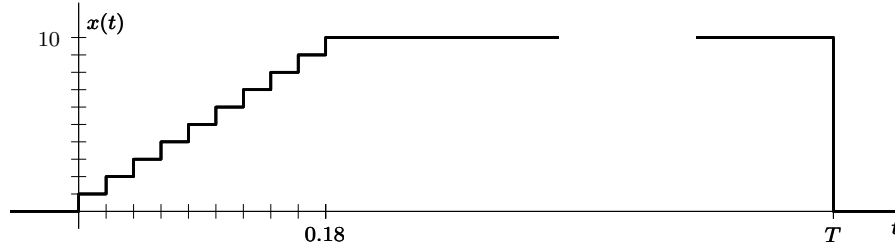
$$E_{x_2} = (T - 0.18)(10)^2 = 100T - 18$$

Figure S1.1-5

Combining, we see that

$$E_x = E_{x_1} + E_{x_2} = 5.7 + 100T - 18 = 300 = E_{\text{orig}}.$$

Solving for T, the duration of the soft-start signal is

$$T = \frac{300 - 5.7 + 18}{100} = 3.123 \text{ seconds.}$$

Solution 1.1-6

We can solve much of this problem by referring to Ex. 1.2 in the text.

(a) The power of a sinusoid of amplitude C is $C^2/2$ regardless of its frequency ($\omega \neq 0$) and phase. Therefore, in this case $P = 5^2 + (10)^2/2 = 75$.

(b) Power of a sum of sinusoids is equal to the sum of the powers of the sinusoids. Therefore, in this case $P = \frac{(10)^2}{2} + \frac{(16)^2}{2} = 178$.

(c) $(10 + 2\sin 3t)\cos 10t = 10\cos 10t + \sin 13t - \sin 3t$. Hence $P = \frac{(10)^2}{2} + \frac{1}{2} + \frac{1}{2} = 51$.

(d) $10\cos 5t \cos 10t = 5(\cos 5t + \cos 15t)$. Hence $P = \frac{(5)^2}{2} + \frac{(5)^2}{2} = 25$.

(e) $10\sin 5t \cos 10t = 5(\sin 15t - \sin 5t)$. Hence $P = \frac{(5)^2}{2} + \frac{(-5)^2}{2} = 25$.

(f) The power of complex signal $e^{j\alpha t}\cos\omega_0 t$ is given as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} |e^{j\alpha t}\cos\omega_0 t|^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} e^{j\alpha t}\cos\omega_0 t e^{-j\alpha t}\cos\omega_0 t \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} \cos^2\omega_0 t \, dt$$

Clearly, the power of $e^{j\alpha t}\cos\omega_0 t$ of is the same as the power of $\cos\omega_0 t$. Thus, $P = 1/2$.

Solution 1.1-7

First, $x(t) = \begin{cases} \frac{2A}{T}t & 0 \leq t < \frac{T}{2} \\ 0 & \frac{T}{2} \leq t < T \\ x(t+T) & \forall t \end{cases}$.   Next, $P_x = \frac{1}{T}\int_0^{T/2}\left(\frac{2A}{T}t\right)^2 dt = \frac{4A^2}{T^2}\int_0^{T/2} t^2 dt =$

$\frac{4A^2}{T^2}\frac{t^3}{3}\Big|_0^{T/2} = \frac{4A^2}{T^2}\frac{T^3}{3(8)} = \frac{A^2}{6}$. Since power is finite, energy must be infinite. Thus,

$$P_x = \frac{A^2}{6} \text{ and } E_x = \infty.$$

Solution 1.1-8

(a) Signal $x(t)$, shown in Fig. S1.1-8, is 1-periodic. Thus, $E_x = \infty$ and

$$P_x = \int_0^1 x^2(t)\,dt = \int_0^1 1\,dt = 1.$$

(b) Signal $y(t)$, shown in Fig. S1.1-8, is 1-periodic. Thus, $E_y = \infty$ and

$$P_y = \int_0^1 y^2(t)\,dt = \int_0^1 1\,dt = 1.$$

(c) Signal $f(t) = x(t) + jy(t)$ is also 1-periodic. Thus, $E_f = \infty$ and

$$P_f = \int_0^1 |f(t)|^2\,dt = \int_0^1 (x(t) + jy(t))(x(t) - jy(t))\,dt = \int_0^1 x^2(t)\,dt + \int_0^1 y^2(t)\,dt = 2.$$

(d) The energy of complex signal $f(t)$ requires integrating $|f(t)|^2$. If $f(t) = x(t) + jy(t)$, where $x(t)$ and $y(t)$ are real signals, then $|f(t)|^2 = (x(t) + jy(t))(x(t) - jy(t)) = x^2(t) + y^2(t)$, and the energy of $f(t)$ is just the sum the individual energies. That is,

$$\begin{aligned}
E_f &= \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} (x(t) + jy(t))(x(t) - jy(t))\,dt \\
&= \int_{-\infty}^{\infty} (x^2(t) + y^2(t))\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \\
&= E_x + E_y
\end{aligned}$$

Following a similar proof, the conclusion for the power of $f(t)$ is the same. Thus,

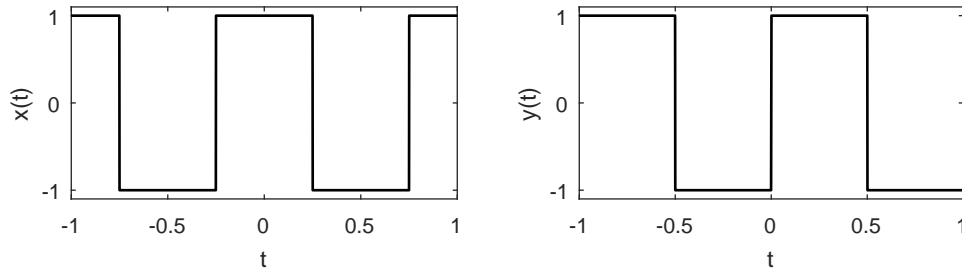for real $x(t)$, real $y(t)$, and $f(t) = x(t) + jy(t)$, $E_f = E_x + E_y$ and $P_f = P_x + P_y$.

Figure S1.1-8

Solution 1.1-9

(a) By definition, $E[Tx_1(t)] = \int_{t=-\infty}^{\infty} (Tx_1(t))^2\,dt = \int_{t=-\infty}^{\infty} T^2 x_1^2(t)\,dt = T^2 \int_{t=-\infty}^{\infty} x_1^2(t)\,dt = T^2 E[x_1(t)]$.

$$E[Tx_1(t)] = T^2 E[x_1(t)].$$

(b) By definition, $E[x_1(t - T)] = \int_{t=-\infty}^{\infty} (x_1(t - T))^2\,dt$. Substituting $t' = t - T$ and $dt' = dt$ yields $\int_{t'=\infty}^{\infty} x_1^2(t')\,dt' = E[x_1(t)]$.

$$E[x_1(t - T)] = E[x_1(t)].$$

(c) By definition, $E\left[x_1(t) + x_2(t)\right] = \int_{t=-\infty}^{\infty} \left(x_1(t) + x_2(t)\right)^2 dt = \int_{t=-\infty}^{\infty} \left(x_1^2(t) + 2x_1(t)x_2(t) + x_2^2(t)\right) dt.$ However, $x_1(t)$ and $x_2(t)$ are non-overlapping so their product $x_1(t)x_2(t)$ must be zero. Thus, $E\left[x_1(t) + x_2(t)\right] = \int_{t=-\infty}^{\infty} \left(x_1^2(t) + x_2^2(t)\right) dt = \int_{t=-\infty}^{\infty} x_1^2(t)dt + \int_{t=-\infty}^{\infty} x_2^2(t)dt = E\left[x_1(t)\right] + E\left[x_2(t)\right].$

$$\text{If } (x_1(t) \neq 0) \Rightarrow (x_2(t) = 0) \text{ and } (x_2(t) \neq 0) \Rightarrow (x_1(t) = 0),$$

$$\text{Then, } E\left[x_1(t) + x_2(t)\right] = E\left[x_1(t)\right] + E\left[x_2(t)\right].$$

(d) By definition, $E\left[x_1(Tt)\right] = \int_{t=-\infty}^{\infty} x_1^2(Tt)dt.$

First, consider the case $T > 0$. Substituting $t' = Tt$ and $dt' = Tdt$ yields $E\left[x_1(Tt)\right] = \int_{t'=-\infty}^{\infty} x_1^2(t')\frac{dt'}{T} = \frac{1}{T}\int_{t'=-\infty}^{\infty} x_1^2(t')dt' = \frac{E[x_1(t)]}{T} = \frac{E[x_1(t)]}{|T|}.$

Next, consider the case $T < 0$. Substituting $t' = Tt$ and $dt' = Tdt$ yields $E\left[x_1(Tt)\right] = \int_{t'=\infty}^{-\infty} x_1^2(t')\frac{dt'}{T} = \frac{-1}{T}\int_{t'=-\infty}^{\infty} x_1^2(t')dt' = \frac{E[x_1(t)]}{-T}.$ For $T < 0$, we know $T = -|T|$. Making this substitution yields $E\left[x_1(Tt)\right] = \frac{E[x_1(t)]}{|T|}.$

Since energy is the same whether $T < 0$ or $T > 0$, we know

$$E\left[x_1(Tt)\right] = \frac{E\left[x_1(t)\right]}{|T|}.$$

Solution 1.1-10

To solve this problem, we use the results of Prob. 1.1-9. Also, consider signal $y(t) = t(u(t) - u(t-1))$ which has energy equal to $E\left[y(t)\right] = \int_0^1 t^2 dt = 1/3.$

To determine $E\left[x(t)\right]$, consider dividing $x(t)$ into three non-overlapping pieces: a first piece $x_a(t)$ from $(-2 \leq t < -1)$, a second piece $x_b(t)$ from $(-1 \leq t < 0)$, and a third piece $x_c(t)$ from $(0 \leq t < 3)$. Since the pieces are non-overlapping, the total energy $E\left[x(t)\right] = E\left[x_a(t)\right] + E\left[x_b(t)\right] + E\left[x_c(t)\right].$

Using the properties of energy, we know that shifting or reflecting a signal does not affect its energy. Notice that $y(t/3)$ is the same as a flipped and shifted version of $x_c(t)$. Thus, $E\left[x_c(t)\right] = E\left[y(t/3)\right] = 3(1/3) = 1.$ Also, it is possible to combine $x_a(t)$ with a flipped and shifted version of $x_b(t)$ to equal a flipped and shifted version of $2y(t/2)$. Thus, $E\left[x_a(t) + x_b(t)\right] = E\left[2y(t/2)\right] = 4(2)(1/3) = 8/3.$

Thus,

$$E\left[x(t)\right] = 11/3.$$

Solution 1.1-11

(a)

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=m}^{n} \sum_{r=m}^{n} D_k D^*_r e^{j(\omega_k - \omega_r)t}\, dt$$

The integrals of the cross-product terms (when $k \neq r$) are finite because the integrands are periodic signals (made up of sinusoids). These terms, when divided by $T \to \infty$, yield zero. The remaining terms $(k = r)$ yield

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=m}^{n} |D_k|^2\, dt = \sum_{k=m}^{n} |D_k|^2$$

(b) Prob. 1.1-6(a)

$$x(t) = 5 + 10\cos(100t + \pi/3)$$
$$= 5 + 5e^{j(100t + \frac{\pi}{3})} + 5e^{-j(100t + \frac{\pi}{3})}$$
$$= 5 + 5e^{j\pi/3}e^{j100t} + 5e^{-j\pi/3}e^{-j100t}$$

Hence,

$$P_x = 5^2 + |5e^{j\pi/3}|^2 + |5e^{-j\pi/3}|^2 = 25 + 25 + 25 = 75.$$

Thought of another way, note that $D_0 = 5, D_{\pm 1} = 5$ and thus $P_x = 5^2 + 5^2 + 5^2 = 75$.

Prob. 1.1-6(b)

$$x(t) = 10\cos(100t + \pi/3) + 16\sin(150t + \pi/5)$$
$$= 5e^{j\pi/3}e^{j100t} + 5e^{-j\pi/3}e^{-j100t} - j8e^{j\pi/5}e^{j150t} + j8e^{-j\pi/5}e^{-j150t}$$

Hence,

$$P_x = |5e^{j\pi/3}|^2 + |5e^{-j\pi/3}|^2 + |-j8e^{j\pi/5}|^2 + |j8e^{-j\pi/5}|^2 = 25 + 25 + 64 + 64 = 178.$$

Thought of another way, note that $D_{\pm 1} = 5$ and $D_{\pm 2} = 8$. Hence, $P_x = 5^2 + 5^2 + 8^2 + 8^2 = 178$.

Prob. 1.1-6(c)
To begin, we note that $(10 + 2\sin 3t)\cos 10t = 10\cos 10t + \sin 13t - \sin 3t$. In this case, $D_{\pm 1} = 5$, $D_{\pm 2} = 0.5$ and $D_{\pm 3} = 0.5$. Hence, $P = 5^2 + 5^2 + (0.5)^2 + (0.5)^2 + (0.5)^2 + (0.5)^2 = 51$.

Prob. 1.1-6(d)
To begin, we note that $10\cos 5t \cos 10t = 5(\cos 5t + \cos 15t)$. In this case, $D_{\pm 1} = 2.5$ and $D_{\pm 2} = 2.5$. Hence, $P = (2.5)^2 + (2.5)^2 + (2.5)^2 + (2.5)^2 = 25$.

Prob. 1.1-6(e)
$10\sin 5t \cos 10t = 5(\sin 15t - \sin 5t)$. In this case, $D_{\pm 1} = 2.5$ and $D_{\pm 2} = 2.5$. Hence, $P = (2.5)^2 + (2.5)^2 + (2.5)^2 + (2.5)^2 = 25$.

Prob. 1.1-6(f)
In this case, $e^{j\alpha t}\cos\omega_0 t = \frac{1}{2}\left[e^{j(\alpha+\omega_0)t} + e^{j(\alpha-\omega_0)t}\right]$. Thus, $D_{\pm 1} = 0.5$ and $P = (1/2)^2 + (1/2)^2 = 1/2$.

Solution 1.1-12

First, notice that $x(t) = x^2(t)$ and that the area of each pulse is one. Since $x(t)$ has an infinite number of pulses, the corresponding energy must also be infinite. To compute the power, notice that N pulses requires an interval of width $\sum_{i=0}^{N} 2(i+1) = N^2 + 3N$. As $N \to \infty$, power is computed by the ratio of area to width, or $P = \lim_{N\to\infty} \frac{N}{N^2+3N} = 0$. Thus,

$$P = 0 \text{ and } E = \infty.$$

Solution 1.2-1

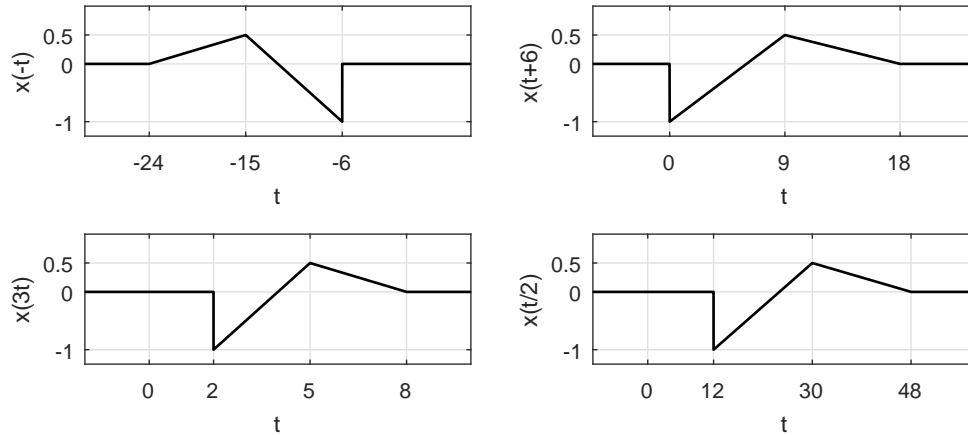Figure S1.2-1 shows **(a)** $x(-t)$, **(b)** $x(t+6)$, **(c)** $x(3t)$, and **(d)** $x(t/2)$.

Figure S1.2-1

Solution 1.2-2

Figure S1.2-2 shows **(a)** $x(t-4)$, **(b)** $x(t/1.5)$, **(c)** $x(-t)$, **(d)** $x(2t-4)$, and **(e)** $x(2-t)$.
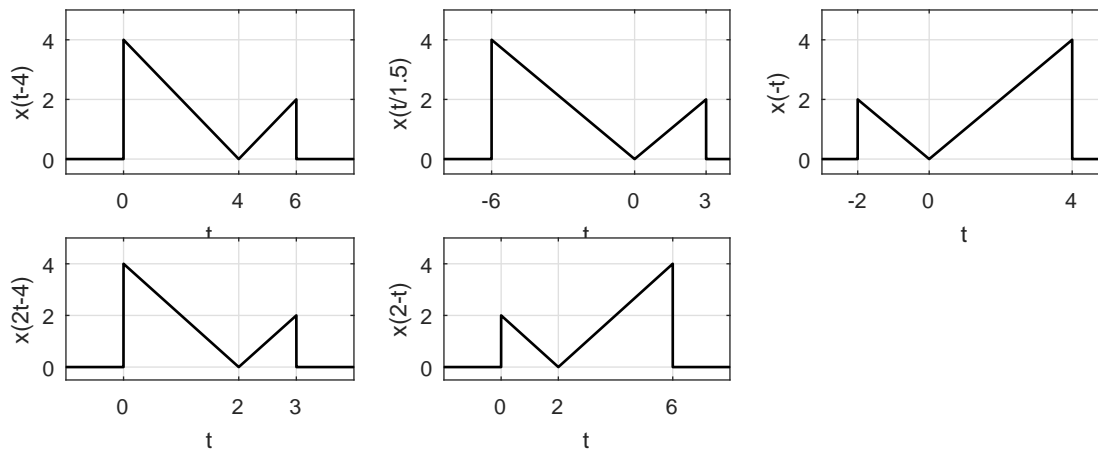

Figure S1.2-2

Solution 1.2-3

(a) $x_1(t)$ can be formed by shifting $x(t)$ to the left by 1 plus a time-inverted version of $x(t)$ shifted to left by 1. Thus,

$$x_1(t) = x(t+1) + x(-t+1) = x(t+1) + x(1-t).$$

(b) $x_2(t)$ can be formed by time-expanding $x(t)$ by factor 2 to obtain $x(t/2)$ Now, left-shift $x(t/2)$ by unity to obtain $x(\frac{t+1}{2})$. We now add to this a time-inverted version of $x(\frac{1-t}{2})$ to obtain $x_2(t)$. Thus,

$$x_2(t) = x(\tfrac{t+1}{2}) + x(\tfrac{1-t}{2}).$$

(c) Observe that $x_3(t)$ is composed of two parts:
First, a rectangular pulse to form the base is constructed by time-expanding $x_2(t)$ by a factor of

2. This is obtained by replacing t with $t/2$ in $x_2(t)$. Thus, we obtain $x_2(t/2) = x(\frac{t+2}{4}) + x(\frac{2-t}{4})$. Second, the two triangles on top of the rectangular base are constructed by time-expanded (factor of 2) and shifted versions of $x(t)$ according to $x(t/2) + x(-t/2)$. Thus,

$$x_3(t) = x(\tfrac{t+2}{4}) + x(\tfrac{2-t}{4}) + x(t/2) + x(-t/2).$$

(d) $x_4(t)$ can be obtained by time-expanding $x_1(t)$ by a factor 2 and then multiplying it by $4/3$ to obtain $\frac{4}{3}x_1(t/2) = \frac{4}{3}\left[x(\frac{t+2}{2}) + x(\frac{2-t}{2})\right]$. From this, we subtract a rectangular pedestal of height $1/3$ and width 4. This is obtained by time-expanding $x_2(t)$ by 2 and multiplying it by $1/3$ to yield $\frac{1}{3}x_2(t/2) = \frac{1}{3}\left[x(\frac{t+2}{4}) + x(\frac{2-t}{4})\right]$. Hence,

$$x_4(t) = \tfrac{4}{3}\left[x(\tfrac{t+2}{2}) + x(\tfrac{2-t}{2})\right] - \tfrac{1}{3}\left[x(\tfrac{t+2}{4}) + x(\tfrac{2-t}{4})\right].$$

(e) $x_5(t)$ is a sum of three components: (i) $x_2(t)$ time-compressed by a factor 2, (ii) $x(t)$ left-shifted by 1.5, and (iii) $x(t)$ time-inverted and then right shifted by 1.5. Hence,

$$x_5(t) = x(t+0.5) + x(0.5-t) + x(t+1.5) + x(1.5-t).$$

Solution 1.2-4

$$E_{-x} = \int_{-\infty}^{\infty} [-x(t)]^2\, dt = \int_{-\infty}^{\infty} x^2(t)\, dt = E_x$$

$$E_{x(-t)} = \int_{-\infty}^{\infty} [x(-t)]^2\, dt = \int_{-\infty}^{\infty} x^2(x)\, dx = E_x$$

$$E_{x(t-T)} = \int_{-\infty}^{\infty} [x(t-T)]^2\, dt = \int_{-\infty}^{\infty} x^2(x)\, dx = E_x,$$

$$E_{x(at)} = \int_{-\infty}^{\infty} [x(at)]^2\, dt = \frac{1}{a}\int_{-\infty}^{\infty} x^2(x)\, dx = E_x/a$$

$$E_{x(at-b)} = \int_{-\infty}^{\infty} [x(at-b)]^2\, dt = \frac{1}{a}\int_{-\infty}^{\infty} x^2(x)\, dx = E_x/a,$$

$$E_{x(t/a)} = \int_{-\infty}^{\infty} [x(t/a)]^2\, dt = a\int_{-\infty}^{\infty} x^2(x)\, dt = aE_x$$

$$E_{ax(t)} = \int_{-\infty}^{\infty} [ax(t)]^2\, dt = a^2\int_{-\infty}^{\infty} x^2(t)\, dt = a^2 E_x$$

Comment: Multiplying a signal by constant a increases the signal energy by a factor a^2.

Solution 1.2-5

(a) Calling $y(t) = 2x(-3t+1) = t(u(-t-1) - u(-t+1))$, MATLAB is used to sketch $y(t)$.

```
>>  u = @(t) 1.0*(t>=0); t = -1.5:.001:1.5; y = @(t) t.*(u(-t-1)-u(-t+1));
>>  subplot(121); plot(t,y(t)); axis([-1.5 1.5 -1.1 1.1]);
>>  xlabel('t'); ylabel('2x(-3t+1)'); grid on
```

(b) Since $y(t) = 2x(-3t+1)$, $0.5*y(-t/3+1/3) = 0.5(2)x(-3(-t/3+1/3)+1) = x(t)$. MATLAB is used to sketch $x(t)$.

```
>>  t = -3:.001:5; x = @(t) 0.5*y(-t/3+1/3);
>>  subplot(122); plot(t,x(t)); axis([-3 5 -1.1 1.1]);
>>  xlabel('t'); ylabel('x(t)'); grid on
```
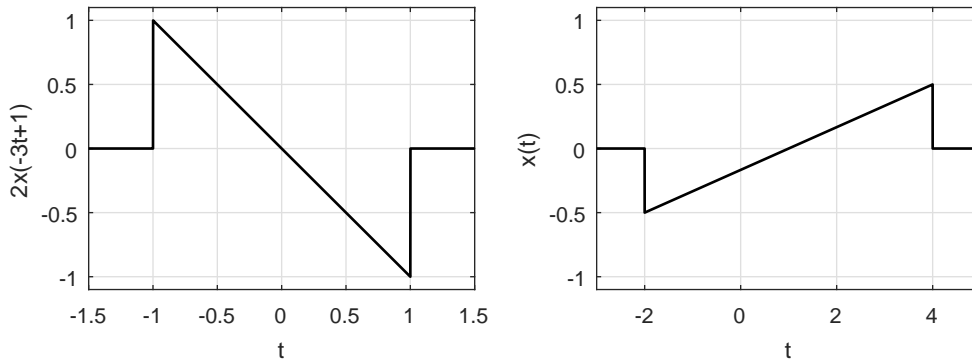
Figure S1.2-5

Solution 1.2-6

MATLAB is used to compute each sketch. Notice that the unit step is in the exponent of the function $x(t)$.

(a)
```
>>  u = @(t) 1.0*(t>=0); t = [-1:.001:1]; x = @(t) 2.^(-t.*u(t));
>>  subplot(121); plot(t,x(t),'k'); grid on;
>>  axis([-1 1 0 1.1]); xlabel('t'); ylabel('x(t)');
```

(b)
```
>>  subplot(122); plot(t,0.5*x(1-2*t),'k'); grid on;
>>  axis([-1 1 0 1.1]); xlabel('t'); ylabel('y(t)');
```
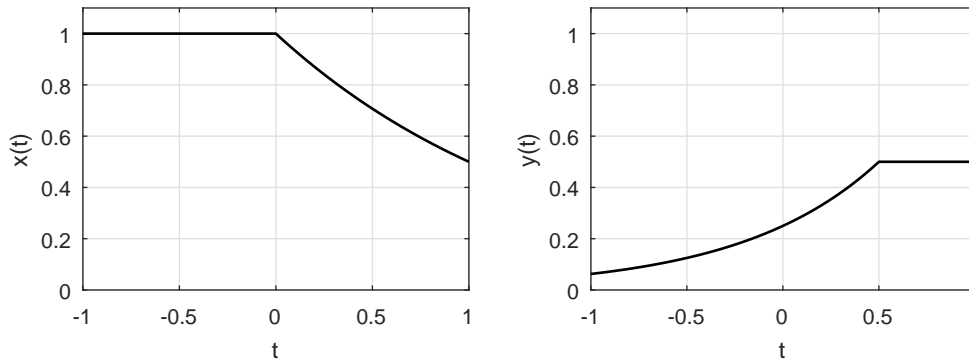

Figure S1.2-6

Solution 1.2-7

(a) Here, we are looking for constants a, b, and c to produce $z(t) = ax(bt + c)$. Signal $z(t)$ is three times taller than $y(t)$, which is $x(t)$ scaled by $-\frac{1}{2}$. Thus, $3(-\frac{1}{2}) = a$ or $a = -\frac{3}{2}$. Next, we pick two corresponding points of $y(t)$ and $z(t)$ to determine b and c.

$$-3t + 2|_{t=-1} = bt + c|_{t=8} \quad \Rightarrow \quad 8b + c = 5$$

and

$$-3t + 2|_{t=0} = bt + c|_{t=2} \quad \Rightarrow \quad 2b + c = 2$$

This system of equations is easily solved with MATLAB.

```
>>  inv([8 1;2 1])*[5;2]
    ans = 0.5
          1.0
```

Thus,

$$a = -\frac{3}{2}, \quad b = \frac{1}{2}, \quad c = 1.$$

(b) Differentiating the plot of $z(t)$, we obtain a signal $v(t)$ such that $z(t) = \int_{-\infty}^{t} v(\tau)\, d\tau$. The result, shown in Fig. S1.2-7, is expressed mathematically as

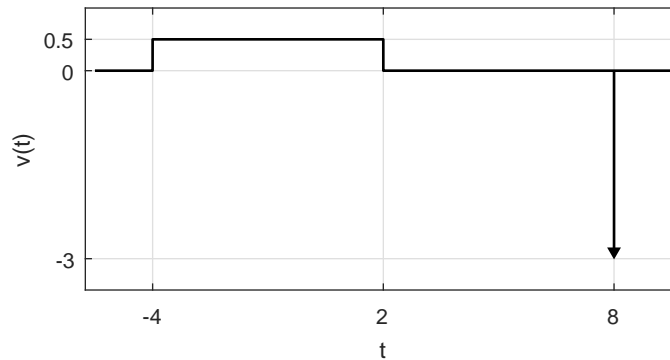$$v(t) = \frac{1}{2}u(t+4) - \frac{1}{2}u(t-2) - 3\delta(t-8).$$

Figure S1.2-7

Solution 1.3-1

There are an infinite number of possible answers to this problem. Let us consider a simple example to demonstrate the overall logic of the problem.

Consider a simple circuit where a series of three AA batteries connect to an LED through a push-button switch. A simple real-world signal is the voltage $v(t)$ measured at the LED when a person presses the button for 1 second. Assuming the button is pressed at time $t = 0$, signal $v(t)$ is reasonably modeled as

$$v(t) = 4.5\left[u(t) - u(t-1)\right].$$

By inspection, we see that

$v(t)$ is (a) continuous-time, (b) analog, (c) aperiodic, (d) energy, (e) causal, and (f) deterministic.

It is not possible to devise a real-world signal that is opposite in all six of the characteristics of $v(t)$. To understand why, we note that any practical real-world signal must be finite in duration (nothing lasts for ever in the physical world) and of finite energy (it is impractical to generate the infinite energy needed for a power signal). Since practical signals are finite duration, no real-world signal is truly periodic. Thus, we cannot devise a real-world signal that has the needed (opposite to $v(t)$) characteristics of being periodic and a power signal. It is possible, for a real-world signal to have the other four opposite characteristics of discrete, digital, noncausal, and random. For example, recording the average yearly temperature, rounded to the nearest degree Celsius, from 1000 BC to 2000 AD would result in a discrete-time, digital, aperiodic, energy, noncausal, and random signal.

Solution 1.3-2

(a) The leftmost edge of the right-sided signal $x(t)$ occurs when the argument t is equal to 1, $t = 1$. The same edge occurs for signal $x(-2t + a)$ when its argument $-2t + a$ also equals 1,

or $a = 1 + 2t$. To be borderline anticausal, the edge of the left-sided signal $x(-2t + a)$ must occur at $t = 0$, which requires $a = 1 + 2t|_{t=0} = 1$. Thus,

$$a = 1 \text{ causes } x(-2t + a) \text{ to be borderline anticausal.}$$

(b) To test for periodicity, we look to see if $y(t + T_y) = y(t)$ for some value T_y. To begin, we notice that

$$y(t + T_y) = \sum_{k=-\infty}^{\infty} x(0.5t + 0.5T_y - 10k).$$

For $T_y = 20l$ and any integer l, we see that

$$y(t + T_y) = \sum_{k=-\infty}^{\infty} x(0.5t + 10l - 10k) = \sum_{k=-\infty}^{\infty} x(0.5t - 10(k - l)).$$

Letting $k' = k - l$, we obtain

$$y(t + T_y) = \sum_{k'=-\infty}^{\infty} x(0.5t - 10k') = y(t).$$

Thus, $y(t)$ is periodic. Setting $l = 1$ yields the fundamental period of $T_y = 20$. In summary,

$$y(t) \text{ is periodic with fundamental period } T_y = 20.$$

Solution 1.3-3

(a) False. Figure 1.11b is an example of a signal that is continuous-time but digital.

(b) False. Figure 1.11c is discrete-time but analog.

(c) False. e^{-t} is neither an energy nor a power signal.

(d) False. $e^{-t}u(t)$ has infinite duration but is an energy signal.

(e) False. $u(t)$ is a power signal that is causal.

(f) True. A periodic signal, which repeats for all t, cannot be 0 for $t > 0$ like an anticausal signal.

Solution 1.3-4

(a) True. Every bounded periodic signal is a power signal.

(b) False. Signals with bounded power are not necessarily periodic. For example, $x(t) = \cos(t)u(t)$ is non-periodic but has a bounded power of $P_x = 0.25$.

(c) True. If an energy signal $x(t)$ has energy E, then the energy of $x(at)$ is $\frac{E}{a}$ (a real and positive).

(d) False. If a power signal $x(t)$ has power P, then the power of $x(at)$ is generally not $\frac{P}{a}$. A counter-example provides a simple proof. Consider the case of $x(t) = u(t)$, which has $P = 0.5$. Letting $a = 2$, $x(at) = x(2t) = u(2t) = u(t)$, which still has power $P = 0.5$ and not $P/a = P/2$.

Solution 1.3-5

(a) For periodicity, $x_1(t) = \cos(t) = \cos(t + T_1) = x_1(t + T_1)$. Since cosine is a 2π-periodic function, $T_1 = 2\pi$. Similarly, $x_2(t) = \sin(\pi t) = \sin(\pi t + \pi T_2) = \sin(\pi(t + T_2)) = x_2(t + T_2)$. Thus, $\pi T_2 = 2\pi k$. The smallest possible value is $T_2 = 2$. Thus,

$$T_1 = 2\pi \text{ and } T_2 = 2.$$

(b) Periodicity requires $x_3(t) = x_3(t + T_3)$ or $\cos(t) + \sin(\pi t) = \cos(t + T_3) + \sin(\pi t + \pi T_3)$. This requires $T_3 = 2\pi k_1$ and $\pi T_3 = 2\pi k_2$ for some integers k_1 and k_2. Combining, periodicity thus requires $T_3 = 2\pi k_1 = 2k_2$ or $\pi = k_1/k_2$. However, π is irrational. Thus, no suitable k_1 and k_2 exist, and $x_3(t)$ cannot be periodic.

(c)

$$P_{x_1} = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t)dt = \frac{1}{2\pi} \left(0.5(t + \sin(t)\cos(t))|_{t=0}^{2\pi} \right) = \frac{1}{2\pi} \left(\frac{1}{2} \right) 2\pi = \frac{1}{2}$$

$$P_{x_2} = \frac{1}{2} \int_0^2 \sin^2(\pi t)dt = \frac{1}{2} \left(\frac{1}{2\pi} (\pi t - \sin(\pi t)\cos(\pi t)) \Big|_{t=0}^2 \right) = \frac{1}{2} \left(\frac{1}{2\pi} \right) 2\pi = \frac{1}{2}$$

$$P_{x_3} = \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^T (\cos(t) + \sin(\pi t))^2 \, dt$$

$$= \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^T \left(\cos^2(t) + \sin^2(t) + \cos(t)\sin(\pi t) \right) dt$$

$$= P_{x_1} + Px_2 + \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^T 0.5 \left(\sin(\pi t - t) + \sin(\pi t + t) \right) dt$$

$$= P_{x_1} + Px_2 + 0 = 1$$

Thus,

$$P_{x_1} = \frac{1}{2}, \quad P_{x_2} = \frac{1}{2}, \quad \text{and} \quad P_{x_3} = 1.$$

Solution 1.3-6

No, $f(t) = \sin(\omega t)$ is not guaranteed to be a periodic function for an arbitrary constant ω. Specifically, if ω is purely imaginary then $f(t)$ is in the form of hyperbolic sine, which is not a periodic function. For example, if $\omega = j$ then $f(t) = j\sinh(t)$. Only when ω is constrained to be real will $f(t)$ be periodic.

Solution 1.3-7

(a) $E_{y_1} = \int_{-\infty}^\infty y_1^2(t)dt = \int_{-\infty}^\infty \frac{1}{9} x^2(2t)dt$. Performing the change of variable $t' = 2t$ yields $\int_{-\infty}^\infty \frac{1}{9} x^2(t') \frac{dt'}{2} = \frac{E_x}{18}$. Thus,

$$E_{y_1} = \frac{E_x}{18} \approx \frac{1.0417}{18} = 0.0579.$$

(b) Since $y_2(t)$ is just a $(T_{y_2} = 4)$-periodic replication of $x(t)$, the power is easily obtained as

$$P_{y_2} = \frac{E_x}{T_{y_2}} = \frac{E_x}{4} \approx= 0.2604.$$

(c) Notice, $T_{y_3} = T_{y_2}/2 = 2$. Thus, $P_{y_3} = \frac{1}{2} \int_{T_{y_3}} y_3^2(t)dt = \frac{1}{2} \int_{T_{y_3}} \frac{1}{9} y_2^2(2t)dt$. Performing the change of variable $t' = 2t$ yields $P_{y_3} = \frac{1}{2} \int_{T_{y_2}} \frac{1}{9} y_2^2(t') \frac{dt'}{2} = \frac{1}{36} \int_0^4 x^2(t')dt' = \frac{E_x}{36}$. Thus,

$$P_{y_3} = \frac{E_x}{36} \approx 0.0289.$$

Solution 1.3-8

For all parts, $y_1(t) = y_2(t) = t^2$ over $0 \le t \le 1$.

(a) To ensure $y_1(t)$ is even, $y_1(t) = t^2$ over $-1 \le t \le 0$. Since $y_1(t)$ is $(T_1 = 2)$-periodic, $y_1(t) = y_1(t+2)$ for all t. Thus, $y_1(t) = \begin{cases} t^2 & -1 \le t \le 1 \\ y_1(t+2) & \forall t \end{cases}$. $P_{y_1} = \frac{1}{T_1} \int_{-1}^{1} (t^2)^2 dt = 0.5 \frac{t^5}{5} \Big|_{t=-1}^{1} = 1/5$. Thus,

$$P_{y_1} = 1/5.$$

A sketch of $y_1(t)$ over $-3 \le t \le 3$ is created using MATLAB and shown in Fig. S1.3-8.

```
>>  t = [-3:.001:3]; y_1 = @(mt) (mt<=1).*(mt.^2) + (mt>1).*((mt-2).^2);
>>  subplot(121); plot(t,y_1(mod(t,2)),'k'); grid on
>>  xlabel('t'); ylabel('y_1(t)'); axis([-3 3 -.1 1.1]);
```

(b) Let

$$y_2(t) = \begin{cases} k & 1 \le t < 1.5 \\ t^2 & 0 \le t < 1 \\ -y_2(-t) & \forall t \\ y_2(t+3) & \forall t \end{cases}.$$

With this form, $y_2(t)$ is odd and $(T_2 = 3)$-periodic. The constant k is determined by constraining the power to be unity, $P_{y_2} = 1 = \frac{1}{3}\left(k^2 + \frac{2}{5}t^5\big|_{t=0}^{1}\right)$. Solving for k yields $k^2 = 3 - 2/5 = 13/5$ or $k = \sqrt{13/5}$. Thus,

$$y_2(t) = \begin{cases} \sqrt{13/5} & 1 \le t < 1.5 \\ t^2 & 0 \le t < 1 \\ -y_2(-t) & \forall t \\ y_2(t+3) & \forall t \end{cases}.$$

A sketch of $y_2(t)$ over $-3 \le t \le 3$ is created using MATLAB and shown in Fig. S1.3-8.

```
>>  y_2 = @(mt) (mt<1).*(mt.^2)-(mt>=2).*((mt-3).^2)+...
>>        ((mt>=1)&(mt<1.5))*sqrt(13/5)-...
>>        ((mt>=1.5)&(mt<2))*sqrt(13/5);
>>  subplot(122); plot(t,y_2(mod(t,3)),'k'); grid on
>>  xlabel('t'); ylabel('y_2(t)'); axis([-3 3 -2 2]);
```

(c) Define $y_3(t) = y_1(t) + jy_2(t)$. To be periodic, $y_3(t)$ must equal $y_3(t + T_3)$ for some value T_3. This implies that $y_1(t) = y_1(t + T_3)$ and $y_2 = y_2(t + T_3)$. Since $y_1(t)$ is $(T_1 = 2)$-periodic, T_3 must be an integer multiple of T_1. Similarly, since $y_2(t)$ is $(T_2 = 3)$-periodic, T_3 must be an integer multiple of T_2. Thus, periodicity of $y_3(t)$ requires $T_3 = T_1 k_1 = 2k_1 = T_2 k_2 = 3k_2$, which is satisfied letting $k_1 = 3$ and $k_2 = 2$. Thus,

$$y_3(t) \text{ is periodic with } T_3 = 6.$$

(d) Noting $y_3(t)y_3^*(t) = y_1^2(t) + y_2^2(t)$, $P_{y_3} = \frac{1}{T_3} \int_{T_3} \left(y_1^2(t) + y_2^2(t)\right) dt = P_{y_1} + P_{y_2}$. Thus,

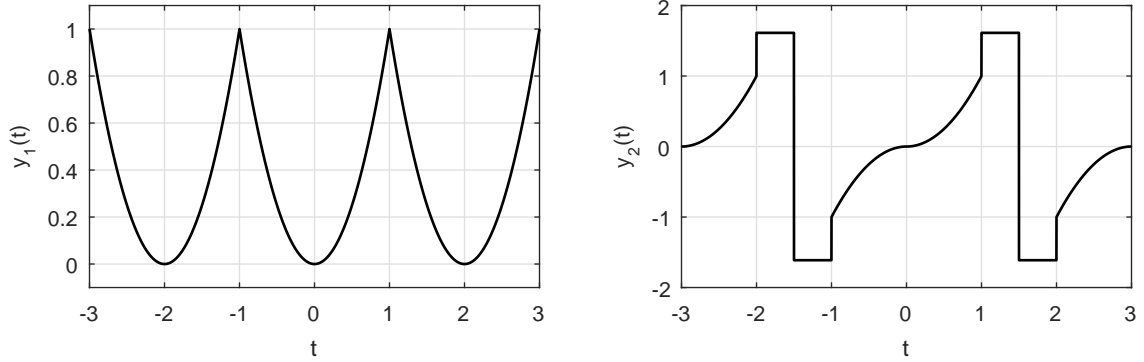$$P_{y_3} = 1 + \frac{1}{5} = \frac{6}{5}.$$

Figure S1.3-8

Solution 1.4-1

Figure S1.4-1 shows **(a)** $u(t-5) - u(t-7)$, **(b)** $u(t-5) + u(t-7)$, **(c)** $t^2[u(t-1) - u(t-2)]$, and **(d)** $(t-4)[u(t-2) - u(t-4)]$.
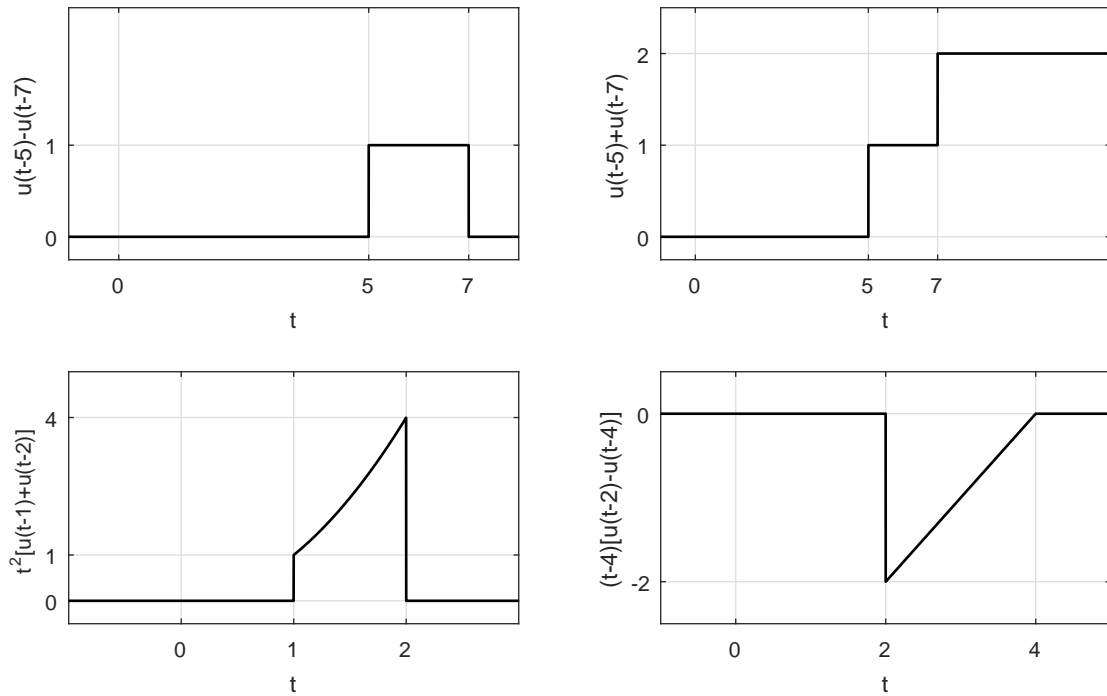

Figure S1.4-1

Solution 1.4-2

(a)

$$x_1(t) = (4t+1)[u(t+1) - u(t)] + (-2t+4)[u(t) - u(t-2)]$$
$$= (4t+1)u(t+1) - 6tu(t) + 3u(t) + (2t-4)u(t-2)$$

(b)

$$x_2(t) = t^2[u(t) - u(t-2)] + (2t-8)[u(t-2) - u(t-4)]$$
$$= t^2 u(t) - (t^2 - 2t + 8)u(t-2) - (2t-8)u(t-4)$$

Solution 1.4-3

(a) Signal $w(t)$ is a unit-duration ramp. Signal $x(t)$ is a periodic replication of a compressed-by-2 version of $w(t)$ interlaced with a periodic replication of a compressed-by-2, negated, and shifted version of $w(t)$. Both $w(t)$ and $x(t)$ are shown in Fig. S1.4-3. By inspection, it is clear that the fundamental period of $x(t)$ is $T_0 = 1$.

(b) To sketch $y(t) = \frac{d}{dt}x(1 - 0.5t)$, we first plot $x(1 - 0.5t)$ and then graphically differentiate the waveform to obtain $y(t)$. Both $x(1 - 0.5t)$ and $y(t)$ are shown in Fig. S1.4-3.

(c) To assist in finding the energy E_z and power P_z of the signal $z(t) = x(0.5 - 1.5t)[u(t) - u(t - 1)]$, we first sketch $z(t)$. As shown in Fig. S1.4-3, $z(t)$ is a finite-duration signal, which means it is an energy signal and $P_z = 0$. Further, $z(t)$ is comprised of three triangular pieces. The energy of the first triangular piece is

$$E_{\text{tri}} = \int_0^{\frac{1}{3}} (1 - 3t)^2 \, dt = \left. \frac{(1 - 3t)^3}{-9} \right|_{t=0}^{\frac{1}{3}} = \frac{1}{9}.$$

Since shifting and reflecting a signal do not impact energy, the energy of the third triangle equals the energy of the first. The second triangle, which is like the first scaled by $-\frac{1}{2}$, has energy $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ as large as the first triangle. Since the triangles are non-overlapping, E_z is just the sum of the energies of the three pieces. That is, $E_z = \frac{1}{9} + \frac{1}{36} + \frac{1}{9} = \frac{1}{4}$. Thus,

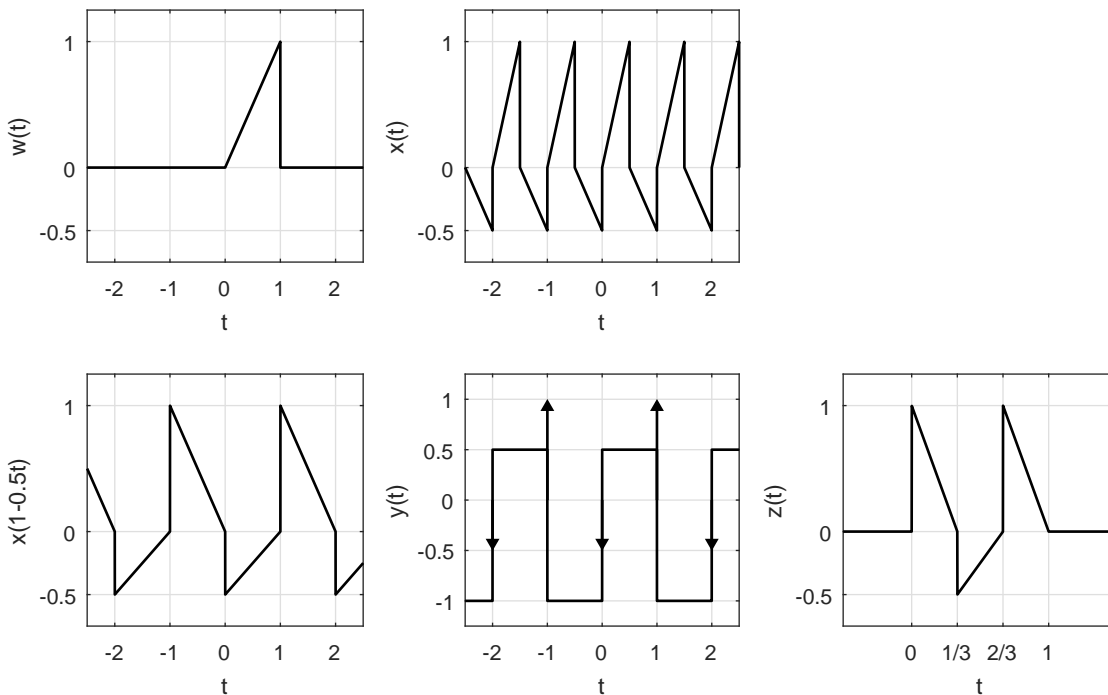signal $z(t)$ has energy $E_z = \frac{1}{4}$ and power $P_z = 0$.

Figure S1.4-3

Solution 1.4-4

(a) To sketch $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$, we first sketch $x(t)$ (see Fig. S1.4-4). Now, $y(t)$ can be obtained graphically as an accumulation of area of $x(t)$ as we move from left to right. Alternatively, we can analytically compute and then plot $y(t)$.

$$
y(t) = \begin{cases}
0 & t < 1 \\
\int_{1}^{t} d\tau = t - 1 & 1 \le t < 2.5 \\
1.5 & 2.5 \le t < 4 \\
1.5 - \int_{-\infty}^{4} 2\delta(\tau - 4)\, d\tau = 1.5 - 2 = -0.5 & 4 \le t < 6 \\
-0.5 + \int_{-\infty}^{6} 2\delta(\tau - 6)\, d\tau = -0.5 + 1 = 0.5 & t \ge 6
\end{cases}
$$

(b) If we adjust the weight of the right-most delta function from 1 to $\frac{1}{2}$, then $y(t)$ will be end at $t = 6$ and have finite duration and finite energy. That is,

if $\delta(t - 6)$ in $x(t)$ is changed to $\frac{1}{2}\delta(t - 6)$, then $y(t)$ will have finite energy.

(c) Signal $z(t) = \int_{t}^{\infty} x(\tau)\, d\tau$, which is the accumulation of area of $x(t)$ as we move from right to left, can be sketched by inspection of $x(t)$ (see Fig. S1.4-4). Alternatively, we can analytically compute and then plot $z(t)$.

$$
z(t) = \begin{cases}
0 & t > 6 \\
\int_{6}^{\infty} \delta(\tau - 6) d\tau = 1 & 4 < t \le 6 \\
1 - \int_{4}^{-\infty} 2\delta(\tau - 4)\, d\tau = 1 - 2 = -1 & 2.5 < t \le 4 \\
-1 + \int_{t}^{2.5} d\tau = -1 + (2.5 - t) = 1.5 - t & 1 < t \le 2.5 \\
0.5 & t \le 1
\end{cases}
$$

(d) There are two ways to determine real constants A and B so that $w(t) = x\left(\frac{t-A}{B}\right)$ has a region of support $[-2, 2]$. In the first way, we map the leftmost edge of $x(t)$ to the leftmost edge of $w(t)$, and map the rightmost edge of $x(t)$ to the rightmost edge of $w(t)$. This requires that

$$z(-2) = x(\tfrac{-2-A}{B}) = x(1) \quad \Rightarrow \quad A + B = -2$$

and

$$z(2) = x(\tfrac{2-A}{B}) = x(6) \quad \Rightarrow \quad A + 6B = 2.$$

Solving this pair of equations yields

$$A = -\frac{14}{5} \quad \text{and} \quad B = \frac{4}{5}.$$

The resulting signal $w(t)$ is shown in Fig. S1.4-4.

The second way to solve this problem is to map the leftmost edge of $x(t)$ to the rightmost edge of $w(t)$, and map the rightmost edge of $x(t)$ to the leftmost edge of $w(t)$. This requires that

$$z(2) = x(\tfrac{2-A}{B}) = x(1) \quad \Rightarrow \quad A + B = 2$$

and

$$z(-2) = x(\tfrac{-2-A}{B}) = x(6) \quad \Rightarrow \quad A + 6B = -2.$$

Solving this pair of equations yields

$$A = \frac{14}{5} \quad \text{and} \quad B = -\frac{4}{5}.$$

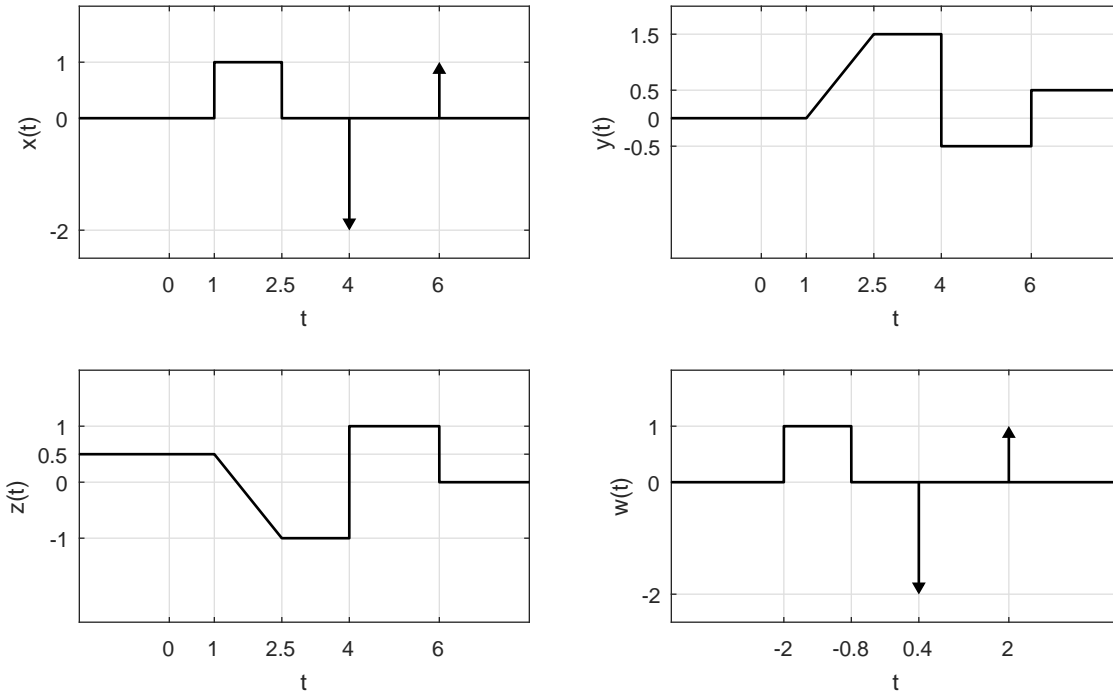The resulting $w(t)$ is just a reflection of the first solution.

Figure S1.4-4

Solution 1.4-5

Using the fact that $f(x)\delta(x) = f(0)\delta(x)$, we have

(a) 0

(b) $\frac{2}{9}\delta(\omega)$

(c) $\frac{1}{2}\delta(t)$

(d) $-\frac{1}{5}\delta(t-1)$

(e) $\frac{1}{2-j3}\delta(\omega+3)$

(f) $k\delta(\omega)$ (use L' Hôpital's rule)

Solution 1.4-6

In these problems remember that impulse $\delta(x)$ is located at $x = 0$. Thus, an impulse $\delta(t-\tau)$ is located at $\tau = t$, and so on.

(a) The impulse is located at $\tau = t$ and $x(\tau)$ at $\tau = t$ is $x(t)$. Therefore

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau = x(t).$$

(b) The impulse $\delta(\tau)$ is at $\tau = 0$ and $x(t-\tau)$ at $\tau = 0$ is $x(t)$. Therefore

$$\int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)\, d\tau = x(t).$$

Using similar arguments, we obtain

(c) 1

(d) 0

(e) e^3

(f) 5

(g) $x(-1)$

(h) $-e^2$

Solution 1.4-7

In this problem we assume that the constant a is real and positive constant a. Letting $t' = at$ and $dt' = a dt$, we see that

$$\int_{-\infty}^{\infty} \delta(at)\, dt = \int_{-\infty}^{\infty} \delta(t') \frac{dt'}{a} = \frac{1}{a}$$

In this way, we see that time-scaling a delta function by (positive) factor a reciprocally changes the strength of the delta function as $\frac{1}{a}$.

Solution 1.4-8

(a) Recall that the derivative of a function at the jump discontinuity is equal to an impulse of strength equal to the amount of discontinuity. Hence, dx/dt contains impulses $4\delta(t+4)$ and $2\delta(t-2)$. In addition, the derivative is -1 over the interval $(-4, 0)$, and is 1 over the interval $(0, 2)$. The derivative is zero for $t < -4$ and $t > 2$. The result dx/dt is shown in Fig. S1.4-8.

(b) Graphically differentiating Fig. P1.4-2a we obtain

$$\frac{dx_1(t)}{dt} = 4[u(t+1) - u(t)] - 2[u(t) - u(t-2)] = 4u(t+1) - 6u(t) + 2u(t-2).$$

Differentiating again, we obtain,

$$\frac{d^2x_1(t)}{dt^2} = 4\delta(t+1) - 6\delta(t) + 2\delta(t-2).$$

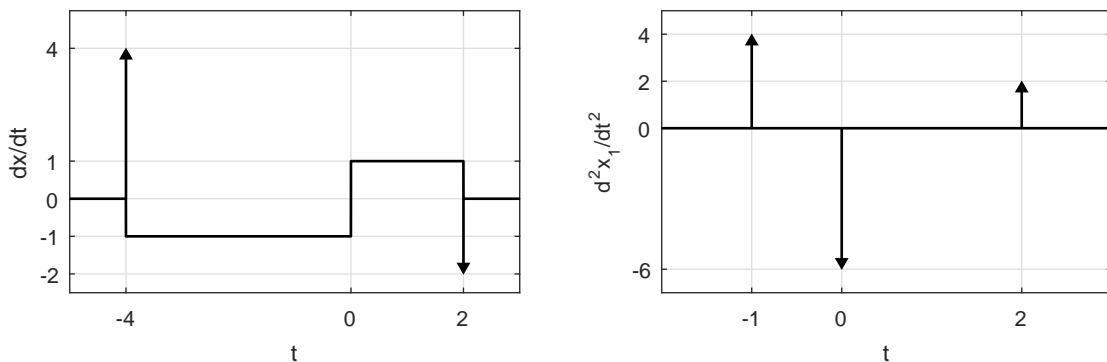This result is also shown in Fig. S1.4-8.

Figure S1.4-8

Solution 1.4-9

For convenience, define $y(t) = \int_{-\infty}^{t} x(t)\, dt$. For sketches, refer to Fig. S1.4-9.

(a) Recall that the area under an impulse of strength k is k. Over the interval $0 \leq t < 1$, we have

$$y_a(t) = \int_0^t 1\, dx = t \qquad 0 \leq t < 1.$$

Over the interval $0 \leq t < 3$, we have

$$y_a(t) = \int_0^1 1\, dx + \int_1^t (-1)\, dx = 2 - t \qquad 1 \leq t < 3.$$

At $t = 3$, the impulse (of strength unity) yields an additional term of unity. Thus (assuming $\epsilon \to 0$),

$$y_a(t) = \int_0^1 1\, dx + \int_1^{3-\epsilon} (-1)\, dx + \int_{3-\epsilon}^t \delta(x - 3)\, dx = 1 + (-2) + 1 = 0 \qquad t > 3$$

Putting the pieces together, we have

$$y_a(t) = \begin{cases} t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

(b) By inspection,

$$y_b(t) = \int_0^t [1 - \delta(x-1) - \delta(x-2) - \delta(x-3) - \cdots]\, dx = tu(t) - u(t-1) - u(t-2) - u(t-3) - \ldots$$
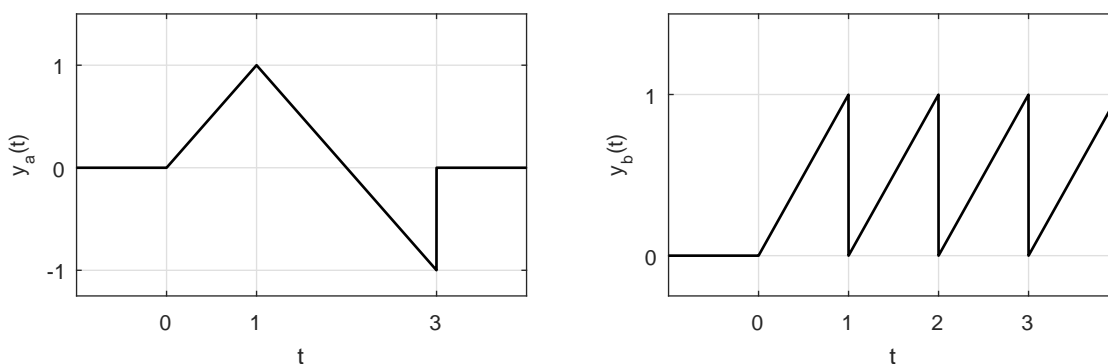

Figure S1.4-9

Solution 1.4-10

Changing the variable t to $-x$, we obtain

$$\int_{-\infty}^{\infty} \phi(t)\delta(-t)\, dt = -\int_{\infty}^{-\infty} \phi(-x)\delta(x)\, dx = \int_{-\infty}^{\infty} \phi(-x)\delta(x)\, dx = \phi(0).$$

This shows that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt = \int_{-\infty}^{\infty} \phi(t)\delta(-t)\, dt = \phi(0).$$

Therefore

$$\delta(t) = \delta(-t).$$

Solution 1.4-11

Letting $at = x$, we obtain (for $a > 0$)

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{a} \int_{-\infty}^{\infty} \phi(\frac{x}{a})\delta(x)\, dx = \frac{1}{a}\phi(0)$$

Similarly for $a < 0$, we show that this integral is $-\frac{1}{a}\phi(0)$. Therefore

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{|a|}\phi(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt$$

Therefore

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Solution 1.4-12

$$\int_{-\infty}^{\infty} \dot{\delta}(t)\phi(t)\, dt = \phi(t)\delta(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \dot{\phi}(t)\delta(t)\, dt$$

$$= 0 - \int \dot{\phi}(t)\delta(t)\, dt = -\dot{\phi}(0)$$
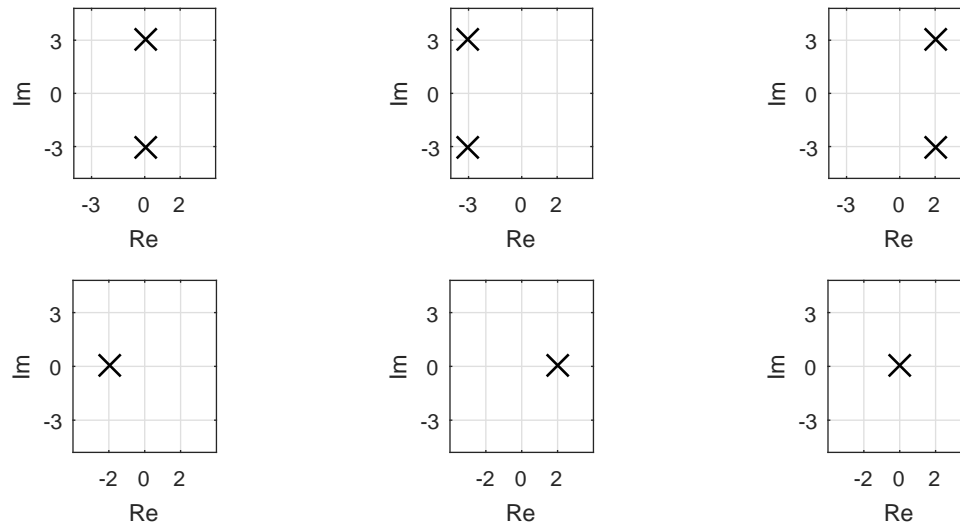
Solution 1.4-13

For sketches, refer to Fig. S1.4-13.

Figure S1.4-13

(a) $s_{1,2} = \pm j3$

(b) $e^{-3t}\cos 3t = 0.5[e^{-(3+j3)t} + e^{-(3-j3)t}]$. Therefore the frequencies are $s_{1,2} = -3 \pm j3$.

(c) Using the argument in (b), we find the frequencies $s_{1,2} = 2 \pm j3$

(d) $s = -2$

(e) $s = 2$

(f) $5 = 5e^{0t}$ so that $s = 0$.

Solution 1.5-1

(a)

$$x_{\mathrm{e}}(t) = 0.5[u(t) + u(-t)] = \begin{cases} 0.5 & t \neq 0 \\ 1 & t = 0 \end{cases}$$

$$x_{\mathrm{o}}(t) = 0.5[u(t) - u(-t)] = \begin{cases} 0.5 & t > 0 \\ 0 & t = 0 \\ -0.5 & t < 0 \end{cases}$$
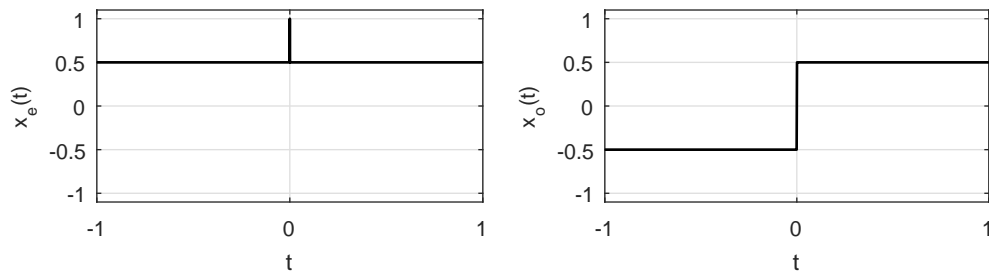
These component signals are shown in Fig. S1.5-1a.

Figure S1.5-1a

(b)

$$x_{\mathrm{e}}(t) = 0.5[tu(t) - tu(-t)] = 0.5|t|$$
$$x_{\mathrm{o}}(t) = 0.5[tu(t) + tu(-t)] = 0.5t$$
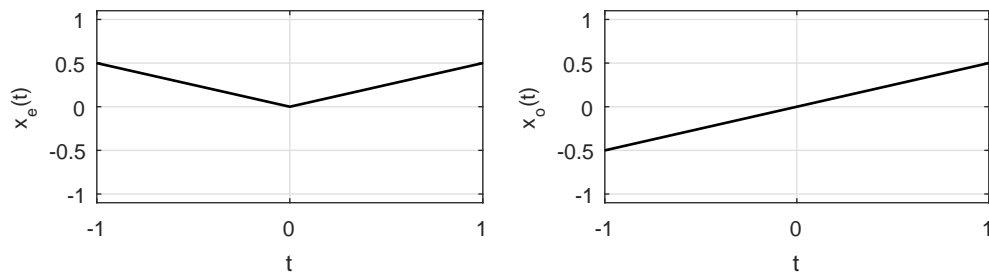
These component signals are shown in Fig. S1.5-1b.

Figure S1.5-1b

(c)

$$x_{\mathrm{e}}(t) = 0.5[\sin \omega_0 t + \sin(-\omega_0 t)] = 0$$
$$x_{\mathrm{o}}(t) = 0.5[\sin \omega_0 t - \sin(-\omega_0 t)] = \sin \omega_0 t$$

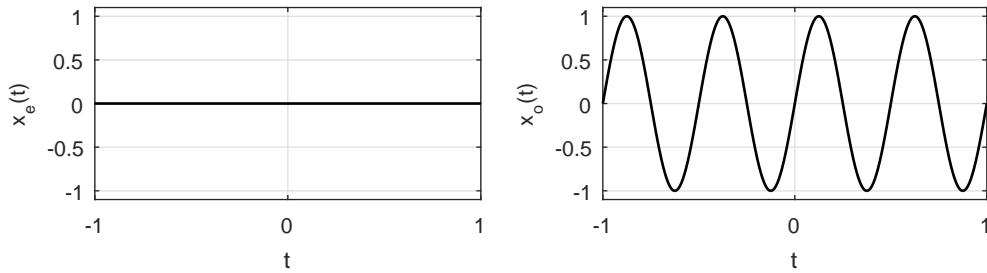These component signals are shown in Fig. S1.5-1c for an example frequency $\omega_0 = 2\pi 2$.

Figure S1.5-1c

(d)

$$x_e(t) = 0.5[\cos \omega_0 t + \cos(-\omega_0 t)] = \cos \omega_0 t$$
$$x_o(t) = 0.5[\cos \omega_0 t - \cos(-\omega_0 t)] = 0$$

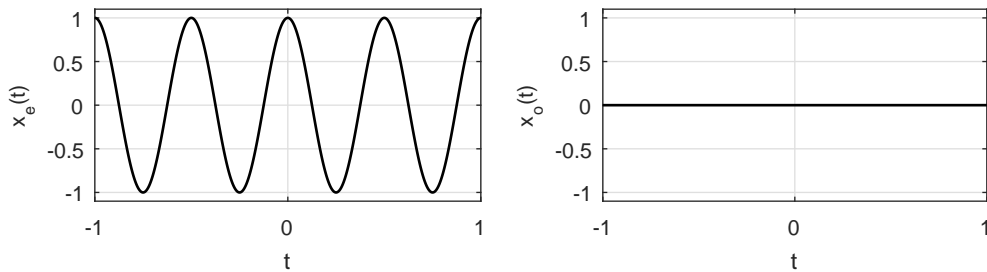These component signals are shown in Fig. S1.5-1d for an example frequency $\omega_0 = 2\pi 2$.

Figure S1.5-1d

(e) Using the results of parts (c) and (d) as well as the fact that $\cos(\omega_0 t + \theta) = \cos \omega_0 t \cos \theta - \sin \omega_0 t \sin \theta$, we see that

$$x_e(t) = \cos \theta \cos \omega_0 t$$
$$x_o(t) = -\sin \theta \sin \omega_0 t$$

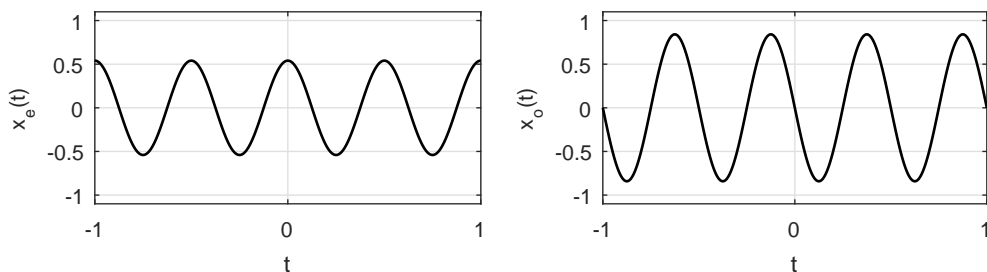These component signals are shown in Fig. S1.5-1e for $\omega_0 = 2\pi 2$ and $\theta = 1$.

Figure S1.5-1e

(f)

$$x_e(t) = 0.5[\sin \omega_0 t\, u(t) + \sin(-\omega_0 t)u(-t)] = 0.5[\sin \omega_0 t\, u(t) - \sin \omega_0 t\, u(-t)]$$
$$x_o(t) = 0.5[\sin \omega_0 t\, u(t) - \sin(-\omega_0 t)u(-t)] = 0.5[\sin \omega_0 t\, u(t) + \sin \omega_0 t\, u(-t)] = 0.5 \sin \omega_0 t$$

These component signals are shown in Fig. S1.5-1f for an example frequency $\omega_0 = 2\pi 2$.
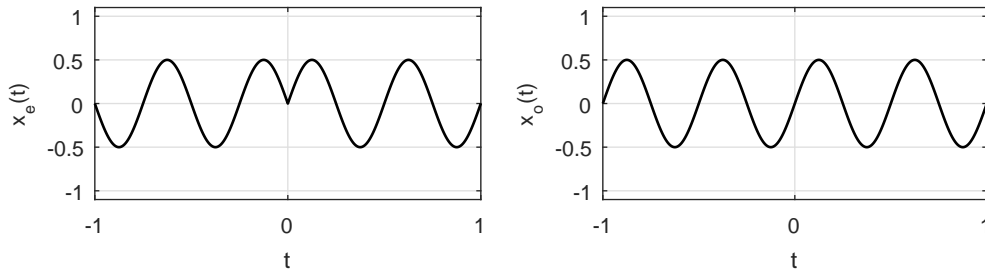
Figure S1.5-1f

(g)

$$x_e(t) = 0.5[\cos\omega_0 t\, u(t) + \cos(-\omega_0 t)u(-t)] = \begin{cases} 0.5\cos\omega_0 t & t \neq 0 \\ 1 & t = 0 \end{cases}$$

$$x_o(t) = 0.5[\cos\omega_0 t\, u(t) - \cos(-\omega_0 t)u(-t)] = 0.5[\cos\omega_0 t\, u(t) - \cos\omega_0 t\, u(-t)]$$

These component signals are shown in Fig. S1.5-1g for an example frequency $\omega_0 = 2\pi 2$.

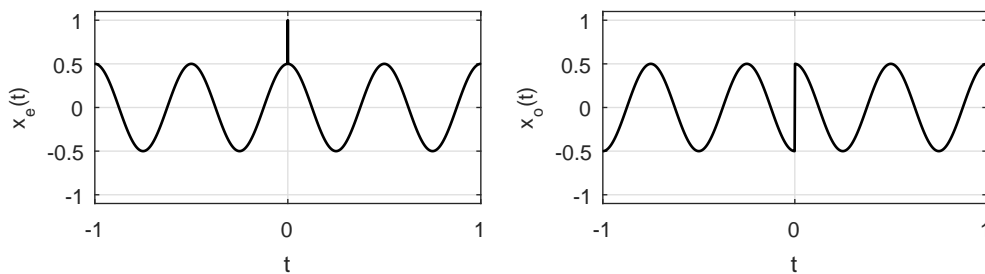Figure S1.5-1g

Solution 1.5-2

MATLAB makes it easy to compute and plot the desired signals. To determine suitable ranges of t for the plots, however, we need to compute the edges of the desired waveforms.

(a) Since $x(t)$ has edges at $t = -1$ and $t = 3$, the signal $x_o(t)$ has edges at ± 3. Thus, the signal $x_o(1 - 2t)$ has edges at $1 - 2t = \pm 3$ or at $t = -1$ and $t = 2$. We chose the slightly wider interval of $-2 \leq t \leq 3$ for our plot.

```
>>  u = @(t) 1.0*(t>=0); x = @(t) 2*u(t+1)-u(t-2)-u(t-3);
>>  t = -2:.001:3; xo = @(t) (x(t)-x(-t))/2;
>>  subplot(121); plot(t,xo(1-2*t)); xlabel('t'); ylabel('x_o(1-2t)');
>>  axis([-2 3 -1.25 1.25]); set(gca,'xtick',-1:.5:2,'ytick',-1:.5:1); grid on
```

(b) Since $x(t)$ has edges at $t = -1$ and $t = 3$, the signal $x_e(t)$ has edges at ± 3. Thus, the signal $x_e(2 + t/3)$ has edges at $2 + t/3 = \pm 3$ or at $t = -15$ and $t = 3$. We chose the slightly wider interval of $-17 \leq t \leq 5$ for our plot.

```
>>  t = -17:.001:5; xe = @(t) (x(t)+x(-t))/2;
>>  subplot(122); plot(t,xe(2+t/3)); xlabel('t'); ylabel('x_e(2+t/3)');
>>  axis([-17 5 -.25 2.25]); set(gca,'xtick',-15:3:3,'ytick',0:.5:2); grid on
```
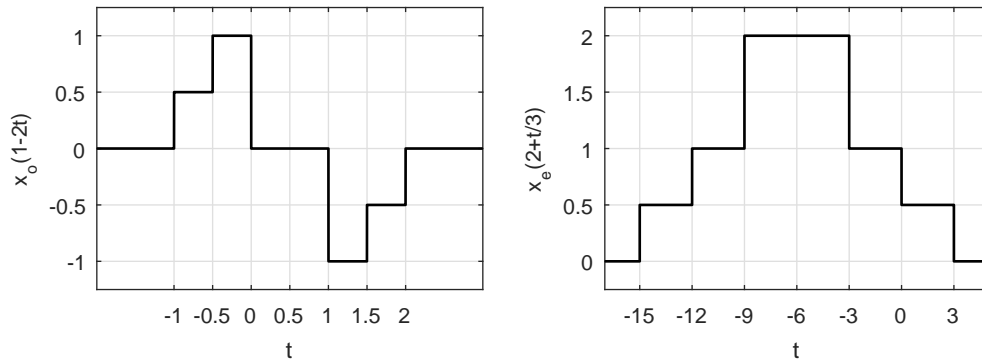
Figure S1.5-2

Solution 1.5-3

(a)

$$x_e(t) = \frac{1}{2}[e^{-2t}u(t) + e^{2t}u(-t)]$$

$$x_o(t) = \frac{1}{2}[e^{-2t}u(t) - e^{2t}u(-t)]$$

(b) $E_{x_e} = \int_{-\infty}^{\infty} x_e^2(t)dt$. Because $e^{-2t}u(t)$ and $e^{2t}u(-t)$ are disjoint in time, the cross-product term in $x_e^2(t)$ is zero. Hence,

$$E_{x_e} = \int_{-\infty}^{\infty} x_e^2(t)dt = \frac{1}{4}\left[\int_0^{\infty} e^{-4t}dt + \int_{-\infty}^0 e^{4t}dt\right] = \frac{1}{8}.$$

Using a similar argument, we have

$$E_{x_o} = \frac{1}{8}.$$

Also,

$$E_x = \int_0^{\infty} e^{-4t}dt = \frac{1}{4}.$$

Hence,

$$E_x = E_{x_e} + E_{x_o}.$$

(c) To generalize this result, we first consider causal $x(t)$. In this case, $x(t)$ and $x(-t)$ are disjoint. Moreover, energy or $x(t)$ is identical to that of $x(-t)$. Hence,

$$E_{x_e} = \frac{1}{4}\left[\int_0^{\infty} |x(t)|^2 dt + \int_{-\infty}^0 |x(-t)|^2 dt\right] = \frac{1}{2}E_x.$$

Using a similar argument, it follows that $E_{x_o} = \frac{1}{2}E_x$. Hence, for causal signals,

$$E_x = E_{x_e} + E_{x_o}$$

Identical arguments hold for anti-causal signals. Thus, for anti-causal signal $x(t)$

$$E_x = E_{x_e} + E_{x_o}$$

Now, every signal can be expressed as a sum of a causal and an anti-causal signal. Also, the signal energy is equal to the sum of energies of the causal and the anti-causal components. Hence, it follows that for a general case

$$E_x = E_{x_e} + E_{x_o}$$

Solution 1.5-4

(a)

$$x_e(t)x_o(t) = \frac{1}{4}[x(t) + x(-t)][x(t) - x(-t)]$$

$$= \frac{1}{4}[|x(t)|^2 - |x(-t)|^2]$$

Since the areas under $|x(t)|^2$ and $|x(-t)|^2$ are identical, it follows that

$$\int_{-\infty}^{\infty} x_e(t)x_o(t)dt = 0$$

(b)

$$\int_{-\infty}^{\infty} x_e(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} x(t)dt + \frac{1}{2}\int_{-\infty}^{\infty} x(-t)dt$$

Because the areas under $x(t)$ and $x(-t)$ are identical, it follows that

$$\int_{-\infty}^{\infty} x_e(t)dt = \int_{-\infty}^{\infty} x(t)dt.$$

Solution 1.5-5

$x_o(t) = 0.5(x(t) - x(-t)) = 0.5(\sin(\pi t)u(t) - \sin(-\pi t)u(-t)) = 0.5\sin(\pi t)(u(t) + u(-t))$. Since $\sin(0) = 0$, this reduces to $x_o(t) = 0.5\sin(\pi t)$, which is a $(T = 2)$-periodic signal. Therefore,

$$x_o(t) = 0.5\sin(\pi t) \text{ is a periodic signal.}$$

Solution 1.5-6

$x_e(t) = 0.5(x(t) + x(-t)) = 0.5(\cos(\pi t)u(t) + \cos(-\pi t)u(-t)) = 0.5\cos(\pi t)(u(t) + u(-t))$. Written another way, $x_e(t) = \begin{cases} 0.5\cos(\pi t) & t \neq 0 \\ 1 & t = 0 \end{cases}$. Since there exists no $T \neq 0$ such that $x_e(t+T) = x_e(t)$,

$$x_e(t) \text{ is not a periodic function.}$$

It is worth pointing out that sometimes the unit step is defined as $u(t) = \begin{cases} 1 & t > 0 \\ 0.5 & t = 0 \\ 0 & t < 0 \end{cases}$. Using this alternate definition, $x_e(t)$ is periodic.

Solution 1.5-7

(a) Using the figure, $x(t) = (t+1)(u(t+1) - u(t)) + (-t+1)(u(t) - u(t-1))$. MATLAB is used to plot $v(t) = 3x\left(-\frac{1}{2}(t+1)\right)$.

```
>>  u = @(t) 1.0*(t>=0);
>>  x = @(t) (t+1).*(u(t+1)-u(t))+(-t+1).*(u(t)-u(t-1));
>>  v = @(t) 3*x(-0.5*(t+1)); t = -4:.001:4;
>>  plot(t,v(t),'k-'); xlabel('t'); ylabel('v(t)');
>>  axis([-4 4 -.25 3.25]); set(gca,'xtick',[-3 -1 1],'ytick',[0 3]); grid on
```
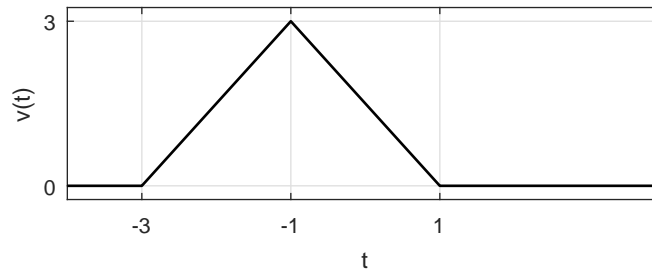
Figure S1.5-7a

(b) Since $v(t)$ is finite duration, $P_v = 0$. Signal is unaffected by shifting, so $v(t)$ is shifted to start at $t = 0$. By symmetry, the energy of the first half is equal to the energy of the second half. Thus, $E_v = 2 \int_0^2 \left(\frac{3}{2}t\right)^2 dt = 2\frac{9}{4}\frac{t^3}{3}\Big|_{t=0}^{t=2} = \frac{24}{2} = 12$. Thus,

$$E_v = 12 \text{ and } P_v = 0.$$

(c) Using the definition $v_e(t) = (v(t) + v(-t))/2$, MATLAB is used to determine and plot $v_e(t)$.

```
>>   ve = @(t) (v(t)+v(-t))/2; t = -4:.001:4;
>>   plot(t,ve(t),'k-'); xlabel('t'); ylabel('v_e(t)');
>>   axis([-4 4 -.25 3.25]); set(gca,'xtick',[-3 -1 1 3],'ytick',[0 1.5]); grid on
```
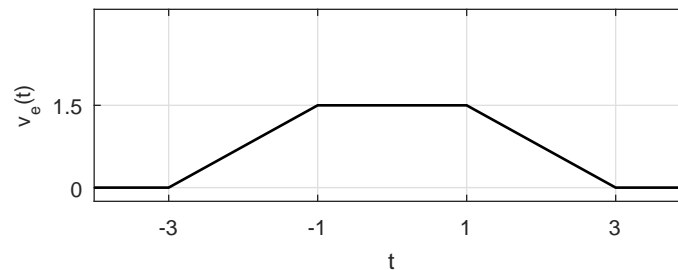

Figure S1.5-7c

Thus,

$$
v_e(t) = \begin{cases}
\frac{3t}{4} + \frac{9}{4} & -3 \le t < -1 \\
\frac{3}{2} & -1 \le t < 1 \\
-\frac{3t}{4} + \frac{9}{4} & 1 \le t < 3 \\
0 & \text{otherwise}
\end{cases}.
$$

(d) Using $v(t)$ computed earlier, MATLAB is used to create the four desired plots.

```
>>   t = [-7:.001:2]; a = 2; b = 3; ax = [-7 2 -.5 9.5];
>>   subplot(221); plot(t,v(a*t+b),'k-'); xlabel('t'); ylabel('v(at+b)');
>>   axis(ax); grid on; set(gca,'xtick',[-3 -2 -1],'ytick',[0 3]);
>>   subplot(222); plot(t,v(a*t)+b,'k-'); xlabel('t'); ylabel('v(at)+b');
>>   axis(ax); grid on; set(gca,'xtick',[-1.5 -.5 .5],'ytick',[0 3 6]);
>>   subplot(223); plot(t,a*v(t+b),'k-'); xlabel('t'); ylabel('av(t+b)');
>>   axis(ax); grid on; set(gca,'xtick',[-6 -4 -2],'ytick',[0 6]);
>>   subplot(224); plot(t,a*v(t)+b,'k-'); xlabel('t'); ylabel('av(t)+b');
>>   axis(ax); grid on; set(gca,'xtick',[-3 -1 1],'ytick',[0 3 9]);
```

(e) Following the same procedure as in part (d), MATLAB is used to create the four desired plots.
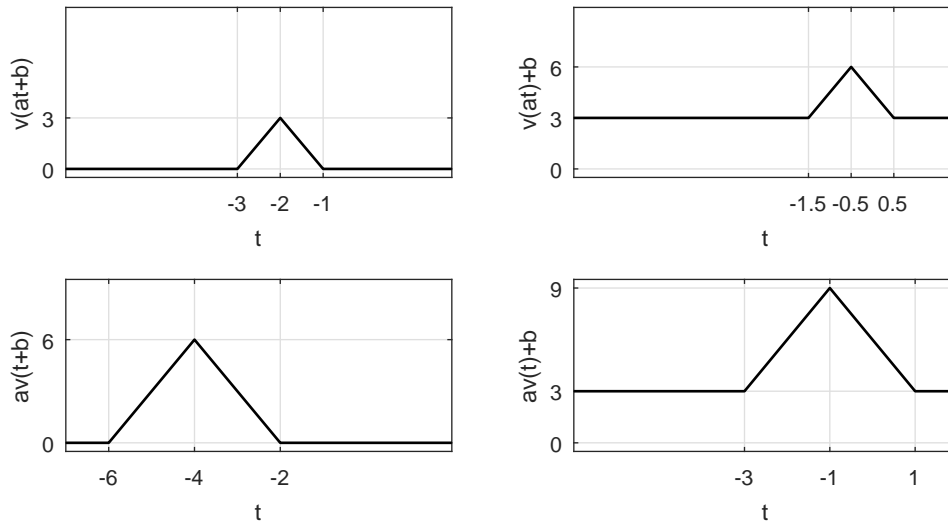
Figure S1.5-7d

```
>>  t = [-3.5:.001:3.5]; a = -3; b = -2; ax = [-3.5 3.5 -11.5 3.5];
>>  subplot(221); plot(t,v(a*t+b),'k-'); xlabel('t'); ylabel('v(at+b)');
>>  axis(ax) ;grid on; set(gca,'xtick',[-1 -1/3 1/3],'ytick',[0 3]);
>>  set(gca,'xticklabel',{'-1','-1/3','1/3'});
>>  subplot(222); plot(t,v(a*t)+b,'k-'); xlabel('t'); ylabel('v(at)+b');
>>  axis(ax); grid on; set(gca,'xtick',[-1/3 1/3 1],'ytick',[-2 1]);
>>  set(gca,'xticklabel',{'-1/3','1/3','1'});
>>  subplot(223); plot(t,a*v(t+b),'k-'); xlabel('t'); ylabel('av(t+b)');
>>  axis(ax); grid on; set(gca,'xtick',[-1 1 3],'ytick',[-9 0]);
>>  subplot(224); plot(t,a*v(t)+b,'k-'); xlabel('t'); ylabel('av(t)+b');
>>  axis(ax); grid on; set(gca,'xtick',[-3 -1 1],'ytick',[-11 -2]);
```
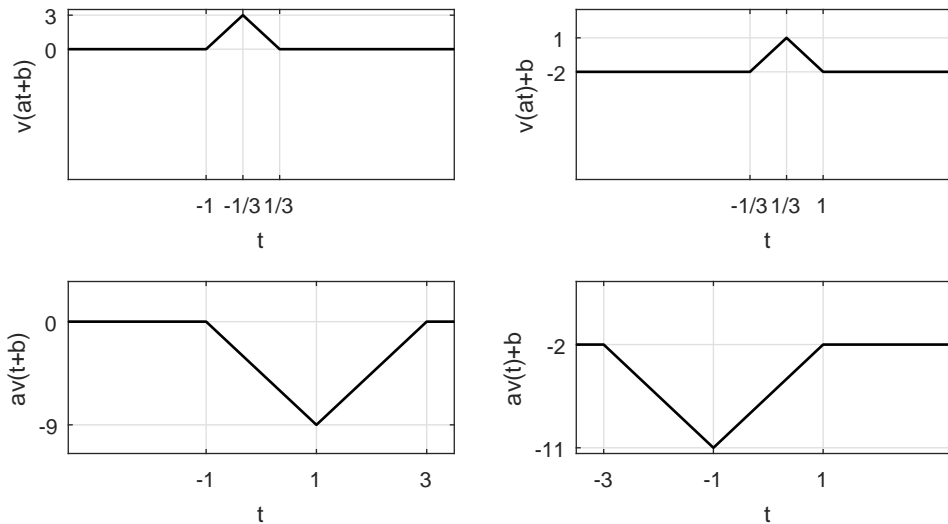

Figure S1.5-7e

Solution 1.5-8

(a) Using the figure, $y(t) = t(u(t) - u(t-1)) + (u(t-1) - u(t-2))$. MATLAB is used to plot $y_o(t) = \frac{y(t) - y(-t)}{2}$.

```
>>  u = @(t) 1.0*(t>=0); y = @(t) t.*(u(t)-u(t-1))+1.0*(u(t-1)-u(t-2));
>>  t = [-3:.001:3]; yo = (y(t)-y(-t))/2;
>>  plot(t,yo,'k-'); xlabel('t'); ylabel('y_o(t)'); axis([-3 3 -.6 .6]);
>>  set(gca,'xtick',[-2 -1 0 1 2],'ytick',[-.5 0 .5]); grid on
```
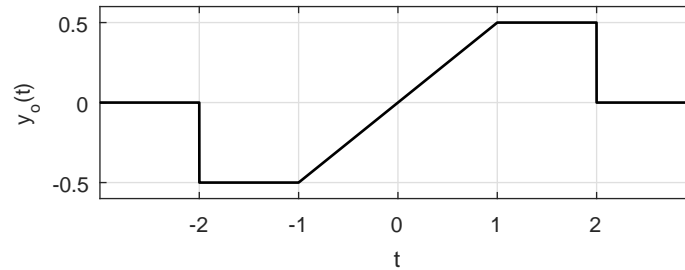

Figure S1.5-8a

Thus,

$$
y_o(t) = \begin{cases} -1/2 & -2 \le t < -1 \\ t/2 & -1 \le t < 1 \\ 1/2 & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} .
$$

(b) Since $y(t) = 0.2x(-2t-3)$, $5y(-0.5t - 1.5) = 5(0.2)x(-2(-0.5t - 1.5) - 3) = x(t)$. MATLAB is used to sketch $x(t)$.

```
>>  u = @(t) 1.0*(t>=0); y = @(t) t.*(u(t)-u(t-1))+1.0*(u(t-1)-u(t-2));
>>  t = [-8:.001:0]; x = @(t) 5*y(-0.5*t-1.5);
>>  plot(t,x(t),'k-'); xlabel('t'); ylabel('x(t)'); axis([-8 0 -.5 5.5]);
>>  set(gca,'xtick',[-7 -5 -3],'ytick',[0 5]); grid on
```


Figure S1.5-8b

Thus,

$$
x(t) = \begin{cases} 5 & -7 \le t < -5 \\ -5(t+3)/2 & -5 \le t < -3 \\ 0 & \text{otherwise} \end{cases} .
$$

Solution 1.5-9

For convenience, let us define the problem's graphed signal as $y(t) = -0.5x(-3t + 2)$.

(a) Since $y(t) = -0.5x(-3t+2)$, $-2y(-t/3+2/3) = -2(-0.5)x(-3(-t/3+2/3)+2) = x(t)$.
    MATLAB is used to sketch $x(t)$.

```
>>  u = @(t) 1.0*(t>=0); y = @(t) 1.0*(u(t+1)-u(t))+(-t+1).*(u(t)-u(t-1));
>>  t = [-2:.001:6]; x = @(t) -2*y(-t/3+2/3);
>>  plot(t,x(t),'k-'); xlabel('t'); ylabel('x(t)'); axis([-2 6 -2.5 0.5]);
>>  set(gca,'xtick',[-1 2 5],'ytick',[-2 0]); grid on
```
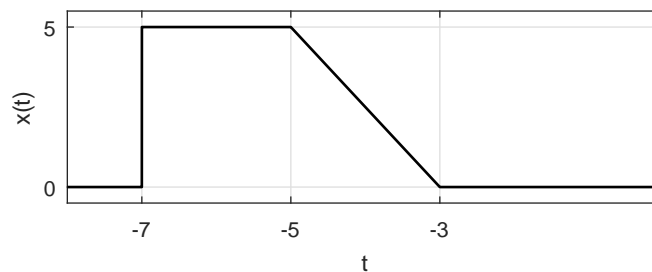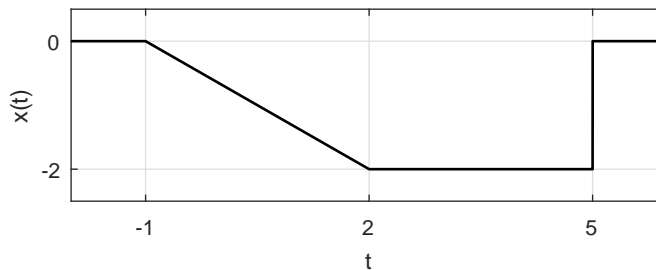

Figure S1.5-9a

Thus,

$$x(t) = \begin{cases} -2(t+1)/3 & -1 \le t < 2 \\ -2 & 2 \le t < 5 \\ 0 & \text{otherwise} \end{cases}.$$

(b) The even portion of $x(t)$ is $x_e(t) = 0.5(x(t) + x(-t))$.

```
>>  t = [-6:.001:6]; xe = @(t) (x(t)+x(-t))/2;
>>  plot(t,xe(t),'k-'); xlabel('t'); ylabel('x_e(t)'); axis([-6 6 -1.25 1.25]);
>>  set(gca,'xtick',[-5 -2 -1 1 2 5],'ytick',[-1 -2/3 0]); grid on
>>  set(gca,'yticklabel',{'-1','-2/3','0'});
```


Figure S1.5-9b

Thus,

$$x_e(t) = \begin{cases} -1 & 2 \le |t| < 5 \\ (-|t| - 1)/3 & 1 \le |t| < 2 \\ -2/3 & |t| < 1 \\ 0 & \text{otherwise} \end{cases}.$$

(c) The odd portion of $x(t)$ is $x_o(t) = 0.5(x(t) - x(-t))$.

```
>>  t = [-6:.001:6]; xo = @(t) (x(t)-x(-t))/2;
>>  plot(t,xo(t),'k-'); xlabel('t'); ylabel('x_o(t)'); axis([-6 6 -1.25 1.25]);
>>  set(gca,'xtick',[-5 -2 -1 1 2 5],'ytick',[-1 -2/3 0 2/3 1]); grid on
>>  set(gca,'yticklabel',{'-1','-2/3','0','2/3','1'});
```
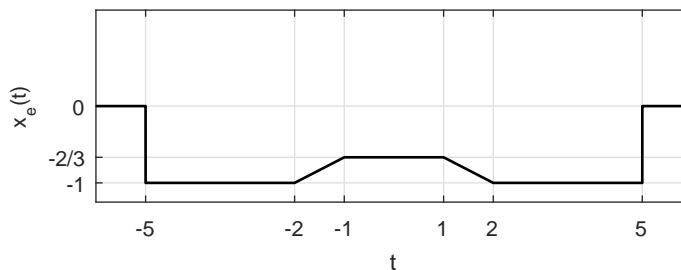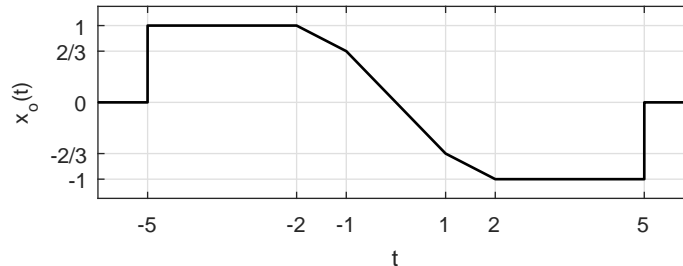

Figure S1.5-9c

Thus,

$$
x_\mathrm{o}(t) = \begin{cases}
1 & -5 \le t < -2 \\
(-t+1)/3 & -2 \le t < -1 \\
-2t/3 & -1 \le t < 1 \\
-(t+1)/3 & 1 \le t < 2 \\
-1 & 2 \le t < 5 \\
0 & \text{otherwise}
\end{cases}.
$$

Solution 1.5-10

Notice, $w_{cs}^*(-t) = 0.5(w(-t) + w^*(t))^* = 0.5(w^*(-t) + w(t)) = w_{cs}(t)$. In Cartesian form, this becomes $w_{cs}^*(-t) = x(-t) - jy(-t) = x(t) + jy(t) = w_{cs}(t)$. Equating the real portions yields $x(-t) = x(t)$, and equating the imaginary portions yields $-y(-t) = y(t)$. Thus, by definition, the real portion of $w_{cs}(t)$ is even and the imaginary portion of $w_{cs}(t)$ is odd.

Solution 1.5-11

Notice, $-w_{ca}^*(-t) = -0.5(w(-t) - w^*(t))^* = 0.5(-w^*(-t) + w(t)) = w_{ca}(t)$. In Cartesian form, this becomes $-w_{ca}^*(-t) = -x(-t) + jy(-t) = x(t) + jy(t) = w_{ca}(t)$. Equating the real portions yields $-x(-t) = x(t)$, and equating the imaginary portions yields $y(-t) = y(t)$. Thus, by definition, the real portion of $w_{ca}(t)$ is odd and the imaginary portion of $w_{ca}(t)$ is even.

Solution 1.5-12

In this problem, $w(t) = e^{j(t+\pi/4)}$.

(a) Using the definition in Prob. 1.5-10,

$$
w_{cs}(t) = \frac{w(t) + w^*(-t)}{2} = \frac{e^{j(t+\frac{\pi}{4})} + e^{-j(-t+\frac{\pi}{4})}}{2} = e^{jt}\frac{e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}}}{2} = e^{jt}\cos(\pi/4).
$$

Expressed in standard rectangular form,

$$
w_{cs}(t) = \cos(\pi/4)\cos(t) + j\cos(\pi/4)\sin(t) = \frac{1}{\sqrt{2}}\cos(t) + j\frac{1}{\sqrt{2}}\sin(t).
$$

(b) Using the definition in Prob. 1.5-11,

$$
w_{ca}(t) = \frac{w(t) - w^*(-t)}{2} = \frac{e^{j(t+\frac{\pi}{4})} - e^{-j(-t+\frac{\pi}{4})}}{2} = e^{jt}\frac{e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}}}{2} = je^{jt}\sin(\pi/4).
$$

Expressed in standard polar form,

$$w_{\text{ca}}(t) = \sin(\pi/4)e^{j(t+\frac{\pi}{2})} = \frac{1}{\sqrt{2}}e^{j(t+\frac{\pi}{2})}.$$

As expected, $w(t) = w_{\text{cs}}(t) + w_{\text{ca}}(t)$.

Solution 1.5-13

Complex signal $w(t)$ is defined in the text over $(0 \le t \le 1)$. Assigning certain properties to $w(t)$ allows us to plot $w(t)$ over $(-1 \le t \le 1)$.

(a) If $w(t)$ is even, $w(t) = w(-t)$, it is even in both the real and imaginary components. Thus, the graph folds back on itself and appears unchanged. Consider, for example, point (2,1), which now corresponds to both $t = 1$ and $t = -1$. Figure S1.5-13a illustrates the case when $w(t)$ is even.
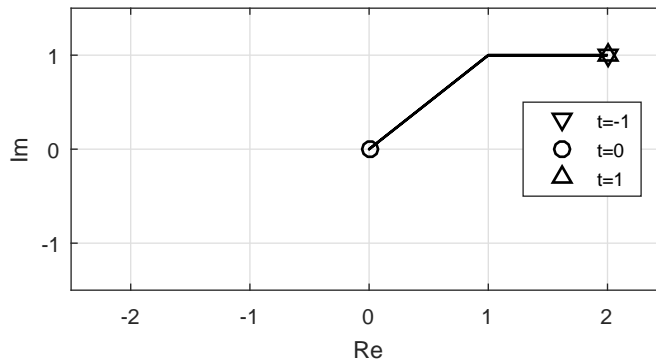
Figure S1.5-13a

(b) If $w(t)$ is odd, $w(t) = -w(-t)$, it is odd in both the real and imaginary components. Thus, the graph reflects about both the real and imaginary axes. Figure S1.5-13b illustrates the case when $w(t)$ is odd.
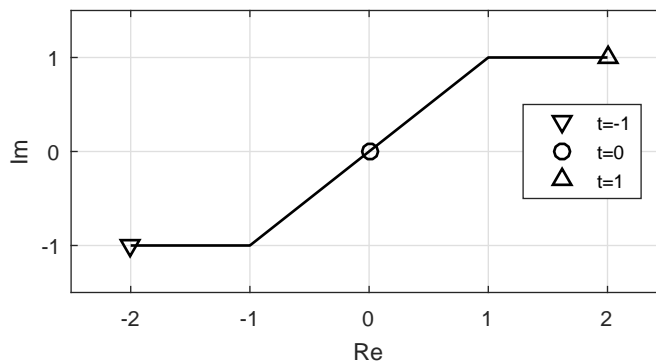
Figure S1.5-13b

(c) If $w(t)$ is conjugate symmetric, $w(t) = w^*(-t)$, it is even in the real component and odd in the imaginary component. Thus, the graph reflects about the real axis. Figure S1.5-13c illustrates the case when $w(t)$ is conjugate-symmetric.
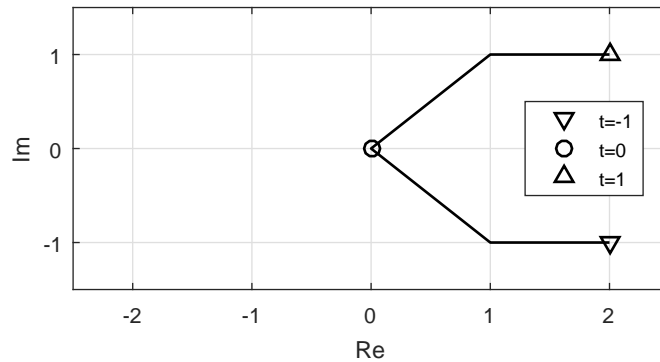
Figure S1.5-13c

(d) If $w(t)$ is conjugate antisymmetric, $w(t) = -w^*(-t)$, it is odd in the real component and even in the imaginary component. Thus, the graph reflects about the imaginary axis. Figure S1.5-13d illustrates the case when $w(t)$ is conjugate-antisymmetric.
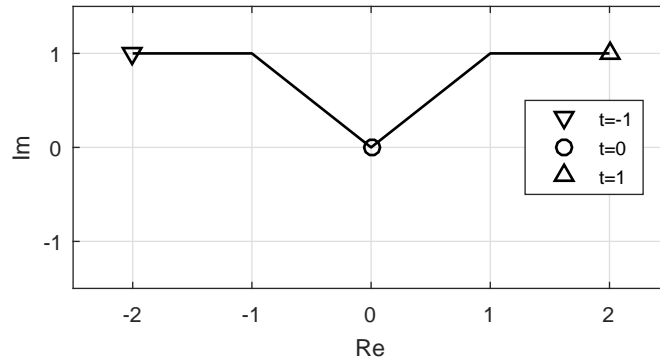
Figure S1.5-13d

(e) Since $w(t)$ is only given over $(0 \le t \le 1)$, $w(3t)$ can be determined only for $(0 \le t \le 1/3)$. Since the function does not change, only the time at which it occurs, the complex-plane graph of $w(3t)$ looks identical to the original complex-plane graph of $w(t)$ with the exception that the points are assigned different times. For example, point $(2,1)$ occurs now at $t = 1/3$ (see Fig. S1.5-13e).
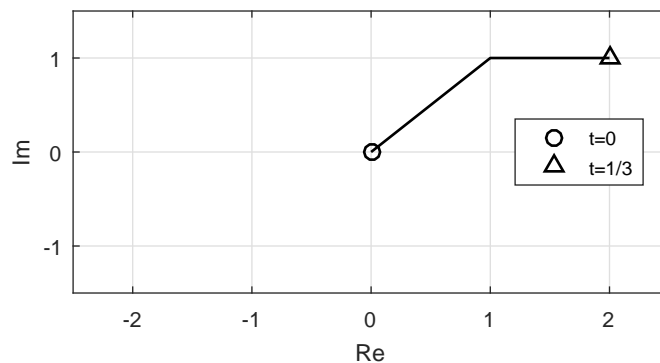
Figure S1.5-13e

Solution 1.5-14

(a) We know $x(t) = t^2(1+j)$ over $(1 \leq t \leq 2)$. Since $x(t)$ is skew-Hermitian, $x(t) = -x^*(-t)$ and thus $x(t) = -t^2(1-j)$ over $(-2 \leq t \leq -1)$. To minimize energy, $x(t)$ is set to zero everywhere else. Thus,

$$x(t) = \begin{cases} t^2(1+j) & 1 \leq t \leq 2 \\ -t^2(1-j) & -2 \leq t \leq -1 \\ 0 & \text{otherwise} \end{cases}.$$

(b) MATLAB is used to sketch $y(t) = \text{Re}\{x(t)\}$. See Fig. S1.5-14b.

```
>>  t = [-2.5:.001:2.5]; u = @(t) 1.0*(t>=0);
>>  x = @(t) (t.^2*(1+j)).*(u(t-1)-u(t-2))+(-t.^2*(1-j)).*(u(t+2)-u(t+1));
>>  plot(t,real(x(t)),'k'); xlabel('t'); ylabel('y(t)'); grid on
>>  axis([-2.5 2.5 -4.5 4.5]); set(gca,'xtick',[-2 -1 1 2],'ytick',[-4 -1 0 1 4]);
```
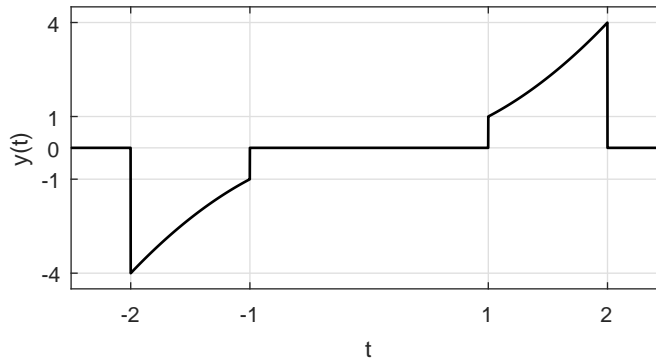

Figure S1.5-14b

(c) MATLAB is used to sketch $z(t) = \text{Real}\{jx(-2t+1)\}$. See Fig. S1.5-14c.

```
>>  t = [-2.5:.001:2.5]; u = @(t) 1.0*(t>=0);
>>  x = @(t) (t.^2*(1+j)).*(u(t-1)-u(t-2))+(-t.^2*(1-j)).*(u(t+2)-u(t+1));
>>  plot(t,real(j*x(-2*t+1)),'k'); xlabel('t'); ylabel('y(t)'); grid on
>>  axis([-2.5 2.5 -4.5 4.5]); set(gca,'xtick',[-.5 0 1 1.5],'ytick',[-4 -1 0]);
```


Figure S1.5-14c

(d) Since $x(t)$ is finite duration, $P_x = 0$. Using symmetry, $E_x = 2\int_1^2 (t^2(1+j))(t^2(1-j))dt =$
$2\int_1^2 2t^4 dt = \frac{4t^5}{5}\Big|_{t=1}^{2} = \frac{4}{5}(32-1) = \frac{124}{5}$. Thus,

$$E_x = \frac{124}{5} = 24.8 \quad \text{and} \quad P_x = 0.$$

Solution 1.6-1

If $x(t)$ and $y(t)$ are the input and output, respectively, of an ideal integrator, then

$$\dot{y}(t) = x(t)$$

and

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = \underbrace{\int_{-\infty}^{0} x(\tau)\,d\tau}_{} + \int_{0}^{t} x(\tau)\,d\tau = \underbrace{y(0)}_{\text{zero-input}} + \underbrace{\int_{0}^{t} x(\tau)\,d\tau}_{\text{zero-state}}$$
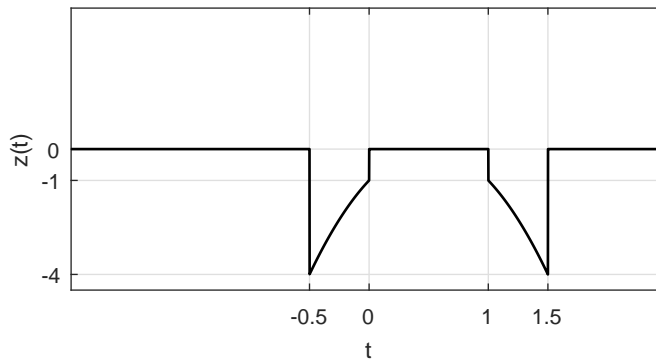

Solution 1.6-2

From Newton's law

$$x(t) = M\frac{dv}{dt}$$

and

$$v(t) = \frac{1}{M}\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{M}\int_{-\infty}^{0} x(\tau)\,d\tau + \frac{1}{M}\int_{0}^{t} x(\tau)\,d\tau = v(0) + \frac{1}{M}\int_{0}^{t} x(\tau)\,d\tau$$

Solution 1.6-3

There are an infinite number of solutions to this problem. We consider a representative solution for each part.

(a) Single-input, single-output (SISO) system: a mono audio signal (single input) is amplified and played on a single speaker speaker (single output).

(b) Multiple-input, single-output (MISO) system: the left and right channels of a stereo audio source (multiple inputs) are averaged together and played on a single speaker (single output).

(c) Single-input, multiple-output (SIMO) system: a mono audio signal (single input) is split, filtered, and sent to separate bass and tweeter speakers (multiple outputs).

(d) Multiple-input, multiple-output (MIMO) system: the left and right channels of a stereo audio source (multiple inputs) are combined and filtered to produce a symphony hall effect when played on a pair of carefully positioned speakers (multiple outputs).

Solution 1.7-1

(a) In this case, $\frac{d}{dt}y(t) + 2y(t) = x^2(t)$. If we let $x_1(t) = kx(t)$, we see that $\frac{d}{dt}y_1(t) + 2y_1(t) = x_1^2(t)$ or $\frac{d}{dt}y_1(t) + 2y_1(t) = k^2 x^2(t) = k^2\left(\frac{d}{dt}y(t) + 2y(t)\right)$. This requires that $y_1(t) = k^2 y(t)$ while linearity requires that $y_1(t) = ky(t)$. Thus,

the system is not linear.

(b) For inputs $x_1(t)$ and $x_2(t)$ and respective outputs $y_1(t)$ and $y_2(t)$, we know that $\frac{d}{dt}y_1(t) + 3ty_1(t) = t^2 x_1(t)$ and $\frac{d}{dt}y_2(t) + 3ty_2(t) = t^2 x_2(t)$. Multiplying the first equation by k_1, the second by k_2, and adding yields

$$\frac{d}{dt}(k_1 y_1(t) + k_2 y_2(t)) + 3t(k_1 y_1(t) + k_2 y_2(t)) = t^2(k_1 x_1(t) + k_2 x_2(t)).$$

This, however, is just the system with input $k_1x_1(t) + k_2x_2(t)$ and output $k_1y_1(t) + k_2y_2(t)$. Since superposition holds,

the system is linear.

(c) In this case, $3y(t) + 2 = x(t)$. The output in response to $x_1(t) = 0$ is $y_1(t) = -\frac{2}{3}$. However, the output in response to $x_2(t) = 2x_1(t) = 0$ is $y_2(t) = \frac{2}{3} \neq 2y_1(t)$. Since superposition does not hold,

the system is not linear.

(d) In this case, $\frac{d}{dt}y(t) + y^2(t) = x(t)$. If we let $x_1(t) = kx(t)$, linearity requires that the output is $y_1(t) = ky(t)$. However, the system acts on $x_1(t) = kx(t)$ according to $\frac{d}{dt}y_1(t) + y_1^2(t) = x_1(t)$ or $\frac{d}{dt}y_1(t) + y_1^2(t) = kx(t) = k\left(\frac{d}{dt}y(t) + y^2(t)\right)$. Since this equality does not hold for $y_1(t) = ky(t)$,

the system is not linear.

(e) In this case, $\left(\frac{d}{dt}y(t)\right)^2 + 2y(t) = x(t)$. If we let $x_1(t) = kx(t)$, linearity requires that the output is $y_1(t) = ky(t)$. However, the system acts on $x_1(t) = kx(t)$ according to $\left(\frac{d}{dt}y_1(t)\right)^2 + 2y_1(t) = x_1(t)$ or $\left(\frac{d}{dt}y_1(t)\right)^2 + 2y_1(t) = kx(t) = k\left(\left(\frac{d}{dt}y(t)\right)^2 + 2y(t)\right) = \left(\frac{d}{dt}\sqrt{k}y(t)\right)^2 + 2ky(t) \neq \left(\frac{d}{dt}y_1(t)\right)^2 + 2y_1(t)$. Thus,

the system is not linear.

(f) For inputs $x_1(t)$ and $x_2(t)$ and respective outputs $y_1(t)$ and $y_2(t)$, we know that $\frac{d}{dt}y_1(t) + \sin(t)y_1(t) = \frac{d}{dt}x_1(t) + 2x_1(t)$ and $\frac{d}{dt}y_2(t) + \sin(t)y_2(t) = \frac{d}{dt}x_2(t) + 2x_2(t)$. Multiplying the first equation by k_1, the second by k_2, and adding yields

$$\frac{d}{dt}\left(k_1y_1(t) + k_2y_2(t)\right) + \sin(t)\left(k_1y_1(t) + k_2y_2(t)\right) = \frac{d}{dt}\left(k_1x_1(t) + k_2x_2(t)\right) + 2\left(k_1x_1(t) + k_2x_2(t)\right).$$

This, however, is just the system with input $k_1x_1(t) + k_2x_2(t)$ and output $k_1y_1(t) + k_2y_2(t)$. Since superposition holds,

the system is linear.

(g) In this case, $\frac{d}{dt}y(t) + 2y(t) = x(t)\frac{d}{dt}x(t)$. If we let $x_1(t) = kx(t)$, we see that $\frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t)\frac{d}{dt}x_1(t)$ or $\frac{d}{dt}y_1(t) + 2y_1(t) = kx(t)\frac{d}{dt}kx(t) = k^2\left(\frac{d}{dt}y(t) + 2y(t)\right)$. This requires that $y_1(t) = k^2y(t)$ while linearity requires that $y_1(t) = ky(t)$. Thus,

the system is not linear.

(h) For inputs $x_1(t)$ and $x_2(t)$ and respective outputs $y_1(t)$ and $y_2(t)$, we know that $y_1(t) = \int_{-\infty}^{t} x_1(\tau)\,d\tau$ and $y_2(t) = \int_{-\infty}^{t} x_2(\tau)\,d\tau$. Multiplying the first equation by k_1, the second by k_2, and adding yields

$$k_1y_1(t) + k_2y_2(t) = \int_{-\infty}^{t} \left(k_1x_1(\tau) + k_2x_2(\tau)\right)\,d\tau.$$

This, however, is just the system with input $k_1x_1(t) + k_2x_2(t)$ and output $k_1y_1(t) + k_2y_2(t)$. Since superposition holds,

the system is linear.

Solution 1.7-2

(a) In this case, $y(t) = x(t-2)$. Letting $x_1(t) = x(t-T)$, we know that $y_1(t) = x_1(t-2)$ or $y_1(t) = x(t-T-2) = y(t-T)$. Since delayed input $x_1(t) = x(t-T)$ produces delayed output $y_1(t) = y(t-T)$,

$$\text{the system is time invariant.}$$

(b) The input $x(t)$ yields the output $y(t) = x(-t)$. For $T > 0$, delayed input $x_1(t) = x(t-T)$ produces $y_1(t) = x_1(-t)$ or $y_1(t) = x(-t-T) = x(-(t+T)) = y(t+T)$. Since delayed input $x_1(t) = x(t-T)$ produces an advanced (not delayed!) output $y_1(t) = y(t+T)$,

$$\text{the system is time variant.}$$

(c) The input $x(t)$ yields the output $y(t) = x(at)$. For $T > 0$, delayed input $x_1(t) = x(t-T)$ produces $y_1(t) = x_1(at)$ or $y_1(t) = x(at-T) = x(a(t-\frac{T}{a})) = y(t-\frac{T}{a})$. Since T-delayed input $x_1(t) = x(t-T)$ produces $\frac{T}{a}$-delayed output $y_1(t) = y(t-\frac{T}{a})$,

$$\text{the system is time variant.}$$

(d) In this case, $y(t) = tx(t-2)$. Letting $x_1(t) = x(t-T)$, we know that $y_1(t) = tx_1(t-2)$ or $y_1(t) = tx(t-T-2) \neq y(t-T) = (t-T)x(t-T-2)$. Since delayed input $x_1(t) = x(t-T)$ produces output $y_1(t) \neq y(t-T)$,

$$\text{the system is time variant.}$$

(e) In this case, $y(t) = \int_{-5}^{5} x(\tau)\, d\tau$. Notice that $y(t)$ is a constant for all t and equals the area of $x(t)$ over $-5 \le t \le 5$. Letting $x_1(t) = x(t-T)$, we know that $y_1(t) = \int_{-5}^{5} x_1(\tau)\, d\tau$ or $y_1(t) = \int_{-5}^{5} x(\tau-T)\, d\tau = \int_{-5-T}^{5-T} x(\tau)\, d\tau \neq y(t-T)$. Notice that $y_1(t)$ a constant for all t and equals the area of $x(t)$ over $-5-T \le t \le 5-T$, which is generally different than the constant that is $y(t)$. Since delayed input $x_1(t) = x(t-T)$ does not produce delayed output $y(t-T)$,

$$\text{the system is time variant.}$$

(f) In this case, $y(t) = \left(\frac{d}{dt}x(t)\right)^2$. Letting $x_1(t) = x(t-T)$, we know that $y_1(t) = \left(\frac{d}{dt}x_1(t)\right)^2$ or $y_1(t) = \left(\frac{d}{dt}x(t-T)\right)^2 = y(t-T)$. Since delayed input $x_1(t) = x(t-T)$ produces delayed output $y_1(t) = y(t-T)$,

$$\text{the system is time invariant.}$$

Solution 1.7-3

(a) If $|T(t)| \le M_T < \infty$ and $|V(t)| \le M_V < \infty$ (that is, inputs are bounded), then $|W(t)| \le 35.74 + 0.6215M_T + 35.75M_V^{0.16} + 0.4275M_T M_V^{0.16} < \infty$ (output is bounded). Thus, bounded inputs guarantee a bounded output.

$$\text{Yes. The system is BIBO stable.}$$

(b) Current output $W(t)$ depends only on current inputs $T(t)$ and $V(t)$; no memory is required.

$$\text{Yes. The system is memoryless.}$$

(c) From part (b), we know the system is memoryless. All memoryless systems are necessarily causal (no prediction of the future).

$$\text{Yes. The system is causal.}$$

(d) If we apply $T_1(t) = 1$, then the system output is $W_1(t) = k_1 + k_2$. If we apply $T_2(t) = 2T_1(t) = 2$, then the system output is $W_2(t) = k_1 + 2k_2 \neq 2W_1(t)$. Clearly, the system fails the homogeneity (scaling) property of linearity.

No. The simplified system $W(t) = k_1 + k_2 T(t)$ is not linear.

(e) If we apply $V_1(t) = 1$, then the system output is $W_1(t) = k_3 + k_4$. If we apply $V_2(t) = 2V_1(t) = 2$, then the system output is $W_2(t) = k_3 + k_4 2^{0.16} \neq 2W_1(t)$. Clearly, the system fails the homogeneity (scaling) property of linearity.

No. The simplified system $W(t) = k_3 + k_4 \{V(t)\}^{0.16}$ is not linear.

Solution 1.7-4

(a) By inspection, we see that $|y(t)| \leq V_{\text{ref}} < \infty$ for all possible $x(t)$.

Yes. The system is BIBO stable.

(b) Output y at time t depends on input x at time $t - t_{\text{p}}$, which is t_{p} seconds in the past; no future values of the input are needed to determine the output.

Yes. The system is causal.

(c) We will use a counterexample to prove that the system is not invertible. Since $x_1(t) = 2V_{\text{ref}}$ and $x_2(t) = 3V_{\text{ref}}$ both yield the same output $y(t) = -V_{\text{ref}}$, the output cannot uniquely determine the input.

No. The system is not invertible.

(d) We will use a counterexample to prove that the system is not linear. Although the input $x_1(t) = V_{\text{ref}}$ yields the output $y_1(t) = -V_{\text{ref}}$, the input $x_2(t) = 2V_{\text{ref}} = 2x_1(t)$ yields the output $y_2(t) = -V_{\text{ref}} \neq 2y_1(t)$. Clearly, the system fails the homogeneity (scaling) property of linearity.

No. The system is not linear.

(e) From part (b), we see that the current output $y(t)$ requires a past input value $x(t - t_{\text{p}})$.

No. The system is not memoryless.

(f) System parameters do not change with time, and delaying the input causes a corresponding delay in the output. That is, if $x(t) \to y(t)$, then $x(t - T) \to y(t - T)$.

Yes. The system is time-invariant.

Solution 1.7-5

(a) If $|x(t)| \leq M_x < \infty$, then $|y(t)| \leq |-2x(t)| \leq 2M_x < \infty$. Thus, any bounded input guarantees a bounded output.

Yes. The system is BIBO stable.

(b) The output y at time t depends solely on the input x one second in the past at time $t - 1$; no future values of the input are needed to determine the output.

Yes. The system is causal.

(c) We will use a counterexample to prove that the system is not invertible. Since $x_1(t) = -1$ and $x_2(t) = -2$ both yield the same output $y(t) = 0$, the output cannot uniquely determine the input.

<div align="center">No. The system is not invertible.</div>

(d) We will use a counterexample to prove that the system is not linear. Although input $x_1(t) = 1$ produces output $y_1(t) = -2$, input $x_2(t) = -2x_1(t) = -2$ produces output $y_2(t) = 0 \neq 2y_1(t)$. Clearly, the system fails the homogeneity (scaling) property of linearity.

<div align="center">No. The system is not linear.</div>

(e) From part (b), we see that the current output $y(t)$ requires a past input value $x(t-1)$.

<div align="center">No. The system is not memoryless.</div>

(f) A shift in the input causes a corresponding shift in the output. That is, if $x(t) \rightarrow y(t)$, then $x(t - T) \rightarrow y(t - T)$.

<div align="center">Yes. The system is time-invariant.</div>

Solution 1.7-6

(a) The output $y(t)$ either equals the input $x(t)$ or a delay of the input $x(t-1)$. Thus, if the input is bounded, then the output must also be bounded. That is, if $|x(t)| \leq M_x < \infty$, then $|y(t)| \leq M_x < \infty$. All bounded inputs produce bounded outputs.

<div align="center">Yes. The system is BIBO stable.</div>

(b) To determine the current output $y(t)$, the slope of the input 1 second in the future needs to be known. That is, current output requires knowledge of future input.

<div align="center">No. The system is not causal.</div>

(c) In this system, when the slope of the input changes from positive to negative, up to 1 second of data can be lost. When data is lost, the system cannot be invertible. Let us use a counterexample to show this fact and prove that the system is not invertible. Figure S1.7-6c shows two inputs, $x_1(t)$ and $x_2(t)$, which both have the same output $y(t)$, which is also shown. Since two distinct inputs can yields the same output, it is clearly impossible to always recover input from output, and the system cannot be invertible.

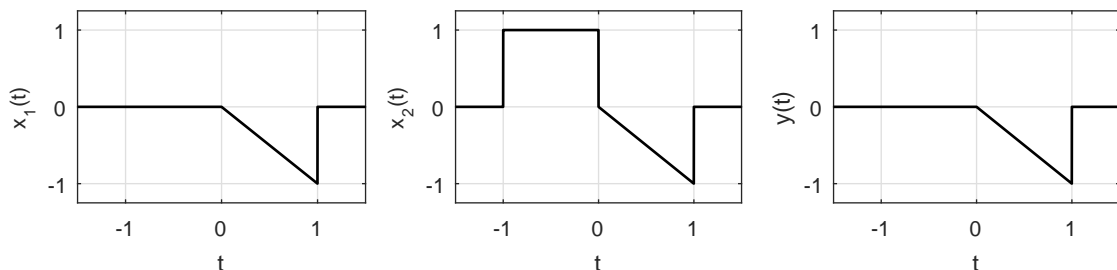<div align="center">No. The system is not invertible.</div>

Figure S1.7-6c

(d) We will use a counterexample to prove that the system is not linear. Figure S1.7-6d shows input $x_1(t)$ and its output $y_1(t)$. Figure S1.7-6d also shows input $x_2(t)$ and its output $y_2(t)$. Although $x_2(t) = -x_1(t)$, we see that $y_2(t) \neq -y_1(t)$, and the system fails the homogeneity (scaling) property of linearity.
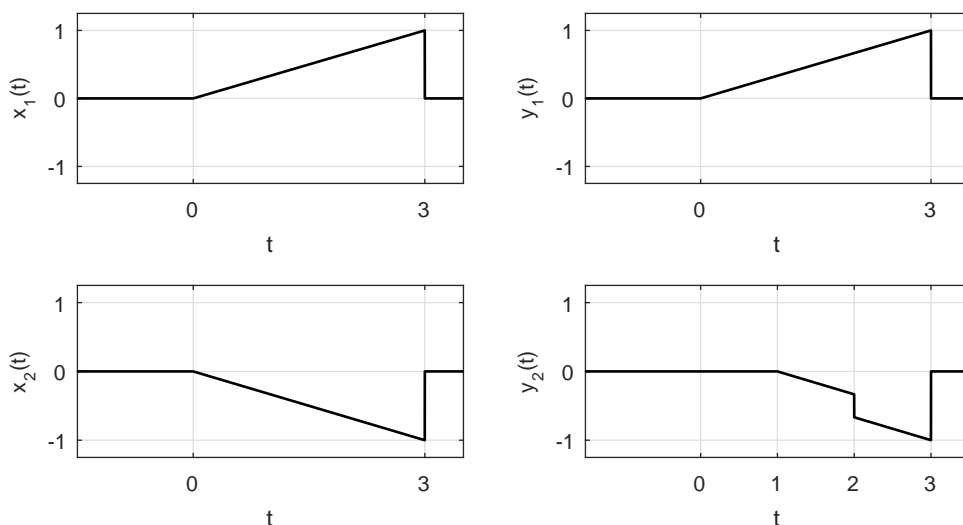
<div align="center">No. The system is not linear.</div>

<div align="center">Figure S1.7-6d</div>

(e) If $\frac{d}{dt}x(t) < 0$, the output y at time $t - 1$ depends on a the input x one second earlier at time $t - 2$. Clearly, memory is required.

<div align="center">No. The system is not memoryless.</div>

(f) A shift in the input causes a corresponding shift in the output. That is, if $x(t) \rightarrow y(t)$, then $x(t - T) \rightarrow y(t - T)$.

<div align="center">Yes. The system is time-invariant.</div>

Solution 1.7-7

(a) We will use a counterexample to prove that the system is not BIBO stable. The bounded input $x(t) = u(t)$ produces the unbounded output $y(t) = tu(t)$. Since the system output is not bounded for all possible inputs, the system is not BIBO stable.

<div align="center">No. The system is not BIBO stable.</div>

(b) By inspection, it is clear that the system only depends on the current input. No future values of the input are involved.

<div align="center">Yes. The system is causal.</div>

(c) Since the system essentially "kills" any input for $t < 0$, such input values are lost and cannot be recovered. Thus, the system is not invertible. To provide a specific example, consider the inputs $x_1(t) = \delta(t + 1)$ and $x_2(t) = \delta(t + 2)$, both of which produce output $y(t) = 0$. Clearly, it is impossible to recover the input from the output in this case.

<div align="center">No. The system is not invertible.</div>

(d) Begin assuming $y_1(t) = r(t)x_1(t)$ and $y_2(t) = r(t)x_2(t)$. Applying $ax_1(t)+bx_2(t)$ to the system yields $y(t) = r(t)\,(ax_1(t) + bx_2(t)) = ar(t)x_1(t) + br(t)x_2(t) = ay_1(t) + by_2(t)$.

<div align="center">Yes. The system is linear.</div>

(e) By inspection, it is clear that the system only depends on the current input.

<div align="center">Yes. The system is memoryless.</div>

(f) Since the system function depends on the independent variable t, it is unlikely that the system is time-invariant. To explicitly verify, let $y(t) = r(t)x(t)$. Next, delay $x(t)$ by τ to obtain a new input $x_2 = x(t-\tau)$. Applying $x_2(t)$ to the system yields $y_2(t) = r(t)x_2(t) = r(t)x(t-\tau) \neq r(t-\tau)x(t-\tau) = y(t-\tau)$. Since, the system operator and the time-shift operator do not commute, the system is not time-invariant.

<div align="center">No. The system is not time-invariant.</div>

Solution 1.7-8

(a) The system returns the time-delayed derivative, or slope, of the input signal. A square-wave is a bounded signal which, due to point discontinuities, has infinite slope at the corresponding instants in time. Thus, a bounded input may not result in a bounded output, and the system cannot be BIBO stable.

<div align="center">No. The system is not BIBO stable.</div>

(b) By inspection, it is clear that the system output does not depend on future input values.

<div align="center">Yes. The system is causal.</div>

(c) Differentiation eliminates any dc component, which is lost forever and cannot be recovered.

<div align="center">No. The system is not invertible.</div>

(d) Begin assuming $y_1(t) = \frac{d}{dt}x_1(t-1)$ and $y_2(t) = \frac{d}{dt}x_2(t-1)$. Applying $ax_1(t) + bx_2(t)$ to the system yields $y(t) = \frac{d}{dt}\,(ax_1(t-1) + bx_2(t-1)) = a\frac{d}{dt}x_1(t-1)+b\frac{d}{dt}x_2(t-1) = ay_1(t)+by_2(t)$.

<div align="center">Yes. The system is linear.</div>

(e) By inspection, it is clear that the system depends on a past value of the input. For example, at $t = 0$, the output $y(0)$ depends on the time-derivative of $x(-1)$, a past value.

<div align="center">No. The system is not memoryless.</div>

(f) To explicitly verify, let $y(t) = \frac{d}{dt}x(t-1)$. Next, delay $x(t)$ by τ to obtain a new input $x_2 = x(t-\tau)$. Applying $x_2(t)$ to the system yields $y_2(t) = \frac{d}{dt}x_2(t) = \frac{d}{dt}x(t-1-\tau) = y(t-\tau)$. Since, the system operator and the time-shift operator commute, the system is time-invariant. In more loose terms, the derivative operator returns the delayed slope of a signal independent of when that signal is applied.

<div align="center">Yes. The system is time-invariant.</div>

Solution 1.7-9

(a) From the definition of the system, we know that $y(t)$ is either $x(t)$ or 0. Correspondingly, if $|x(t)| < \infty$ then $|y(t)| < \infty$, and the system must be BIBO stable.

<div align="center">Yes. The system is BIBO stable.</div>

(b) By inspection, it is clear that the system output only depends on the current input. No future (or past) values are involved.

<div align="center">Yes. The system is causal.</div>

(c) Any negative values of the input are lost and cannot be recovered. Thus, the system is not invertible. For example, $x_1(t) = -1$ and $x_2(t) = -2$ both produce output $y(t) = 0$; it is clearly impossible to always recover an input from its output.

<div align="center">No. The system is not invertible.</div>

(d) Consider two signals: $x_1(t) = 1$ and $x_2(t) = \cos(t)$. The corresponding outputs of these individual signals is $y_1(t) = 1$ and $y_2(t) = \begin{cases} \cos(t) & \text{if} \quad \cos(t) > 0 \\ 0 & \text{if} \quad \cos(t) \le 0 \end{cases}$. However, if we create a third input $x_3(t) = x_1(t) + x_2(t)$, the system output is $y_3(t) = 1 + \cos(t) \ne y_1(t) + y_2(t)$. Since superposition does not apply, the system cannot be linear.

<div align="center">No. The system is not linear.</div>

(e) By inspection, it is clear that the system output only depends on the current input. No memory is involved.

<div align="center">Yes. The system is memoryless.</div>

(f) Consider delaying $x(t)$ by τ to obtain a new input $x_2 = x(t - \tau)$. Applying $x_2(t)$ to the system yields $y_2(t) = \begin{cases} x(t-\tau) & \text{if} \quad x(t-\tau) > 0 \\ 0 & \text{if} \quad x(t-\tau) \le 0 \end{cases} = y(t - \tau)$. Since the system operator and the time-shift operator commute, the system is time-invariant.

<div align="center">Yes. The system is time-invariant.</div>

Solution 1.7-10

(a) Using the sifting property, this system operation is rewritten as $y(t) = 0.5\,(x(t) - x(-t))$.

<div align="center">Thus, this system extracts the odd portion of the input.</div>

(b) If $|x(t)| \le M_x < \infty$, then $|y(t)| = |0.5\,(x(t) - x(-t))| \le 0.5\,(|x(t)| + |-x(-t)|) \le M_x < \infty$. That is, bounded inputs always produce bounded outputs.

<div align="center">Yes. The system is BIBO stable.</div>

(c) By inspection, it is clear that the system output for $t < 0$ depends on future inputs. For example, at time $t = -1$ the output $y(-1) = 0.5(x(-1) - x(1))$ depends on a future value of the input, $x(1)$.

<div align="center">No. The system is not causal.</div>

(d) More than one signal can have the same odd portion. Thus, this system cannot be invertible. For example, $x_1(t) = u(t)$ and $x_2(t) = -u(-t)$ both produce output $y(t) = 0.5u(t) - 0.5u(-t)$; it is clearly impossible to always uniquely recover an input from its output.

No. The system is not invertible.

(e) Let $y_1(t) = 0.5(x_1(t) - x_1(-t))$ and $y_2(t) = 0.5(x_2(t) - x_2(-t))$. Applying $ax_1(t) + bx_2(t)$ to the system yields $y(t) = 0.5\,(ax_1(t) + bx_2(t) - (ax_1(-t) + bx_2(-t))) = 0.5a(x_1(t) - x_1(-t)) + 0.5b(x_2(t) - x_2(-t)) = ay_1(t) + by_2(t)$.

Yes. The system is linear.

(f) By inspection, it is clear that the system output can depend on past or future inputs. For example, at time $t = 1$ the output $y(1) = 0.5(x(1) - x(-1))$ depends on a past value of the input, $x(-1)$.

No. The system is not memoryless.

(g) We will use a counterexample to prove that the system is not time-invariant. Let the input be $x(t) = t(u(t + 1) - u(t - 1))$. Since this input is already odd, the output is just the input, $y(t) = x(t)$. Shifting by a non-zero τ, $x(t - \tau)$ is not odd, and the output is not $y(t - \tau) = x(t - \tau)$. That is, shifting the input does not produce a simple shift in the original output.

No. The system is not time-invariant.

Solution 1.7-11

We construct the table below from the first three rows of data. Let r_j denote the jth row.

Row	$x(t)$	$q_1(0)$	$q_2(0)$	$y(t)$
r_1	0	1	-1	$e^{-t}u(t)$
r_2	0	2	1	$e^{-t}(3t + 2)u(t)$
r_3	$u(t)$	-1	-1	$2u(t)$
$r_4 = \frac{1}{3}(r_1 + r_2)$	0	1	0	$(t + 1)e^{-t}u(t)$
$r_5 = \frac{1}{2}(r_1 + r_3)$	$\frac{1}{2}u(t)$	0	-1	$(\frac{1}{2}e^{-t} + 1)u(t)$
$r_6 = (r_4 + r_5)$	$\frac{1}{2}u(t)$	1	-1	$(1.5e^{-t} + te^{-t} + 1)u(t)$
$r_7 = 2(r_6 + r_1)$	$u(t)$	0	0	$(e^{-t} + 2te^{-t} + 2)u(t)$

In our case, the input $x(t) = u(t + 5) - u(t - 5)$. From r_7 and the superposition and time-invariance properties, we have

$$y(t) = r_7(t + 5) - r_7(t - 5)$$
$$= \left[e^{-(t+5)} + 2(t + 5)e^{-(t+5)} + 2 \right] u(t + 5) - \left[e^{-(t-5)} + 2(t - 5)e^{-(t-5)} + 2 \right] u(t - 5)$$

Solution 1.7-12

If the input is $kx(t)$, the new output $y(t)$ is

$$y(t) = k^2 x^2(t)/\left(k\frac{dx}{dt} \right) = k[x^2(t)/\left(\frac{dx}{dt} \right)]$$

Hence the homogeneity is satisfied. If the input-output pair is denoted by $x_i \rightarrow y_i$, then

$$x_1 \rightarrow y_1 = (x_1)^2/(\dot{x}_1) \quad \text{and} \quad x_2 \rightarrow y_2 = (x_2)^2/(\dot{x}_2)$$

$$\text{But} \quad x_1 + x_2 \rightarrow (x_1 + x_2)^2 / (\dot{x}_1 + \dot{x}_2) \neq y_1 + y_2$$

Solution 1.7-13

From the hint it is clear that when $v_c(0) = 0$, the capacitor may be removed, and the circuit behaves as shown in Fig. S1.7-13. It is clearly zero-state linear. To show that it is zero-input nonlinear, consider the circuit with $x(t) = 0$ (zero-input). The current $y(t)$ has the same direction (shown by arrow) regardless of the polarity of v_c (because the input branch is a short). Thus the system is zero-input nonlinear.
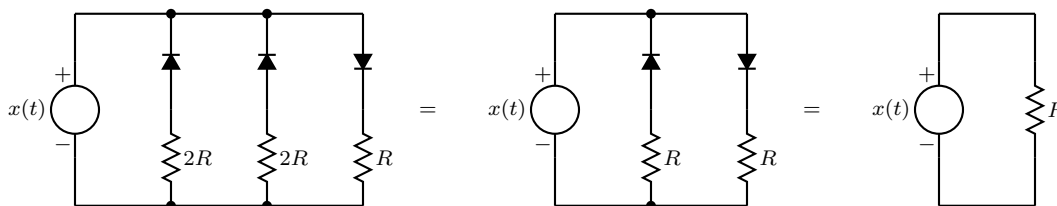

Figure S1.7-13

Solution 1.7-14

The solution is trivial. The input is a current source, which has infinite impedance. Hence, as far as the output $y(t)$ is concerned, the circuit behaves as shown in Fig. S1.7-14. The nonlinear elements are irrelevant in computing the output $y(t)$, and the output $y(t)$ satisfies the linearity conditions. Yet, the circuit is nonlinear because it contains nonlinear elements.
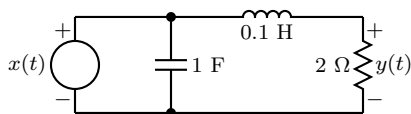

Figure S1.7-14

Solution 1.7-15

(a) $y(t) = x(t - 2)$. Thus, the output $y(t)$ always starts after the input by 2 seconds. Clearly, the system is causal.

(b) $y(t) = x(-t)$. The output $y(t)$ is obtained by time inversion of the input. If the input starts at $t = 0$, the output starts before $t = 0$. Hence, the system is not causal.

(c) $y(t) = x(at)$, $a > 1$. The output $y(t)$ is obtained by time compression of the input by factor a. If the input starts at some time $t > 0$, the output will start before the input. Hence, the system is not causal.

(d) $y(t) = x(at)$, $a < 1$. The output $y(t)$ is obtained by time expansion of the input by factor $1/a$. If the input starts at some time $t < 0$, the output will start before the input. Hence, the system is not causal.

Figure S1.7-15 illustrates each case with an example; the input is shown as a dashed line while the output is shown as a solid line.
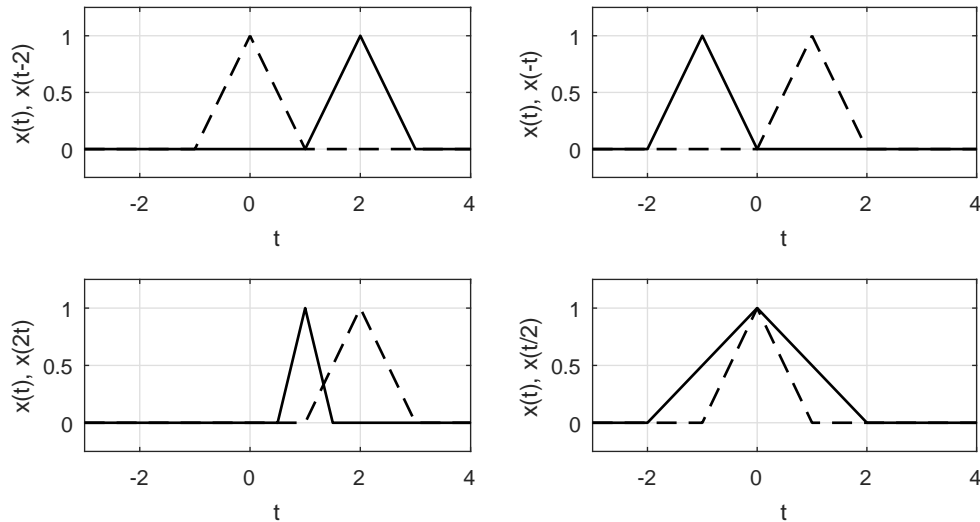
Figure S1.7-15

Solution 1.7-16

(a) Invertible because the input can be obtained by taking the derivative of the output. Hence, the inverse system equation is $y(t) = dx/dt$.

(b) Not invertible for even values of n, because the sign information is lost. However, the system is invertible for odd values of n. The inverse system equation is $y(t) = [x(t)]^{1/n}$.

(c) Not invertible because differentiation operation irretrievable loses the constant part of $x(t)$.

(d) The system $y(t) = x(3t - 6) = x(3[t - 2])$ represents an operation of signal compression by factor 3, and then time delay by 2 seconds. Hence, the input can be obtained from the output by first advancing the output by 2 seconds, and then time-expanding by factor 3. Hence, the inverse system equation is $y(t) = x(\frac{t}{3} + 2)$. Although the system is invertible, it is not realizable because it involves the operation of signal compression and signal advancing (which makes it noncausal). However, if we can accept time delay, we can realize a noncausal system.

(e) Not invertible because cosine is a multiple valued function, and $\cos^{-1}[x(t)]$ is not unique.

(f) Invertible. $x(t) = \ln y(t)$.

Solution 1.7-17

(a) No, Bill is not correct. The $x_1(3t)$ term represents a compression rather than the necessary dilation. One way to construct $x_2(t)$ is $x_2(t) = 2x_1(t/3) - x_1(t-1)$. However, this form is not unique; $x_2(t) = 2x_1(t) + x_1(t-1) + 2x_1(t-2)$ also works and may be more useful.

(b) The output $y_1(t)$ is given for the input signal $x_1(t)$. The expression $x_2(t) = 2x_1(t) + x_1(t-1) + 2x_1(t-2)$ forms $x_2(t)$ from a superposition of scaled and shifted copies of $x_1(t)$. Since the system is linear and time invariant, the operations of scaling, summing, and shifting commute with the system operator. Thus,

$$y_2(t) = 2y_1(t) + y_1(t-1) + 2y_1(t-2).$$

Notice, it is not true that $y_2(t) = 2y_1(t/3) - y_1(t-1)$; linearity and time-invariance do not apply with the time-scaling operation. MATLAB is used to plot $y_2(t)$.

```
>>  u = @(t) 1.0*(t>=0); y1 = @(t) t.*(u(t)-u(t-1))+u(t-1);
>>  t = [-1:.001: 4]; y2 = @(t) 2*y1(t)+y1(t-1)+2*y1(t-2);
>>  plot(t,y2(t),'k-'); xlabel('t'); ylabel('y_2(t)'); grid on;
>>  set(gca,'xtick',[0 1 2 3],'ytick',[0 2 3 5]); axis([-1 4 -.5 5.5]);
```
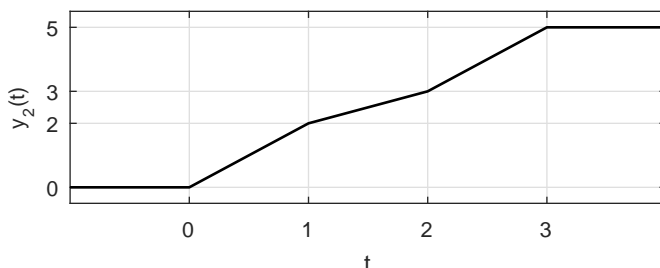

Figure S1.7-17

Solution 1.7-18

(a) Figure S1.7-18 plots input $x(t) = u(t - 0.5) - u(t - 1.5)$ and the corresponding output $y(t) = H\{x(t)\} = 0.5u(t) + 0.5u(t-1) - u(t-2)$, Since the output starts 0.5 second before the input,

$$\text{the system cannot be causal.}$$

By delaying the output by $T \geq 0.5$, the system can be made causal (at least for this input).

(b) Referring to $x(t)$ and $y(t)$ shown in Fig. S1.7-18, since the output duration exceeds the input duration,

$$\text{the system is not memoryless.}$$

(c) To begin, we notice that

$$y(t) = H\{x(t)\} = \tfrac{1}{2}x(t + \tfrac{1}{2}) + x(t - \tfrac{1}{2}).$$

Now, applying $y(t)$ to the system results in output $z(t)$ as

$$\begin{aligned}
z(t) = H\{y(t)\} &= H\{\tfrac{1}{2}x(t + \tfrac{1}{2}) + x(t - \tfrac{1}{2})\} \\
&= \tfrac{1}{2}[\tfrac{1}{2}x(t + 1) + x(t)] + \tfrac{1}{2}x(t) + x(t - 1) \\
&= \tfrac{1}{4}x(t + 1) + x(t) + x(t - 1)
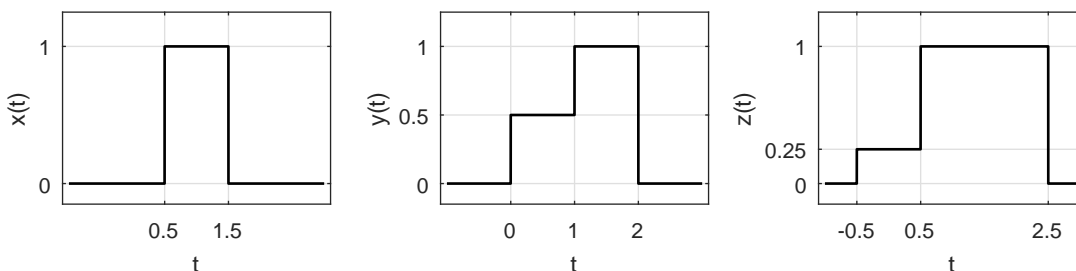\end{aligned}$$

Output $z(t)$ is also shown in Fig. S1.7-18

Figure S1.7-18

Solution 1.8-1

The loop equation for the circuit is

$$3y_1(t) + Dy_1(t) = x(t) \quad \text{or} \quad (D+3)y_1(t) = x(t) \tag{1.8-1a}$$

Also

$$Dy_1(t) = y_2(t) \Longrightarrow y_1(t) = \frac{1}{D}y_2(t) \tag{1.8-1b}$$

Substitution of Eq. (1.8-1b) in Eq. (1.8-1a) yields

$$\frac{(D+3)}{D}y_2(t) = x(t) \quad \text{or} \quad (D+3)y_2(t) = Dx(t)$$

Solution 1.8-2

The currents in the resistor, capacitor and inductor are $2y_2(t)$, $Dy_2(t)$ and $(2/D)y_2(t)$, respectively. Therefore

$$(D + 2 + \frac{2}{D})y_2(t) = x(t)$$

or

$$(D^2 + 2D + 2)y_2(t) = Dx(t) \tag{1.8-2a}$$

Also

$$y_1(t) = Dy_2(t) \quad \text{or} \quad y_2(t) = \frac{1}{D}y_1(t) \tag{1.8-2b}$$

Substituting of Eq. (1.8-2b) in Eq. (1.8-2a) yields

$$\frac{D^2 + 2D + 2}{D}y_1(t) = Dx(t)$$

or

$$(D^2 + 2D + 2)y_1(t) = D^2x(t)$$

Solution 1.8-3

The freebody diagram for the mass M is shown in Fig. S1.8-3. From this diagram it follows that

$$M\ddot{y} = B(\dot{x} - \dot{y}) + K(x - y)$$

or

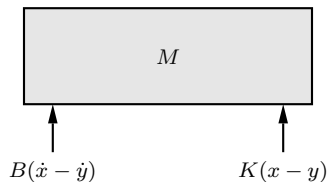$$(MD^2 + BD + K)y(t) = (BD + K)x(t)$$

$B(\dot{x} - \dot{y})$ $K(x - y)$
Figure S1.8-3

Solution 1.8-4

The loop equation for the field coil is

$$(DL_f + R_f)i_f(t) = x(t) \tag{1.8-4a}$$

If $T(t)$ is the torque generated, then

$$T(t) = K_f i_f(t) = (JD^2 + BD)\theta(t) \tag{1.8-4b}$$

Substituting of Eq. (1.8-4a) in Eq. (1.8-4b) yields

$$\frac{K_f}{DL_f + R_f} x(t) = (JD^2 + BD)\theta(t)$$

or

$$(JD^2 + BD)(DL_f + R_f)\theta(t) = K_f x(t)$$

Solution 1.8-5

$$[q_i(t) - q_0(t)]\triangle t = A\triangle h$$

or

$$\dot{h}(t) = \frac{1}{A}[q_i(t) - q_0(t)] \tag{1.8-5a}$$

But

$$q_0(t) = Rh(t) \tag{1.8-5b}$$

Differentiation of Eq. (1.8-5b) yields

$$\dot{q}_0(t) = R\dot{h}(t) = \frac{R}{A}[q_i(t) - q_0(t)]$$

and

$$\left(D + \frac{R}{A}\right)q_0(t) = \frac{R}{A}q_i(t)$$

or

$$(D + a)q_0(t) = aq_i(t) \qquad a = \frac{R}{A}$$

and

$$q_0(t) = \frac{a}{D + a}q_i(t)$$

Substituting this in Eq. (1.8-5a) yields

$$\dot{h}(t) = \frac{1}{A}\left(1 - \frac{a}{D + a}\right)q_i(t) = \frac{D}{A(D + a)}q_i(t)$$

or

$$\left(D + \frac{R}{A}\right)h(t) = \frac{1}{A}q_i(t)$$

Solution 1.8-6

(a) The order of the system is zero; there are no energy storage components such as capacitors or inductors.

(b) Using KVL on the left loop yields $x(t) = R_1 y_1(t) + R_2(y_1(t) - y_2(t)) = 3y_1(t) - 2y_2(t)$. KVL on the middle loop yields $0 = R_2(y_2(t) - y_1(t)) + R_3 y_2(t) + R_4(y_2(t) - y_3(t)) = -2y_1(t) + 9y_2(t) - 4y_3(t)$. Finally, KVL on the right loop yields $R_4(y_3(t) - y_2(t)) + (R_5 + R_6)y_3(t) = -4y_2(t) + 15y_3(t)$. Combining together yields

$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 15 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \\ 0 \end{bmatrix}.$$

(c) Cramer's rule suggests

$$
y_3(t) = \frac{\begin{vmatrix} 3 & -2 & x(t) \\ -2 & 9 & 0 \\ 0 & -4 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 15 \end{vmatrix}}.
$$

MATLAB computes the denominator determinant.

```
>> det([3 -2 0;-2 9 -4;0 -4 15])
   ans = 297
```

The numerator determinant is easy computed by hand as $0 + 0 + 8x(t) = 8x(t)$. Thus,

$$
y_3(t) = \frac{8}{297}x(t) = \frac{8}{297}\left(2 - |\cos(t)|\right)u(t-1).
$$

Solution 1.10-1

From Fig. P1.8-2 we obtain

$$
x(t) = q_1/2 + \dot{q}_1 + q_2
$$

Moreover, the capacitor voltage $q_1(t)$ equals the voltage across the inductor, which is $\frac{1}{2}\dot{q}_2$. Hence, the state equations are

$$
\dot{q}_1 = -q_1/2 - q_2 - x \quad \text{and} \quad \dot{q}_2 = 2q_1
$$

Solution 1.10-2

The capacitor current $C\dot{q}_3 = \frac{1}{2}\dot{q}_3$ is $q_1 - q_2$. Therefore,

$$
\dot{q}_3 = 2q_1 - 2q_2 \tag{1.10-2a}
$$

The two loop equations are

$$
2q_1 + \dot{q}_1 + q_3 = x \Longrightarrow \dot{q}_1 = -2q_1 - q_3 + x \tag{1.10-2b}
$$

and

$$
-q_3 + \frac{1}{3}\dot{q}_2 + q_2 = 0 \Longrightarrow \dot{q}_2 = -3q_2 + 3q_3 \tag{1.10-2c}
$$

Equations Eq. (1.10-2a), Eq. (1.10-2b) and Eq. (1.10-2c) are the state equations.

Next, we express the current and voltage of every element in terms of the state variables:

Element	Current [amps]	Voltage [volts]
1 H inductor	q_1	$\dot{q}_1 = x(t) - 2q_1 - q_3$
$\frac{1}{3}$ H inductor	q_2	$\frac{1}{3}\dot{q}_2 = -q_2 + q_3$
$\frac{1}{2}$ F capacitor	$q_1 - q_2$	q_3
1 Ω resistor	q_2	q_2

At the instant t, $q_1 = 5$, $q_2 = 1$, $q_3 = 2$ and $x = 10$. Substituting these values in the above results yields

Element	Current [amps]	Voltage [volts]
1 H inductor	5	-2
$\frac{1}{3}$ H inductor	1	1
$\frac{1}{2}$ F capacitor	4	2
1 Ω resistor	1	1

Solution 1.11-1

We need to use MATLAB to plot the odd portion $x_o(t)$ of the function $x(t) = 2^{-t}\cos(2\pi t)u(t - \pi)$. To begin, we note that $x(t)$ starts at $t = \pi$. Since 2^{-t} decays to less than $1/100$ its original strength in 7 seconds, plotting $x(t)$ to $t = 10$ is sufficient. Since $x_o(t)$ includes $x(t)$ and its reflection, a suitable time range for our plot is $-10 \le t \le 10$. Since $\cos(2\pi t)$ oscillates one cycle per second, a plot density of 20 points per second is sufficient for a reasonable-quality plot.

```
>>  u = @(t) 1.0*(t>=0); t = -10:1/20:10;
>>  x = @(t) 2.^(-t).*cos(2*pi*t).*u(t-pi); xo = @(t) (x(t)-x(-t))/2;
>>  plot(t,xo(t),'k-'); xlabel('t'); ylabel('x_o(t)'); grid on;
>>  axis([-10 10 -1/20 1/20]); set(gca,'xtick',[-10 -pi 0 pi 10]);
```
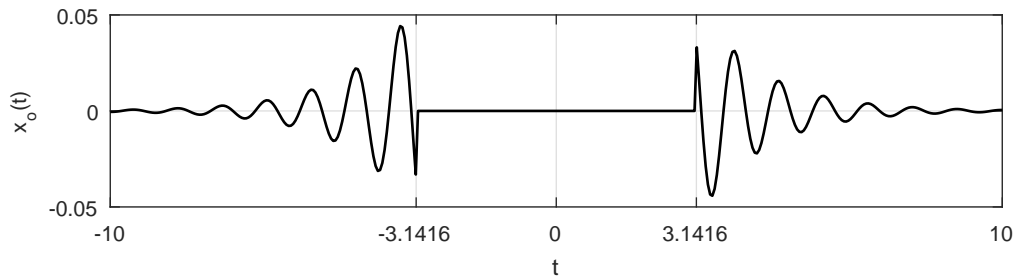

Figure S1.11-1

Solution 1.11-2

We need to use MATLAB to plot the even portion $x_e(t)$ of the function $x(t) = 2^{-t/2}\cos(4\pi t)u(t-0.5)$ over a suitable t using $\Delta t = 0.002$ seconds between points. To begin, we note that $x(t)$ starts at $t = 0.5$. Since $2^{-t/2}$ decays to less than $1/100$ its original strength in about 14 seconds, plotting $x(t)$ to $t = 14$ is sufficient. Since $x_e(t)$ includes $x(t)$ and its reflection, a suitable time range for our plot is $-14 \le t \le 14$.

```
>>  u = @(t) 1.0*(t>=0); t = -14:0.002:14;
>>  x = @(t) 2.^(-t/2).*cos(4*pi*t).*u(t-0.5); xe = @(t) (x(t)+x(-t))/2;
>>  plot(t,xe(t),'k-'); xlabel('t'); ylabel('x_e(t)'); grid on;
>>  axis([-14 14 -0.5 0.5]); set(gca,'xtick',[-14 -7 -0.5 0.5 7 14]);
```
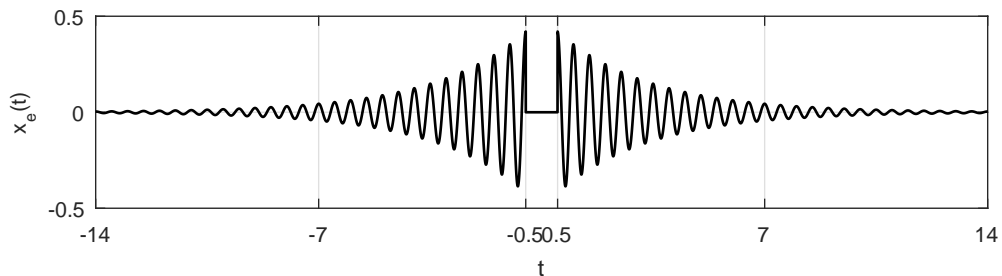

Figure S1.11-2

Solution 1.11-3

In this problem, we define $x(t) = e^{t(1+j2\pi)}u(-t)$ and $y(t) = \mathrm{Re}\left\{2x\left(\frac{-5-t}{2}\right)\right\}$.

(a) Here, we use MATLAB to plot $\mathrm{Re}\left\{x(t)\right\}$ versus $\mathrm{Im}\left\{x(at)\right\}$ for $a = 0.5$, 1, and 2 and $-10 \le t \le 10$.

```
>>  u = @(t) 1.0*(t>=0); t = -10:.001:10;
>>  x = @(t) exp(t*(1+2j*pi)).*u(-t);
>>  subplot(131); plot(real(x(t)),imag(x(0.5*t)),'k-'); grid on
>>  xlabel('Re(x(t))'); ylabel('Im(x(0.5t))'); axis([-1 1 -1 1]);
>>  subplot(132); plot(real(x(t)),imag(x(t)),'k-'); grid on
>>  xlabel('Re(x(t))'); ylabel('Im(x(t))'); axis([-1 1 -1 1]);
>>  subplot(133); plot(real(x(t)),imag(x(2*t)),'k-'); grid on
>>  xlabel('Re(x(t))'); ylabel('Im(x(2t))'); axis([-1 1 -1 1]);
```

As shown by the results in Fig. S1.11-3a, the parameter a greatly impacts the overall waveform shapes.
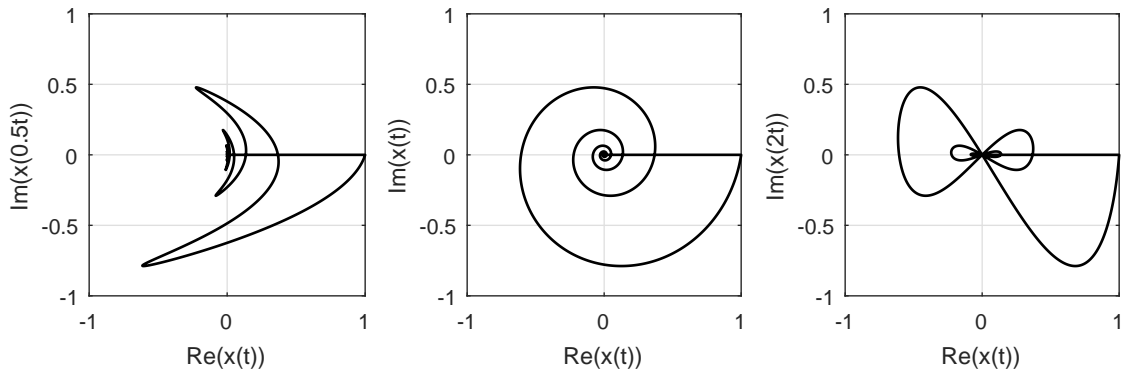

Figure S1.11-3a

(b) Because of the $u(-t)$ term, we know that $x(t)$ has a jump discontinuity at $t = 0$. Since $y(t)$ depends on $x\left(\frac{-5-t}{2}\right)$, $y(t)$ has a jump discontinuity at $\frac{-5-t_0}{2} = 0$ or $t_0 = -5$. We readily verify this calculation when we plot $y(t)$ using MATLAB, the result of which is shown in Fig. S1.11-3b.

```
>>  u = @(t) 1.0*(t>=0); t = -10:.001:10;
>>  x = @(t) exp(t*(1+2j*pi)).*u(-t); y = @(t) real(2*x((-5-t)/2));
>>  plot(t,y(t),'k-'); axis([-10 10 -2.1 2.1]);
>>  grid on; xlabel('t'); ylabel('y(t)');
>>  set(gca,'xtick',-10:5:10,'ytick',[-2:.5:2]);
```

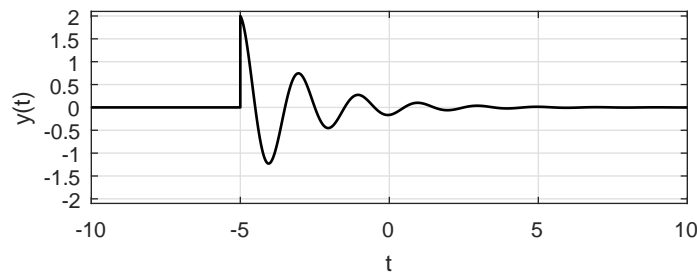As shown by the results in Fig. S1.11-3a, the parameter a greatly impacts the overall waveform shapes.

Figure S1.11-3b

(c) To determine the energy of $x(t)$, we use the quad function over a suitable range of time.

```
>>  u = @(t) 1.0*(t>=0); x = @(t) exp(t*(1+2j*pi)).*u(-t);
>>  x2 = @(t) x(t).*conj(x(t)); Ex = quad(x2,-100,0)
    Ex = 0.5000
```

Thus, we see that

$$E_x = \frac{1}{2}$$

(d) To determine the energy of $y(t)$, we use the quad function over a suitable range of time.

```
>>  u = @(t) 1.0*(t>=0); x = @(t) exp(t*(1+2j*pi)).*u(-t);
>>  y = @(t) real(2*x((-5-t)/2)); y2 = @(t) y(t).*conj(y(t));
>>  Ey = quad(y2,-5,100)
    Ey = 2.0494
```

Thus, we see that
$$E_y = 2.0494$$

Solution 1.11-4

Noting that $u(\frac{t}{2} + 1) = u(t + 2)$ and that $\delta(t/2) = 2\delta(t)$, we rewrite $x(t)$ as

$$x(t) = u(t + 2) - u(t - 1) - 2\delta(t).$$

Signal $x(t)$ is shown in Fig. S1.11-4.

(a) Due to the piecewise nature of $x(t)$, $y(t) = \int_{-\infty}^{t-3} x(\tau)\, d\tau$ is also a piecewise function.

$$y(t) = \begin{cases} 0 & t - 3 < -2 \text{ or } t < 1 \\ \int_{-2}^{t-3} 1\, d\tau = t - 1 & 1 \le t < 3 \\ 0 & t = 3 \\ \int_{0}^{t-3} 1\, d\tau = t - 3 & 3 \le t < 4 \\ 1 & t \ge 4 \end{cases}$$

More compactly, we see that

$$y(t) = (t - 1)(u(t - 1) - u(t - 4)) - 2u(t - 3) + 3u(t - 4)$$

This same result, shown in Fig. S1.11-4, can be obtained by graphically integrating (left to right) and then shifting $x(t)$.

(b) Due to the piecewise nature of $x(t)$, $z(t) = \int_{t}^{\infty} x(\tau)\, d\tau$ is also a piecewise function.

$$z(t) = \begin{cases} 0 & t > 1 \\ \int_{t}^{1} 1\, d\tau = 1 - t & 0 < t \le 1 \\ -1 & t = 0 \\ -1 + \int_{t}^{0} 1\, d\tau = -1 - t & -2 < t < 0 \\ 1 & t \le -2 \end{cases}$$

More compactly, we see that

$$z(t) = (1 - t)(u(-t + 1) - u(-t - 2)) - 2u(-t) + 3u(-t - 2)$$

This result, shown in Fig. S1.11-4, can also be obtained by graphically integrating (right to left) $x(t)$.

(c) To obtain $w(t) = \frac{d}{dt}(y(t) + z(t))$, we differentiate and then add the results from parts (a) and (b). Signal $w(t)$, shown in Fig. S1.11-4, is thus

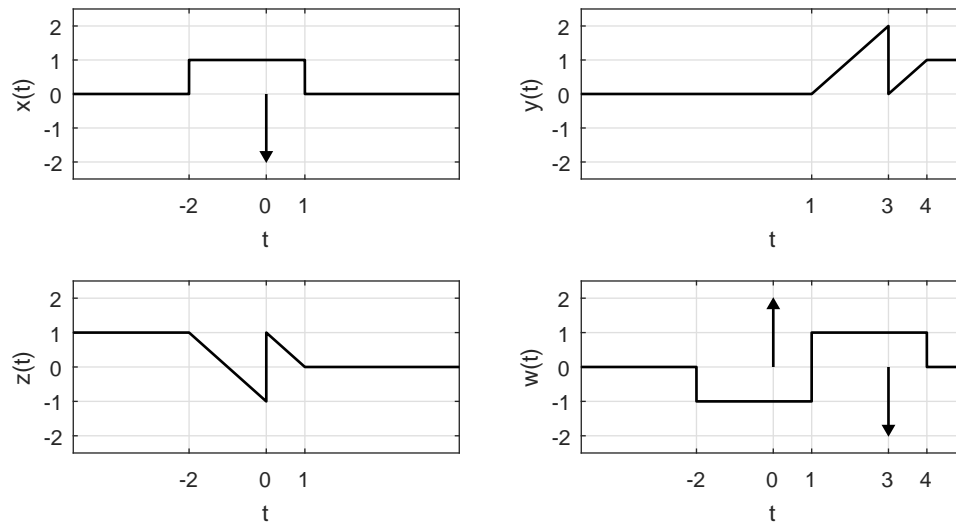$$w(t) = -u(t + 2) + 2u(t - 1) - u(t - 4) + 2\delta(t) - 2\delta(t - 3)$$

Figure S1.11-4

Chapter 2 Solutions

(a) Here, the characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. Solving, we obtain

$$\lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2j.$$

Let us designate $\lambda_1 = -1 - 2j$ and $\lambda_2 = \lambda_1^* = -1 + 2j$.

From the initial conditions, we see that

$$y_{\mathrm{zir}}(0) = 2 = c_1 + c_2 \quad \text{and} \quad \dot{y}_{\mathrm{zir}}(0) = 0 = c_1\lambda_1 + c_2\lambda_2.$$

Using the second equation, we see that $c_1 = -c_2\lambda_2/\lambda_1$. Substituting this result into the first equation yields

$$c_2\left(\frac{\lambda_1 - \lambda_2}{\lambda_1}\right) = c_2\frac{-4j}{-1-2j} = 2 \Rightarrow c_2 = \frac{-2-4j}{-4j} = 1 - \frac{j}{2}.$$

Since the system is real, we know $c_1 = c_2^*$. Putting everything together, we obtain

$$c_1 = 1 + \tfrac{j}{2},\ c_2 = 1 - \tfrac{j}{2},\ \lambda_1 = -1 - 2j,\ \text{and}\ \lambda_2 = -1 + 2j.$$

(b) The characteristic equation is again $\lambda^2 + 2\lambda + 5 = 0$. From part (a), we thus know that $\lambda_1 = -1 - 2j$ and $\lambda_2 = \lambda_1^* = -1 + 2j$.

From the initial conditions, we see that

$$y_{\mathrm{zir}}(0) = 4 = c_1 + c_2 \quad \text{and} \quad \dot{y}_{\mathrm{zir}}(0) = -1 = c_1\lambda_1 + c_2\lambda_2.$$

The unknown coefficients are solved with MATLAB

```
>>  c = inv([1 1;-1-2j -1+2j])*[4;-1]
    c = 2.0000 + 0.7500i
        2.0000 - 0.7500i
```

Putting everything together, we obtain

$$c_1 = 2 + \tfrac{3j}{4},\ c_2 = 2 - \tfrac{3j}{4},\ \lambda_1 = -1 - 2j,\ \text{and}\ \lambda_2 = -1 + 2j.$$

(c) Here, the characteristic equation is $\lambda(\lambda + 2) = 0$. Thus,

$$\lambda_1 = 0 \text{ and } \lambda_2 = -2.$$

From the initial conditions, we see that

$$y_{\mathrm{zir}}(0) = 1 = c_1 + c_2 \quad \text{and} \quad \dot{y}_{\mathrm{zir}}(0) = 2 = -2c_2.$$

Using the second equation, we see that $c_2 = -1$. Substituting this result into the first equation yields $c_1 = 2$. Putting everything together, we obtain

$$c_1 = 2, \; c_2 = -1, \; \lambda_1 = 0, \text{ and } \lambda_2 = -2.$$

(d) In this case, the characteristic equation is $\lambda^2 + 2\lambda + 10 = 0$. Solving, we obtain

$$\lambda = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3j$$

Let us designate $\lambda_1 = -1 + 3j$ and $\lambda_2 = \lambda_1^* = -1 - 3j$.

From the initial conditions, we see that

$$y_{\text{zir}}(0) = 1 = c_1 + c_2 \quad \text{and} \quad \dot{y}_{\text{zir}}(0) = 1 = c_1\lambda_1 + c_2\lambda_2$$

The unknown coefficients are solved with MATLAB

```
>>  c = inv([1 1;-1+3j -1-3j])*[1;1]
    c = 0.5000 - 0.3333i
        0.5000 + 0.3333i
```

Putting everything together, we obtain

$$c_1 = \tfrac{1}{2} - \tfrac{j}{3}, \; c_2 = \tfrac{1}{2} + \tfrac{j}{3}, \; \lambda_1 = -1 + 3j, \text{ and } \lambda_2 = -1 - 3j.$$

(e) By inspection, the characteristic equation is $(\lambda + 3)(\lambda + \tfrac{1}{2}) = 0$. Thus, $\lambda_1 = -\tfrac{1}{2}$ and $\lambda_2 = -3$.

From the initial conditions, we see that

$$y_{\text{zir}}(0) = 3 = c_1 + c_2 \quad \text{and} \quad \ddot{y}_{\text{zir}}(0) = -8 = c_1\lambda_1^2 + c_2\lambda_2^2.$$

The unknown coefficients are solved with MATLAB

```
>>  c = inv([1 1;1/4 9])*[3;-8]
    c = 4.0000
       -1.0000
```

Putting everything together, we obtain

$$c_1 = 4, \; c_2 = -1, \; \lambda_1 = -\tfrac{1}{2}, \text{ and } \lambda_2 = -3.$$

(f) In this case, the characteristic equation is $\lambda^2 + 4\lambda + 13 = 0$. Solving, we obtain

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3j.$$

Let us designate $\lambda_1 = -2 + 3j$ and $\lambda_2 = \lambda_1^* = -2 - 3j$.

From the initial conditions, we see that

$$y_{\text{zir}}(0) = 3 = c_1 + c_2 \quad \text{and} \quad \ddot{y}_{\text{zir}}(0) = -15 = c_1\lambda_1^2 + c_2\lambda_2^2.$$

The unknown coefficients are solved with MATLAB

```
>>  c = inv([1 1;(-2+3j)^2 (-2-3j)^2])*[3;-15]
    c = 1.5000 - 0.0000i
        1.5000 - 0.0000i
```

Putting everything together, we obtain

$$c_1 = c_2 = \tfrac{3}{2}, \ \lambda_1 = -2 + 3j, \text{ and } \lambda_2 = -2 - 3j.$$

In this case, notice that the zero-input response simplifies to $y_{\text{zir}}(t) = 3e^{-2t}\cos(3t)$.

Solution 2.2-2

(a) Since the highest-order derivative acting on the output is three,

$$\text{the system order is 3.}$$

(b) The characteristic equation is $(\lambda+1)(\lambda^2-1) = (\lambda+1)(\lambda+1)(\lambda-1) = 0$. Thus, the characteristic roots are

$$\lambda_1 = -1, \ \lambda_2 = -1, \text{ and } \lambda_3 = 1.$$

(c) The zero-input response and its first two derivatives are:

$$y_{\text{zir}}(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 e^t$$
$$y'_{\text{zir}}(t) = -c_1 e^{-t} + c_2[-t e^{-t} + e^{-t}] + c_3 e^t$$
$$y''_{\text{zir}}(t) = c_1 e^{-t} + c_2[t e^{-t} - 2e^{-t}] + c_3 e^t$$

From the initial conditions, we see that:

$$y_{\text{zir}}(0) = 1 = c_1 + c_3$$
$$y'_{\text{zir}}(0) = 1 = -c_1 + c_2 + c_3$$
$$y''_{\text{zir}}(0) = 1 = c_1 - 2c_2 + c_3$$

The unknown coefficients are solved with MATLAB.

```
>>  c = inv([1 0 1;-1 1 1;1 -2 1])*[1;1;1]
    c = 0
        0
        1
```

Somewhat surprisingly for a third-order system, the zero-input response is comprised of a single mode:

$$y_{\text{zir}}(t) = e^t$$

Solution 2.2-3

(a) From the differential equation, the characteristic equation is

$$\lambda^3 + 9\lambda = \lambda(\lambda^2 + 9) = 0.$$

Clearly, the characteristic roots are $\lambda_1 = 0$, $\lambda_2 = 3j$, and $\lambda_3 = -3j$.

(b) By inspection,

$$\text{the three characterstic modes are } e^{0t} = 1, \ e^{j3t}, \text{ and } e^{-j3t}.$$

(c) The form of the zero-input response is

$$y_{\text{zir}}(t) = c_1 + c_2 e^{j3t} + c_3 e^{-j3t}.$$

From the initial conditions, we see that:

$$y_{\text{zir}}(0) = 4 = c_1 + c_2 + c_3$$
$$y'_{\text{zir}}(0) = -18 = 3jc_2 - 3jc_3$$
$$y''_{\text{zir}}(0) = 0 = -9c_2 - 9c_3$$

The unknown coefficients are solved with MATLAB.

```
>>  c = inv([1 1 1;0 3j -3j;0 -9 -9])*[4;-18;0]
    c = 4.0000 + 0.0000i
        0.0000 + 3.0000i
        0.0000 - 3.0000i
```

Thus,

$$y_{\text{zir}}(t) = 4 + 3je^{j3t} - 3je^{-j3t} = 4 - 6\sin(3t).$$

Solution 2.2-4

The characteristic polynomial is $\lambda^2 + 5\lambda + 6$. The characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$. Also $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3)$. Therefore the characteristic roots are $\lambda_1 = -2$ and $\lambda_2 = -3$. The characteristic modes are e^{-2t} and e^{-3t}. Therefore,

$$y_0(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

and

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

Setting $t = 0$, and substituting initial conditions $y_0(0) = 2$, $\dot{y}_0(0) = -1$ in this equation yields

$$\left.\begin{array}{r} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = -1 \end{array}\right\} \implies \begin{array}{l} c_1 = 5 \\ c_2 = -3 \end{array}.$$

Therefore,

$$y_0(t) = 5e^{-2t} - 3e^{-3t}.$$

Solution 2.2-5

The characteristic polynomial is $\lambda^2 + 4\lambda + 4$. The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$. Also $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$, so that the characteristic roots are -2 and -2 (repeated twice). The characteristic modes are e^{-2t} and te^{-2t}. Therefore

$$y_0(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

and

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}.$$

Setting $t = 0$ and substituting initial conditions yields

$$\left.\begin{array}{r} 3 = c_1 \\ -4 = -2c_1 + c_2 \end{array}\right\} \implies \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}.$$

Therefore,

$$y_0(t) = (3 + 2t)e^{-2t}.$$

Solution 2.2-6

The characteristic polynomial is $\lambda(\lambda + 1) = \lambda^2 + \lambda$. The characteristic equation is $\lambda(\lambda + 1) = 0$. The characteristic roots are 0 and -1. The characteristic modes are 1 and e^{-t}. Therefore,

$$y_0(t) = c_1 + c_2 e^{-t}$$

and

$$\dot{y}_0(t) = -c_2 e^{-t}.$$

Setting $t = 0$, and substituting initial conditions yields

$$\left. \begin{array}{l} 1 = c_1 + c_2 \\ 1 = -c_2 \end{array} \right\} \implies \begin{array}{l} c_1 = 2 \\ c_2 = -1 \end{array}.$$

Therefore,

$$y_0(t) = 2 - e^{-t}.$$

Solution 2.2-7

The characteristic polynomial is $\lambda^2 + 9$. The characteristic equation is $\lambda^2 + 9 = 0$ or $(\lambda + j3)(\lambda - j3) = 0$. The characteristic roots are $\pm j3$. The characteristic modes are e^{j3t} and e^{-j3t}. Therefore,

$$y_0(t) = c \cos(3t + \theta)$$

and

$$\dot{y}_0(t) = -3c \sin(3t + \theta).$$

Setting $t = 0$, and substituting initial conditions yields

$$\left. \begin{array}{l} 0 = c \cos\theta \\ 6 = -3c \sin\theta \end{array} \right\} \implies \left. \begin{array}{l} c \cos\theta = 0 \\ c \sin\theta = -2 \end{array} \right\} \implies \begin{array}{l} c = 2 \\ \theta = -\pi/2 \end{array}.$$

Therefore,

$$y_0(t) = 2 \cos(3t - \frac{\pi}{2}) = 2 \sin 3t.$$

Solution 2.2-8

The characteristic polynomial is $\lambda^2 + 4\lambda + 13$. The characteristic equation is $\lambda^2 + 4\lambda + 13 = 0$ or $(\lambda + 2 - j3)(\lambda + 2 + j3) = 0$. The characteristic roots are $-2 \pm j3$. The characteristic modes are $c_1 e^{(-2+j3)t}$ and $c_2 e^{(-2-j3)t}$. Therefore,

$$y_0(t) = ce^{-2t} \cos(3t + \theta)$$

and

$$\dot{y}_0(t) = -2ce^{-2t} \cos(3t + \theta) - 3ce^{-2t} \sin(3t + \theta).$$

Setting $t = 0$, and substituting initial conditions yields

$$\left. \begin{array}{l} 5 = c \cos\theta \\ 15.98 = -2c \cos\theta - 3c \sin\theta \end{array} \right\} \implies \left. \begin{array}{l} c \cos\theta = 5 \\ c \sin\theta = -8.66 \end{array} \right\} \implies \begin{array}{l} c = 10 \\ \theta = -\pi/3 \end{array}.$$

Therefore,

$$y_0(t) = 10e^{-2t} \cos(3t - \frac{\pi}{3}).$$

Solution 2.2-9

The characteristic polynomial is $\lambda^2(\lambda + 1)$ or $\lambda^3 + \lambda^2$. The characteristic equation is $\lambda^2(\lambda + 1) = 0$. The characteristic roots are 0, 0 and -1 (0 is repeated twice). Therefore,

$$y_0(t) = c_1 + c_2 t + c_3 e^{-t}.$$

Further,
$$\dot{y}_0(t) = c_2 - c_3 e^{-t}$$

and
$$\ddot{y}_0(t) = c_3 e^{-t}.$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 4 = c_1 + c_3 \\ 3 = c_2 - c_3 \\ -1 = c_3 \end{array}\right\} \implies \begin{array}{l} c_1 = 5 \\ c_2 = 2 \\ c_3 = -1 \end{array}.$$

Therefore,
$$y_0(t) = 5 + 2t - e^{-t}.$$

Solution 2.2-10

The characteristic polynomial is $(\lambda + 1)(\lambda^2 + 5\lambda + 6)$. The characteristic equation is $(\lambda + 1)(\lambda^2 + 5\lambda + 6) = 0$ or $(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$. The characteristic roots are -1, -2 and -3. The characteristic modes are e^{-t}, e^{-2t} and e^{-3t}. Therefore,

$$y_0(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

Further,
$$\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t} - 3c_3 e^{-3t}$$

and
$$\ddot{y}_0(t) = c_1 e^{-t} + 4c_2 e^{-2t} + 9c_3 e^{-3t}.$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 2 = c_1 + c_2 + c_3 \\ -1 = -c_1 - 2c_2 - 3c_3 \\ 5 = c_1 + 4c_2 + 9c_3 \end{array}\right\} \implies \begin{array}{l} c_1 = 6 \\ c_2 = -7 \\ c_3 = 3 \end{array}.$$

Therefore,
$$y_0(t) = 6e^{-t} - 7e^{-2t} + 3e^{-3t}.$$

Solution 2.2-11

The zero-input response for a LTIC system is given as $y_0(t) = 2e^{-t} + 3$. Since two modes are visible, the system must have, at least, the characteristic roots $\lambda_1 = 0$ and $\lambda_2 = -1$.

(a) No, it is not possible for the system's characteristic equation to be $\lambda + 1 = 0$ since the required mode at $\lambda = 0$ is missing.

(b) Yes, it is possible for the system's characteristic equation to be $\sqrt{3}(\lambda^2 + \lambda) = 0$ since this equation has the two required roots $\lambda_1 = 0$ and $\lambda_2 = -1$.

(c) Yes, it is possible for the system's characteristic equation to be $\lambda(\lambda + 1)^2 = 0$. This equation supports a general zero-input response of $y_0(t) = c_1 + c_2 e^{-t} + c_3 t e^{-t}$. By letting $c_1 = 3$, $c_2 = 2$, and $c_3 = 0$, the observed zero-input response is possible.

Solution 2.2-12

(a) We know that $i_c(t) = C\dot{v}_c(t)$. Using Kirchoff's current law, we see that

$$\frac{x(t) - v_c(t)}{R} + \frac{y(t) - 0}{R_f} = 0 \quad \Rightarrow \quad v_c(t) = \frac{R}{R_f}y(t) + x(t).$$

Also,

$$i_c(t) = -\frac{y(t)}{R_f} = C\dot{v}_c(t) = C\left(\frac{R}{R_f}\dot{y}(t) + \dot{x}(t)\right).$$

Thus,

$$\frac{RC}{R_f}\dot{y}(t) + \frac{1}{R_f}y(t) = -C\dot{x}(t)$$

or

$$\dot{y}(t) + \frac{1}{RC}y(t) = -\frac{R_f}{R}\dot{x}(t).$$

Therefore,

$$a_1 = \frac{1}{RC}, b_0 = -\frac{R_f}{R}, \text{ and } b_1 = 0.$$

(b) Since $R = 300$ kΩ, $R_f = 1.2$ MΩ, and $C = 5$ μF, we see that

$$\dot{y}(t) + \frac{2}{3}y(t) = -4\dot{x}(t).$$

The characteristic equation is $\lambda + \frac{2}{3} = 0$, the characteristic root is $\lambda = -\frac{2}{3}$, and the zero-input response is $y_0(t) = c_1 e^{-2t/3}$.

We shall use $y_0(0)$ to determine the coefficient c_1. Since the zero-input response requires $x(t) = 0$, we see that

$$i_c(t) = \frac{0 - v_c(t)}{R} = \frac{0 - y_0(t)}{R_f} \quad \Rightarrow \quad y_0(t) = \frac{R_f}{R}v_c(t).$$

Thus,

$$y_0(0) = \frac{R_f}{R}v_c(0) = 4.$$

From this initial condition, we see that $c_1 = 4$ and the zero-input response is

$$y_0(t) = 4e^{-2t/3}.$$

Solution 2.3-1

(a) By inspection, the characteristic equation is $\lambda^2 + 1 = 0$, the characteristic roots are $\pm j$, and the characteristic modes are e^{jt} and e^{-jt}. Since $b_0 = 0$, the impulse response takes the form

$$h(t) = [P(D)y_n(t)]\,u(t) = \left[2D\left(c_1 e^{jt} + c_2 e^{-jt}\right)\right]u(t).$$

Using Eq. (2.18), we see that

$$y_n(0) = 0 = c_1 + c_2 \quad \text{and} \quad \dot{y}_n(0) = 1 = jc_1 - jc_2.$$

Next, we use MATLAB to solve for c_1 and c_2.

```
>>  c = inv([1 1;1j -1j])*[0;1]
    c = 0.0000 - 0.5000i
        0.0000 + 0.5000i
```

Thus,

$$c_1 = \frac{1}{2j} \text{ and } c_2 = -\frac{1}{2j}$$

and

$$h(t) = \left[2D\left(\frac{e^{jt}}{2j} - \frac{e^{-jt}}{2j}\right)\right]u(t) = [2D\,(\sin(t))]\,u(t).$$

Simplifying, we obtain

$$h(t) = 2\cos(t)u(t).$$

(b) By inspection, the characteristic equation is $\lambda^3 + \lambda = 0$, the characteristic roots are 0, j, and $-j$, and the characteristic modes are $e^{0t} = 1$, e^{jt} and e^{-jt}. Since $b_0 = 2$, the impulse response takes the form

$$h(t) = 2\delta(t) + [P(D)y_n(t)]\, u(t) = 2\delta(t) + \left[(2D^3 + 1)\left(c_1 + c_2 e^{jt} + c_3 e^{-jt}\right)\right] u(t).$$

Using Eq. (2.18), we see that

$$y_n(0) = 0 = c_1 + c_2 + c_3$$
$$\dot{y}_n(0) = 0 = jc_2 - jc_3$$
$$\ddot{y}_n(0) = 1 = -c_2 - c_3$$

Next, we use MATLAB to solve for c_1, c_2, and c_3.

```
>>  c = inv([1 1 1;0 1j -1j;0 -1 -1])*[0;0;1]
    c = 1.0000
       -0.5000
       -0.5000
```

Thus,
$$c_1 = 1,\ c_2 = -\tfrac{1}{2}\ \text{and}\ c_3 = -\tfrac{1}{2}$$

and

$$h(t) = 2\delta(t) + \left[(2D^3 + 1)\left(1 - \frac{e^{jt}}{2} - \frac{e^{-jt}}{2}\right)\right] u(t) = 2\delta(t) + \left[(2D^3 + 1)(1 - \cos(t))\right] u(t).$$

Simplifying, we obtain

$$h(t) = 2\delta(t) + \left[-2\sin(t) + 1 - \cos(t)\right] u(t).$$

(c) In this case, the characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0.$$

We determine the characterstic roots using MATLAB. Next, we use MATLAB to solve for c_1 and c_2.

```
>>  lambda = roots([1 2 5])
    lambda = -1.0000 + 2.0000i
             -1.0000 - 2.0000i
```

Thus, the characterstic modes are

$$e^{(-1+j2)t}\ \text{and}\ e^{(-1-j2)t}.$$

Since $b_0 = 0$, the impulse response takes the form

$$h(t) = [P(D)y_n(t)]\, u(t) = \left[8\left(c_1 e^{(-1+j2)t} + c_2 e^{(-1-j2)t}\right)\right] u(t).$$

Using Eq. (2.18), we see that

$$y_n(0) = 0 = c_1 + c_2 \quad \text{and} \quad \dot{y}_n(0) = 1 = (-1 + j2)c_1 + (-1 - j2)c_2.$$

Next, we use MATLAB to solve for c_1 and c_2.

```
>>  c = inv([1 1;-1+2j -1-2j])*[0;1]
    c = -0.0000 - 0.2500i
         0.0000 + 0.2500i
```

Thus,

$$c_1 = \frac{1}{4j} \text{ and } c_2 = -\frac{1}{4j}$$

and

$$h(t) = 8\left(\frac{1}{4j}e^{(-1+j2)t} - \frac{1}{4j}e^{(-1-j2)t}\right)u(t).$$

Simplifying, we obtain

$$h(t) = 4e^{-t}\sin(2t)u(t).$$

Solution 2.3-2

The characteristic equation is $\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0$. The characteristic modes are e^{-t} and e^{-3t}. Therefore,

$$y_n(t) = c_1 e^{-t} + c_2 e^{-3t}$$
$$\dot{y}_n(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$$

Setting $t = 0$, and substituting $y(0) = 0$, $\dot{y}(0) = 1$, we obtain

$$\left.\begin{array}{l} 0 = c_1 + c_2 \\ 1 = -c_1 - 3c_2 \end{array}\right\} \implies \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{array}.$$

Therefore,

$$y_n(t) = \frac{1}{2}(e^{-t} - e^{-3t})$$

and

$$h(t) = [P(D)y_n(t)]u(t) = [(D+5)y_n(t)]u(t) = [\dot{y}_n(t) + 5y_n(t)]u(t).$$

Simplifying, we see that

$$h(t) = (2e^{-t} - e^{-3t})u(t).$$

Solution 2.3-3

The characteristic equation is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$. Thus,

$$y_n(t) = c_1 e^{-2t} + c_2 e^{-3t}$$
$$\dot{y}_n(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

Setting $t = 0$, and substituting $y(0) = 0$, $\dot{y}(0) = 1$, we obtain

$$\left.\begin{array}{l} 0 = c_1 + c_2 \\ 1 = -2c_1 - 3c_2 \end{array}\right\} \implies \begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array}.$$

Therefore,

$$y_n(t) = e^{-2t} - e^{-3t}$$

and

$$[P(D)y_n(t)]u(t) = [\ddot{y}_n(t) + 7\dot{y}_n(t) + 11y_n(t)]u(t) = (e^{-2t} + e^{-3t})u(t).$$

Hence

$$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t) = \delta(t) + (e^{-2t} + e^{-3t})u(t).$$

Solution 2.3-4

The characteristic equation is $\lambda + 1 = 0$, and

$$y_n(t) = ce^{-t}.$$

In this case, the initial condition is $y_n^{n-1}(0) = y_n(0) = 1$. Setting $t = 0$ and using $y_n(0) = 1$, we obtain $c = 1$. Further,

$$y_n(t) = e^{-t} \quad \text{and} \quad P(D)y_n(t) = [-\dot{y}_n(t) + y_n(t)]u(t) = 2e^{-t}u(t).$$

Hence,

$$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t) = -\delta(t) + 2e^{-t}u(t).$$

Solution 2.3-5

The characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. Therefore,

$$y_n(t) = (c_1 + c_2t)e^{-3t}$$
$$\dot{y}_n(t) = [-3(c_1 + c_2t) + c_2]e^{-3t}$$

Using $y_n(0) = 0$ and $\dot{y}_n(0) = 1$, we obtain

$$\left.\begin{array}{l} 0 = c_1 \\ 1 = -3c_1 + c_2 \end{array}\right\} \implies \begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array}.$$

Consequently,

$$y_n(t) = te^{-3t}$$

and

$$h(t) = [P(D)y_n(t)]u(t) = [2\dot{y}_n(t) + 9y_n(t)]u(t) = (2 + 3t)e^{-3t}u(t).$$

Solution 2.3-6

From the solution to Prob. 2.2-12, the differential equation that describes the op-amp circuit of Fig. P2.2-12 is

$$\dot{y}(t) + \frac{2}{3}y(t) = -4\dot{x}(t).$$

The characteristic equation is $\lambda + \frac{2}{3} = 0$, the characteristic root is $\lambda = -\frac{2}{3}$, and $y_n(t) = c_1e^{-2t/3}$. Letting $t = 0$ and using $y_n(0) = 1$, we see that $c_1 = 1$.

From Eq. (2.17), we know that

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t) = -4\delta(t) + \left[(-4D)\left(e^{-2t/3}\right)\right]u(t).$$

Simplifying, we obtain

$$h(t) = -4\delta(t) + \frac{8}{3}e^{-2t/3}u(t).$$

Solution 2.3-7

(a) Differentiating the integral equation three times yields

$$(D^3 + 3D^2 + 2D)y(t) = (D - 1)x(t).$$

(b) The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 2\lambda = \lambda(\lambda + 1)(\lambda + 2) = 0.$$

The characteristic roots are clearly 0, -1, and -2. Thus,

the characteristic modes are $e^{0t} = 1$, e^{-t}, and e^{-2t}.

(c) Since $b_0 = 0$, the impulse response takes the form

$$h(t) = [P(D)y_n(t)]\, u(t) = \left[(D-1)\left(c_1 + c_2 e^{-t} + c_3 e^{-2t}\right)\right] u(t).$$

Using Eq. (2.18),

$$y_n(0) = 0 = c_1 + c_2 + c_3$$
$$\dot{y}_n(0) = 0 = 0 - c_2 - 2c_3$$
$$\ddot{y}_n(0) = 1 = 0 + c_2 + 4c_3$$

Next, we use MATLAB to solve for c_1, c_2, and c_3.

```
>>  c = inv([1 1 1;0 -1 -2;0 1 4])*[0;0;1]
    c =   0.5000
         -1.0000
          0.5000
```

Thus,

$$h(t) = \left[(D-1)\left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right)\right] u(t).$$

Simplifying, we obtain

$$h(t) = \left[-\frac{1}{2} + 2e^{-t} - \frac{3}{2}e^{-2t}\right] u(t).$$

Solution 2.4-1

(a) The plots of $h_1(\tau)$ and $h_2(t-\tau)$ are shown in Fig. S2.4-1.
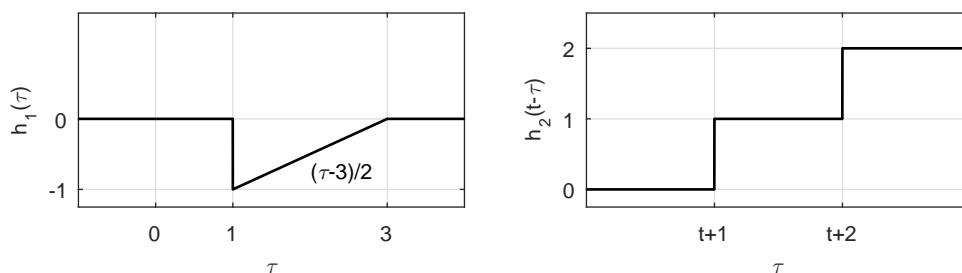
Figure S2.4-1

(b) There are five regions (R1 to R5) for this convolution.

$$
\begin{array}{lll}
\text{R1:} & t < -1 & f(t) = \int_1^3 2\left(\frac{\tau-3}{2}\right) d\tau \\[4pt]
\text{R2:} & -1 \le t < 0 & f(t) = \int_1^{t+2} \frac{\tau-3}{2}\, d\tau + \int_{t+2}^3 2\left(\frac{\tau-3}{2}\right) d\tau \\[4pt]
\text{R3:} & 0 \le t < 1 & f(t) = \int_{t+1}^{t+2} \frac{\tau-3}{2}\, d\tau + \int_{t+2}^3 2\left(\frac{\tau-3}{2}\right) d\tau \\[4pt]
\text{R4:} & 1 \le t < 2 & f(t) = \int_{t+1}^3 \frac{\tau-3}{2}\, d\tau \\[4pt]
\text{R5:} & t \ge 2 & f(t) = \int 0\, d\tau = 0
\end{array}
$$

(c) We use R4 to find $f(1)$.

$$f(1) = \int_2^3 \frac{\tau-3}{2}\, d\tau = \left.\frac{\tau^2}{4} - \frac{3\tau}{2}\right|_{\tau=2}^3 = \frac{9}{4} - \frac{9}{2} - \left(\frac{4}{4} - 3\right) = -\frac{1}{4}$$

Solution 2.4-2

To help visualize this problem, plots of $x(\tau)$ and $h(t-\tau)$ are shown in Fig. S2.4-2.

(a) The last time t_{last} that $y(t)$ is nonzero occurs when the left edge of $h(t-\tau)$, which occurs at $\tau = t-1$, reaches the right edge of $x(\tau)$, which occurs at $\tau = 3\pi/2$. Thus,

$$t_{\text{last}} - 1 = \frac{3\pi}{2} \quad \Rightarrow \quad t_{\text{last}} = \frac{3\pi + 2}{2} \approx 5.71.$$

(b) The signal $y(t)$ will be a maximum (approximately) when $x(\tau)$ (a negative pulse) is centered on the negative pulse of $h(t-\tau)$. That is,

$$t_{\text{max}} \approx \pi.$$
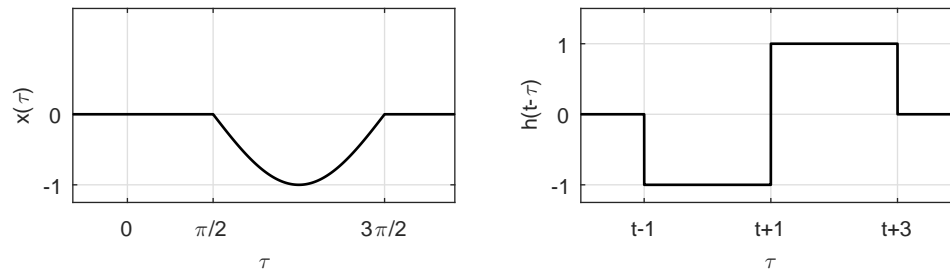
Figure S2.4-2

Solution 2.4-3

To help visualize this problem, plots of $x(\tau)$ and $h(t-\tau)$ are shown in Fig. S2.4-3. The signal $y(t)$ will be a minimum (approximately) when $x(\tau)$ (a positive pulse with center at $\tau = -\frac{3\pi}{2}$) is aligned with the negative pulse of $h(t-\tau)$, which is centered at $\tau = t+\frac{1}{2}$. That is,

$$t_{\text{min}} + \frac{1}{2} \approx -\frac{3\pi}{2} \quad \Rightarrow \quad t_{\text{min}} \approx -\frac{3\pi + 1}{2} \approx -5.21.$$

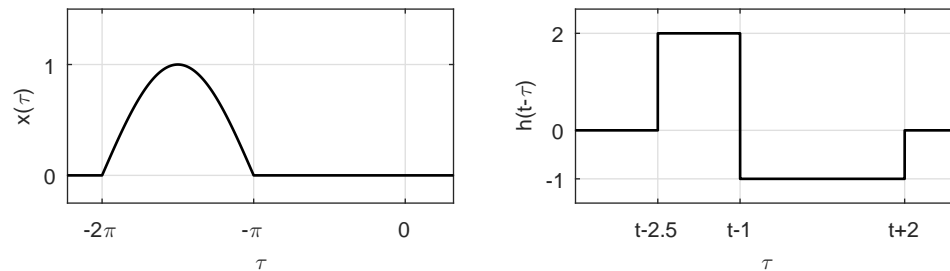
Figure S2.4-3

Solution 2.4-4

To help visualize this problem, plots of $h(\tau)$ and $x(t-\tau)$ are shown in Fig. S2.4-4. There are five regions (R1 to R5) for this convolution.

$$
\begin{aligned}
&\text{R1:} \quad && t < 1 \quad && y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0 \\
&\text{R2:} \quad && 1 \le t < 2 \quad && y(t) = \int_{2}^{t+1} 3\, d\tau = 3\tau\big|_{\tau=2}^{t+1} = 3(t+1-2) = 3t-3 \\
&\text{R3:} \quad && 2 \le t < 4 \quad && y(t) = \int_{t}^{t+1} 3\, d\tau = 3\tau\big|_{\tau=t}^{t+1} = 3(t+1-t) = 3 \\
&\text{R4:} \quad && 4 \le t < 6 \quad && y(t) = \int_{t}^{t+1} 3\, d\tau - \int_{2}^{t-2} \tfrac{3}{2}\, d\tau = 3 - \left(\tfrac{3}{2}\tau\big|_{\tau=2}^{t-2}\right) = 9 - \tfrac{3}{2}t \\
&\text{R5:} \quad && t \ge 6 \quad && y(t) = \int_{t}^{t+1} 3\, d\tau - \int_{t-4}^{t-2} \tfrac{3}{2}\, d\tau = 3 - 3 = 0
\end{aligned}
$$

Thus,

$$y(t) = \begin{cases} 3t - 3 & 1 \leq t < 2 \\ 3 & 2 \leq t < 4 \\ 9 - \frac{3}{2}t & 4 \leq t < 6 \\ 0 & \text{otherwise} \end{cases}.$$
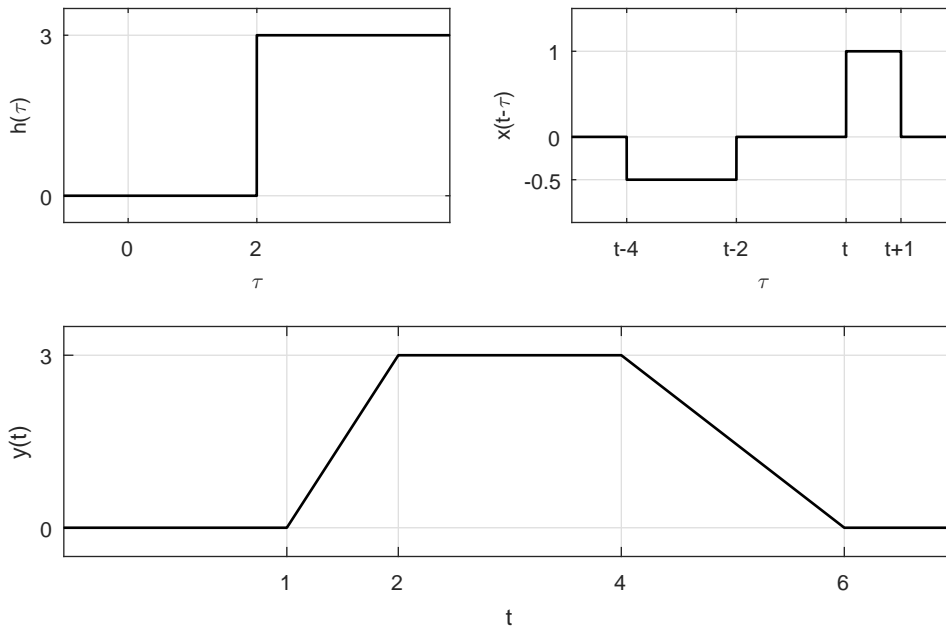
A plot of $y(t)$ is found in Fig. S2.4-4.

Figure S2.4-4

Solution 2.4-5

(a) Written in a more conventional way, we see that

$$h(t) = \begin{cases} 2t - t^2 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Since $h(t) = 0$ for all $t < 0$,

$$\text{yes, the system is causal.}$$

(b) Figure S2.4-5 plots $h(\tau)$ and $x(t - \tau)$. There are three regions (R1 to R3) for this convolution.

R1: $t < 0$ $y_{\text{zsr}}(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0$

R2: $0 \leq t < 2$ $y_{\text{zsr}}(t) = \int_0^t (2\tau - \tau^2) \, d\tau = \tau^2 - \frac{\tau^3}{3}\Big|_{\tau=0}^{t} = t^2 - \frac{t^3}{3}$

R3: $t \geq 2$ $y_{\text{zsr}}(t) = \int_0^2 (2\tau - \tau^2) \, d\tau = \tau^2 - \frac{\tau^3}{3}\Big|_{\tau=0}^{2} = 4 - \frac{8}{3} = \frac{4}{3}$

Thus,

$$y_{\text{zsr}}(t) = \begin{cases} 0 & t < 0 \\ t^2 - \frac{t^3}{3} & 0 \leq t < 2 \\ \frac{4}{3} & t \geq 2 \end{cases}.$$

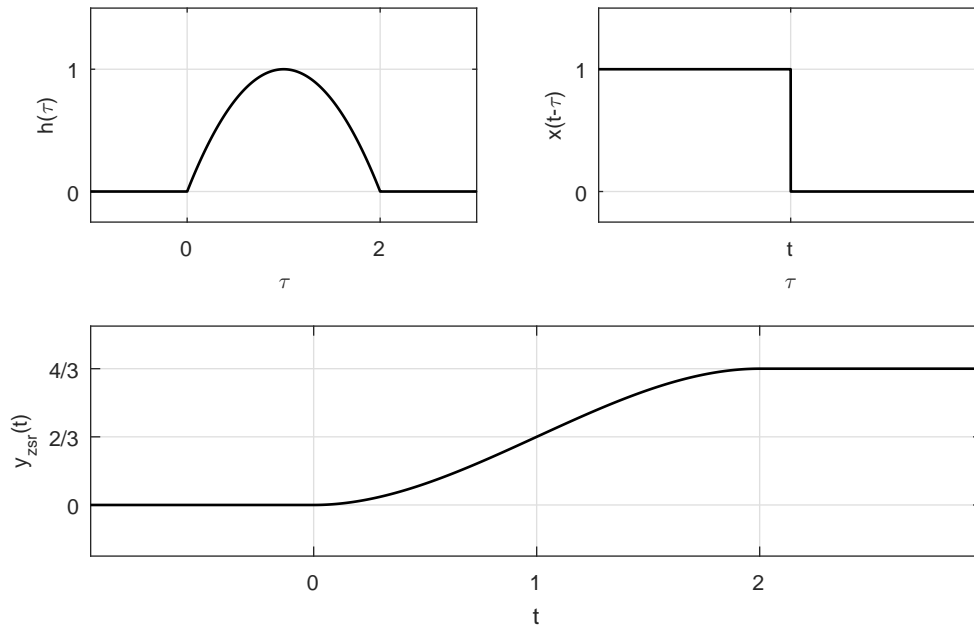A plot of $y_{\text{zsr}}(t)$ is also found in Fig. S2.4-5.

Figure S2.4-5

Solution 2.4-6

(a) Written in a more conventional way, we see that

$$h(t) = \begin{cases} (t+1)^2 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}.$$

Since $h(t) \ne 0$ for all $t < 0$,

no, the system is not causal.

(b) Figure S2.4-6 plots $h(\tau)$ and $x(t-\tau)$. There are three regions (R1 to R3) for this convolution.

R1: $\quad t < -2 \quad\quad y_{\text{zsr}}(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0$

R2: $\quad -2 \le t < 0 \quad y_{\text{zsr}}(t) = \int_{-2}^{t}(\tau^2 + 2\tau + 1)\, d\tau = \frac{\tau^3}{3} + \tau^2 + \tau \Big|_{\tau=-2}^{t} = \frac{t^3}{3} + t^2 + t + \frac{2}{3}$

R3: $\quad t \ge 0 \quad\quad y_{\text{zsr}}(t) = \int_{-2}^{0}(\tau^2 + 2\tau + 1)\, d\tau = \frac{\tau^3}{3} + \tau^2 + \tau \Big|_{\tau=-2}^{0} = \frac{2}{3}$

Thus,

$$y_{\text{zsr}}(t) = \begin{cases} 0 & t < -2 \\ \frac{t^3}{3} + t^2 + t + \frac{2}{3} & -2 \le t < 0 \\ \frac{2}{3} & t \ge 0 \end{cases}.$$

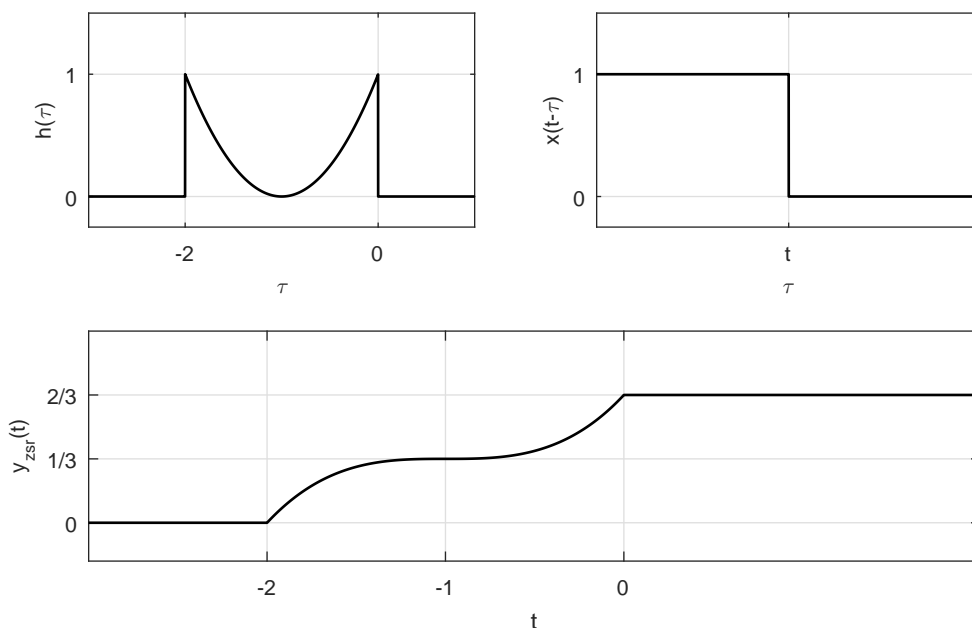A plot of $y_{\text{zsr}}(t)$ is also found in Fig. S2.4-6.

Figure S2.4-6

Solution 2.4-7

Figure S2.4-7 plots $h(\tau)$ and $x(t - \tau)$. There are five regions (R1 to R5) for this convolution.

R1: $t < 2$ $y_{\text{zsr}}(t) = \int_{-1}^{1} 2(-2)\,d\tau + \int_{1.5}^{2}(-3)(-2)\,d\tau = -4\tau|_{\tau=-1}^{1} + 6\tau|_{\tau=1.5}^{2} = -5$

R2: $2 \leq t < 4$ $y_{\text{zsr}}(t) = \int_{t-3}^{1} 2(-2)\,d\tau + \int_{1.5}^{2}(-3)(-2)\,d\tau = -4\tau|_{\tau=t-3}^{1} + 6\tau|_{\tau=1.5}^{2} = 4t - 13$

R3: $4 \leq t < \frac{9}{2}$ $y_{\text{zsr}}(t) = \int_{1.5}^{2}(-3)(-2)\,d\tau = 6\tau|_{\tau=1.5}^{2} = 3$

R4: $\frac{9}{2} \leq t < 5$ $y_{\text{zsr}}(t) = \int_{t-3}^{2}(-3)(-2)\,d\tau = 6\tau|_{\tau=t-3}^{2} = 30 - 6t$

R5: $t \geq 5$ $y_{\text{zsr}}(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0$

Thus,

$$
y_{\text{zsr}}(t) = \begin{cases}
-5 & t < 2 \\
4t - 13 & 2 \leq t < 4 \\
3 & 4 \leq t < \frac{9}{2} \\
30 - 6t & \frac{9}{2} \leq t < 5 \\
0 & t \geq 5
\end{cases} .
$$

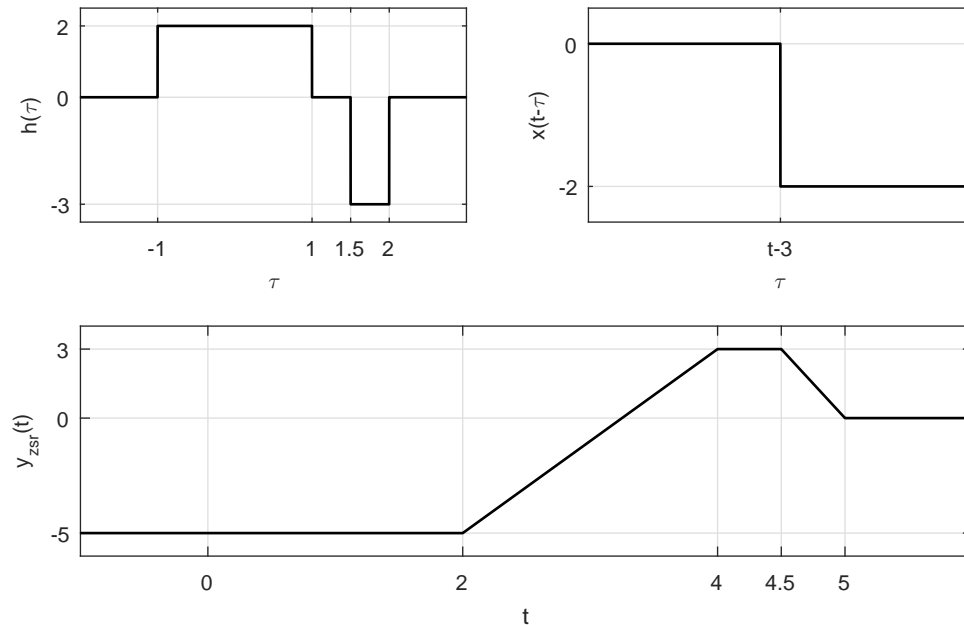A plot of $y_{\text{zsr}}(t)$ is also found in Fig. S2.4-7.

Figure S2.4-7

Solution 2.4-8

(a) Figure S2.4-8a plots $x(\tau)$ and $h(t-\tau)$. There are four regions (R1 to R4) for this convolution. Over the first region of $t < -2$,

$$y_{\text{zsr}}(t) = \int_{t-3}^{t-1} \frac{\tau - t + 3}{2} \, d\tau + \int_{t-1}^{t} 1 \, d\tau = \frac{1}{4}\tau^2 + \frac{(3-t)}{2}\tau \bigg|_{\tau=t-3}^{t-1} + \tau \big|_{\tau=t-1}^{t}$$

$$= \frac{1}{4}(t^2 - 2t + 1 - t^2 + 6t - 9) + \frac{(3-t)}{2}(t - 1 - t + 3) + (t - t + 1)$$

$$= t - 2 - t + 3 + 1 = 2.$$

For the second region, defined on $-2 \leq t < -1$, we have

$$y_{\text{zsr}}(t) = \int_{t-3}^{t-1} \frac{\tau - t + 3}{2} \, d\tau + \int_{t-1}^{-2} 1 \, d\tau = \frac{1}{4}\tau^2 + \frac{(3-t)}{2}\tau \bigg|_{\tau=t-3}^{t-1} + \tau \big|_{\tau=t-1}^{-2}$$

$$= \frac{1}{4}(t^2 - 2t + 1 - t^2 + 6t - 9) + \frac{(3-t)}{2}(t - 1 - t + 3) + (-2 - t + 1)$$

$$= t - 2 - t + 3 - t - 1 = -t.$$

The third region, defined on $-1 \leq t < 1$, has

$$y_{\text{zsr}}(t) = \int_{t-3}^{-2} \frac{\tau - t + 3}{2} \, d\tau = \frac{1}{4}\tau^2 + \frac{(3-t)}{2}\tau \bigg|_{\tau=t-3}^{-2}$$

$$= 1 - (3 - t) - \left[\frac{t^2 - 6t + 9}{4} + \frac{-t^2 + 6t - 9}{2}\right] = \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4}.$$

The fourth and final region, defined over $t \geq 1$, has

$$y_{\text{zsr}}(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0.$$

Thus,

$$y_{zsr}(t) = \begin{cases} 2 & t < -2 \\ -t & -2 \leq t < -1 \\ \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4} & -1 \leq t < 1 \\ 0 & t \geq 1 \end{cases}.$$

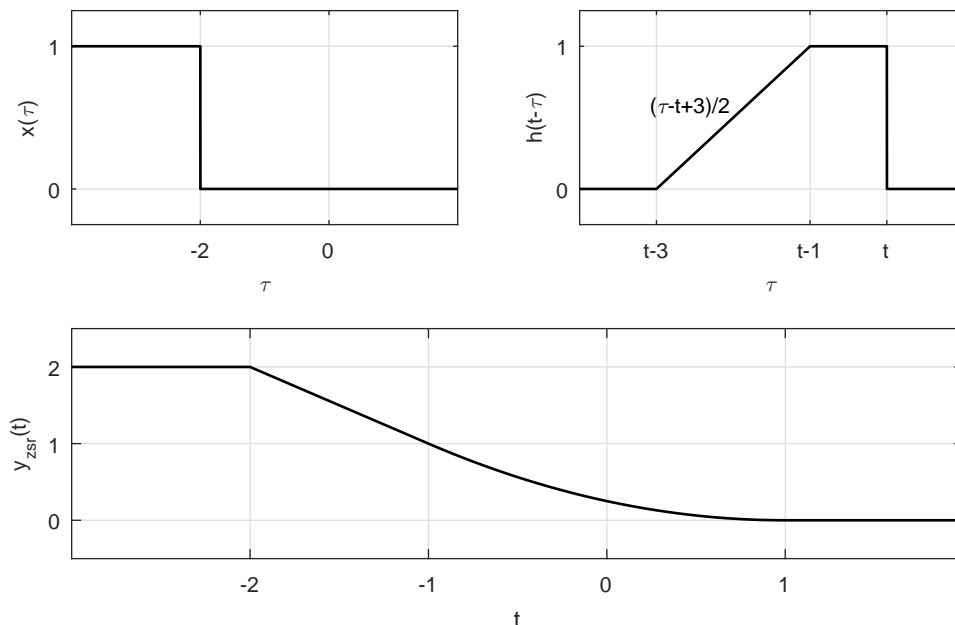A plot of $y_{zsr}(t)$, found using convolution where $h(t)$ is flipped and shifted, is shown in Fig. S2.4-8a.

Figure S2.4-8a

(b) Figure S2.4-8b plots $h(\tau)$ and $x(t - \tau)$. There are four regions (R1 to R4) for this convolution. Over the first region of $t < -2$,

$$y_{zsr}(t) = \int_0^1 1 \, d\tau + \int_1^3 \left(\frac{3 - \tau}{2} \right) d\tau = \tau|_{\tau=0}^1 + \frac{3}{2}\tau - \frac{1}{4}\tau^2 \Big|_{\tau=1}^3$$
$$= 1 + \left(\frac{9}{2} - \frac{9}{4}\right) - \left(\frac{3}{2} - \frac{1}{4}\right) = 2.$$

For the second region, defined on $-2 \leq t < -1$, we have

$$y_{zsr}(t) = \int_{t+2}^1 1 \, d\tau + \int_1^3 \left(\frac{3 - \tau}{2} \right) d\tau = \tau|_{\tau=t+2}^1 + \frac{3}{2}\tau - \frac{1}{4}\tau^2 \Big|_{\tau=1}^3$$
$$= 1 - (t + 2) + \left(\frac{9}{2} - \frac{9}{4}\right) - \left(\frac{3}{2} - \frac{1}{4}\right) = -t.$$

The third region, defined on $-1 \leq t < 1$, has

$$y_{zsr}(t) = \int_{t+2}^3 \left(\frac{3 - \tau}{2} \right) d\tau = \frac{3}{2}\tau - \frac{1}{4}\tau^2 \Big|_{\tau=t+2}^3$$
$$= \frac{9}{4} - \left(\frac{3}{2}t + 3 - \frac{t^2 + 4t + 4}{4} \right) = \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4}.$$

The fourth and final region, defined over $t \geq 1$, has

$$y_{\text{zsr}}(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0.$$

Exactly as found in part (a), we therefore see that

$$y_{\text{zsr}}(t) = \begin{cases} 2 & t < -2 \\ -t & -2 \leq t < -1 \\ \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4} & -1 \leq t < 1 \\ 0 & t \geq 1 \end{cases}.$$

A plot of $y_{\text{zsr}}(t)$, found using convolution where $x(t)$ is flipped and shifted, is shown in Fig. S2.4-8b.
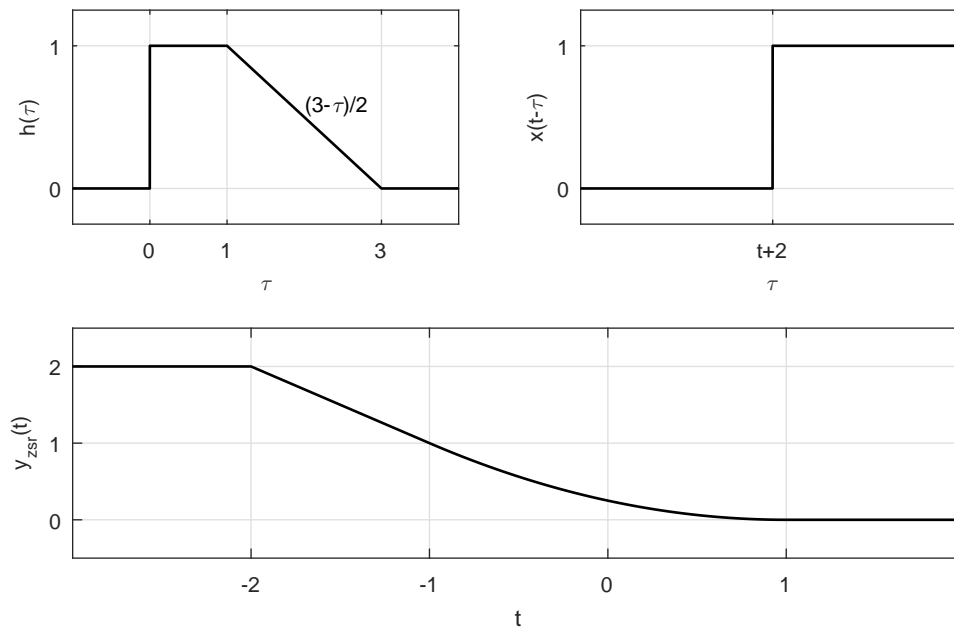

Figure S2.4-8b

Solution 2.4-9

$$A_c = \int_{-\infty}^{\infty} c(t) \, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) g(t - \tau) \, d\tau \right] dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} g(t - \tau) \, dt \right] d\tau = A_g \int_{-\infty}^{\infty} x(\tau) \, d\tau = A_g A_x$$

This property can be readily verified using Exs. 2.10 and 2.12. For Ex. 2.10, we note that

$$\int_{-\infty}^{\infty} e^{-at} \, dt = \frac{1}{a}.$$

Use of this result yields $A_x = 1$, $A_h = 0.5$, and $A_y = 1 - 0.5 = 0.5 = A_x A_h$.

For Ex. 2.12, $A_x = 2$, $A_g = 1.5$, and

$$A_c = \int_{-1}^{1} \frac{1}{6}(t + 1)^2 \, dt + \int_{1}^{2} \frac{2}{3} t \, dt + \int_{2}^{4} -\frac{1}{6}(t^2 - 2t - 8) \, dt$$

$$= \frac{4}{9} + 1 + \frac{14}{9} = 3 = A_x A_g.$$

Solution 2.4-10

$$x(at) * g(at) = \int_{-\infty}^{\infty} x(a\tau)g[a(t-\tau)]\,d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(w)g(at-w)\,dw$$

$$= \frac{1}{a}c(at) \qquad a \geq 0.$$

When $a < 0$, the limits of integration become from ∞ to $-\infty$, which is equivalent to the limits from $-\infty$ to ∞ with a negative sign. Hence, $x(at) * g(at) = |\frac{1}{a}|c(at)$.

Solution 2.4-11

Let $x(t) * g(t) = c(t)$. Using the time scaling property in Prob. 2.4-10 with $a = -1$, we have $x(-t) * g(-t) = c(-t)$. Now, if $x(t)$ and $g(t)$ are both even functions of t, then $x(t) = x(-t)$ and $g(-t) = g(t)$. Clearly $c(t) = c(-t)$. Using a parallel argument, we can show that if both functions are odd, $c(t) = c(-t)$, indicating that $c(t)$ is even. But if one is odd and the other is even, $c(t) = -c(-t)$, indicating that $c(t)$ is odd.

Solution 2.4-12

Figure S2.4-12 plots $x(\tau)$ and $h(t-\tau)$. There are five regions (R1 to R5) for this convolution. The first region, defined for $t < -1$, has

$$y_{\text{zsr}}(t) = \int_{t-1}^{t} (1-t+\tau)\,d\tau = (1-t)\tau + \frac{\tau^2}{2}\bigg|_{\tau=t-1}^{t}$$

$$= (1-t)t + \frac{t^2}{2} - \left(-t^2 + 2t - 1 + \frac{t^2-2t+1}{2}\right) = \frac{1}{2}.$$

The second region, which covers $-1 \leq t < 0$, has

$$y_{\text{zsr}}(t) = \int_{t-1}^{-1} (1-t+\tau)\,d\tau = (1-t)\tau + \frac{\tau^2}{2}\bigg|_{\tau=t-1}^{-1}$$

$$= t - 1 + \frac{1}{2} - \left(-t^2 + 2t - 1 + \frac{t^2-2t+1}{2}\right) = \frac{t^2}{2}.$$

The third region, which covers $0 \leq t < 1$, has

$$y_{\text{zsr}}(t) = \int_{-\infty}^{\infty} 0\,d\tau = 0.$$

The fourth region, which includes $1 \leq t < 2$, has

$$y_{\text{zsr}}(t) = \int_{1}^{t} (1-t+\tau)\,d\tau = (1-t)\tau + \frac{\tau^2}{2}\bigg|_{\tau=1}^{t}$$

$$= t - t^2 + \frac{t^2}{2} - (1 - t + \frac{1}{2}) = -\frac{t^2}{2} + 2t - \frac{3}{2}.$$

The fifth and final region, which includes $t \geq 2$, has

$$y_{\text{zsr}}(t) = \int_{t-1}^{t} (1-t+\tau)\,d\tau = (1-t)\tau + \frac{\tau^2}{2}\bigg|_{\tau=t-1}^{t}$$

$$= t - t^2 + \frac{t^2}{2} - \left(-t^2 + 2t - 1 + \frac{t^2-2t+1}{2}\right) = \frac{1}{2}.$$

Thus,

$$y_{zsr}(t) = \begin{cases} \frac{1}{2} & t < -1 \\ \frac{t^2}{2} & -1 \leq t < 0 \\ 0 & 0 \leq t < 1 \\ -\frac{t^2}{2} + 2t - \frac{3}{2} & 1 \leq t < 2 \\ \frac{1}{2} & t \geq 2 \end{cases}.$$
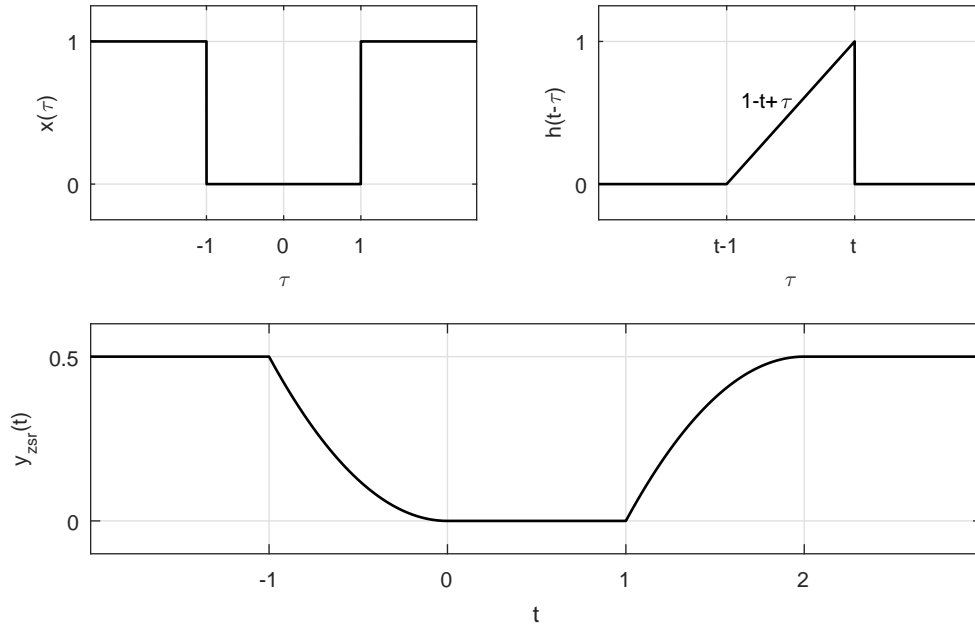
A plot of $y_{zsr}(t)$ is also found in Fig. S2.4-12.

Figure S2.4-12

Solution 2.4-13

$$e^{-at}u(t) * e^{-bt}u(t) = \int_0^t e^{-a\tau}e^{-b(t-\tau)}\, d\tau = e^{-bt}\int_0^t e^{(b-a)\tau}\, d\tau$$

$$= \frac{e^{-bt}}{b-a}e^{(b-a)\tau}\Big|_0^t = \frac{e^{-bt}}{b-a}[e^{(b-a)t} - 1] = \frac{e^{-at} - e^{-bt}}{b-a}$$

Because both functions are causal, their convolution is zero for $t < 0$. Therefore,

$$e^{-at}u(t) * e^{-bt}u(t) = \left(\frac{e^{-at} - e^{-bt}}{b-a}\right)u(t).$$

Solution 2.4-14

Since this problem involves convolving causal functions, we know the results are 0 for $t < 0$.

(a) In this case,

$$u(t) * u(t) = \begin{cases} \int_0^t u(\tau)u(t-\tau)\, d\tau = \int_0^t d\tau = \tau\Big|_0^t = t & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Therefore,

$$u(t) * u(t) = tu(t).$$

(b) Here,

$$e^{-at}u(t) * e^{-at}u(t) = \begin{cases} \int_0^t e^{-a\tau}e^{-a(t-\tau)}\,d\tau = e^{-at}\int_0^t d\tau = te^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Thus,

$$e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t).$$

(c) In this final case,

$$tu(t) * u(t) = \int_0^t \tau u(\tau)u(t-\tau)\,d\tau.$$

The range of integration is $0 \leq \tau \leq t$. Therefore $\tau > 0$ and $\tau - t > 0$ so that $u(\tau) = u(\tau - t) = 1$ and

$$tu(t) * u(t) = \int_0^t \tau\,d\tau = \frac{t^2}{2} \qquad t \geq 0.$$

Combined with the fact that the convolution is 0 for $t < 0$, we see that

$$tu(t) * u(t) = \frac{1}{2}t^2 u(t).$$

Solution 2.4-15

Since this problem involves convolving causal functions, we know the results are 0 for $t < 0$.

(a) In the first case,

$$\sin tu(t) * u(t) = \left(\int_0^t \sin \tau\, u(\tau)u(t-\tau)\,d\tau \right) u(t).$$

Because τ and $t - \tau$ are both nonnegative (when $0 \leq \tau \leq t$), $u(\tau) = u(t-\tau) = 1$, and

$$\sin t\, u(t) * u(t) = \left(\int_0^t \sin \tau\,d\tau \right) u(t) = (1 - \cos t)u(t).$$

(b) Similar to part (a), we see that

$$\cos t\, u(t) * u(t) = \left(\int_0^t \cos \tau\,d\tau \right) u(t) = \sin t\, u(t).$$

Solution 2.4-16

In this problem, we use Table 2.1 to find the desired convolution.

(a) $y(t) = h(t) * x(t) = e^{-t}u(t) * u(t) = (1 - e^{-t})u(t)$

(b) $y(t) = h(t) * x(t) = e^{-t}u(t) * e^{-t}u(t) = te^{-t}u(t)$

(c) $y(t) = e^{-t}u(t) * e^{-2t}u(t) = (e^{-t} - e^{-2t})u(t)$

(d) $y(t) = \sin 3tu(t) * e^{-t}u(t)$
Here we use pair 12 (Table 2.1) with $\alpha = 0$, $\beta = 3$, $\theta = -90°$ and $\lambda = -1$. This yields

$$\phi = \tan^{-1}\left[\frac{-3}{-1} \right] = -108.4°$$

and

$$\sin 3t\, u(t) * e^{-t}u(t) = \frac{(\cos 18.4°)e^{-t} - \cos(3t + 18.4°)}{\sqrt{10}}u(t)$$

$$= \frac{0.9486e^{-t} - \cos(3t + 18.4°)}{\sqrt{10}}u(t)$$

Solution 2.4-17

(a)

$$y(t) = (2e^{-3t} - e^{-2t})u(t) * u(t) = 2e^{-3t}u(t) * u(t) - e^{-2t}u(t) * u(t)$$

$$= \left[\frac{2(1 - e^{-3t})}{3} - \frac{1 - e^{-2t}}{2}\right]u(t)$$

$$= \left(\frac{1}{6} - \frac{2}{3}e^{-3t} + \frac{1}{2}e^{-2t}\right)u(t).$$

(b)

$$(2e^{-3t} - e^{-2t})u(t) * e^{-t}u(t) = 2e^{-3t}u(t) * e^{-t}u(t) - e^{-2t}u(t) * e^{-t}u(t)$$

$$= \left[\frac{2(e^{-t} - e^{-3t})}{2} - \frac{e^{-t} - e^{-2t}}{1}\right]u(t)$$

$$= (e^{-2t} - e^{-3t})u(t).$$

(c)

$$y(t) = (2e^{-3t} - e^{-2t})u(t) * e^{-2t}u(t) = 2e^{-3t}u(t) * e^{-2t}u(t) - e^{-2t}u(t) * e^{-2t}u(t)$$

$$= \left[\frac{2(e^{-2t} - e^{-3t})}{1} - te^{-2t}\right]u(t)$$

$$= [(2 - t)e^{-2t} - 2e^{-3t}]u(t).$$

Solution 2.4-18

$$y(t) = (1 - 2t)e^{-2t}u(t) * u(t) = e^{-2t}u(t) * u(t) - 2te^{-2t}u(t) * u(t)$$

$$= \left[\left(\frac{1 - e^{-2t}}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}e^{-2t} - te^{-2t}\right)\right]u(t)$$

$$= te^{-2t}u(t).$$

Solution 2.4-19

(a) For $y(t) = 4e^{-2t}\cos 3t\, u(t) * u(t)$, we use pair 12 of Table 2.1 with $\alpha = 2$, $\beta = 3$, $\theta = 0$, $\lambda = 0$. Therefore,

$$\phi = \tan^{-1}\left[\frac{-3}{2}\right] = -56.31°$$

and

$$y(t) = 4\left[\frac{\cos(56.31°) - e^{-2t}\cos(3t + 56.31°)}{\sqrt{4 + 9}}\right]u(t)$$

$$= \frac{4}{\sqrt{13}}\left[0.555 - e^{-2t}\cos(3t + 56.31°)\right]u(t).$$

(b) For $y(t) = 4e^{-2t}\cos 3t\, u(t) * e^{-t}u(t)$, we use pair 12 of Table 2.1 with $\alpha = 2$, $\beta = 3$, $\theta = 0$, and $\lambda = -1$. Therefore,

$$\phi = \tan^{-1}\left[\frac{-3}{1}\right] = -71.56°$$

and

$$y(t) = 4 \left[\frac{\cos(71.56°)e^{-t} - e^{-2t}\cos(3t + 71.56°)}{\sqrt{10}} \right] u(t)$$

$$= \frac{4}{\sqrt{10}} \left[0.316e^{-t} - e^{-2t}\cos(3t + 71.56°) \right] u(t)$$

$$= 4 \left[0.1e^{-t} - \frac{1}{\sqrt{10}}e^{-2t}\cos(3t + 71.56°) \right] u(t).$$

Solution 2.4-20

(a) Using pair 4 of Table 2.1,

$$y(t) = e^{-t}u(t) * e^{-2t}u(t) = (e^{-t} - e^{-2t})u(t).$$

(b) Since $e^{-2(t-3)}u(t) = e^6 e^{-2t}u(t)$, we use pair 4 of Table 2.1 to obtain

$$y(t) = e^6 \left[e^{-t}u(t) * e^{-2t}u(t) \right] = e^6(e^{-t} - e^{-2t})u(t).$$

(c) Here, $e^{-2t}u(t-3) = e^{-6}e^{-2(t-3)}u(t-3)$. Using the result in part (a) and the shift property of convolution [Eq. (2.28)], we obtain

$$y(t) = e^{-6} \left[e^{-(t-3)}u(t) - e^{-2(t-3)} \right] u(t-3).$$

(d) In this case, $x(t) = u(t) - u(t-1)$. Now $y_1(t)$, the system response to $x_1(t) = u(t)$, is given by

$$y_1(t) = e^{-t}u(t) * u(t) = (1 - e^{-t})u(t).$$

The system response to $u(t-1)$ is $y_1(t-1)$ because of time-invariance property. Therefore, the response $y(t)$ to $x(t) = u(t) - u(t-1)$ is given by

$$y(t) = y_1(t) - y_1(t-1) = (1 - e^{-t})u(t) - [1 - e^{-(t-1)}]u(t-1).$$
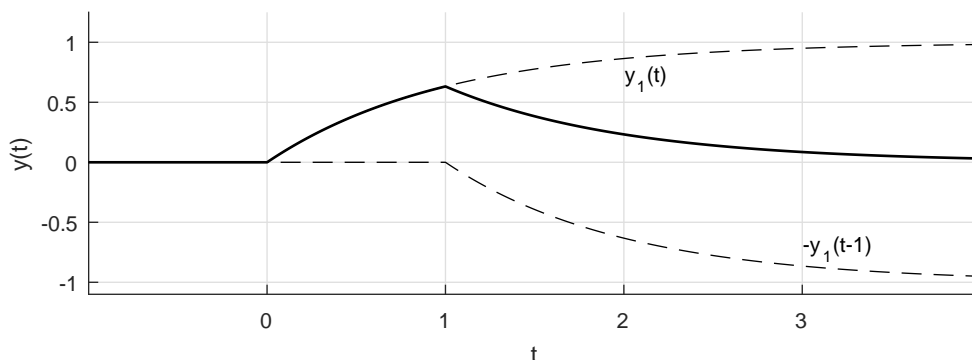
The response is shown in Fig. S2.4-20d.

Figure S2.4-20d

Solution 2.4-21

(a) Here,

$$
\begin{aligned}
y_{\text{zsr}}(t) &= [-\delta(t) + 2e^{-t}u(t)] * e^t u(-t) \\
&= -\delta(t) * e^t u(-t) + 2e^{-t}u(t) * e^t u(-t) \\
&= -e^t u(-t) + [e^{-t}u(t) + e^t u(-t)] \\
&= e^{-t}u(t)
\end{aligned}
$$

(b) The input $x(t) = e^t u(-t)$ and corresponding zero-state response $y_{\text{zsr}}(t)$ are shown in Fig. S2.4-21b. This is quite an interesting case. Although the system is an allpass filter, we see that that output is dramatically different in appearance than the input. In fact, we see that our anticausal input generates a completely causal output!
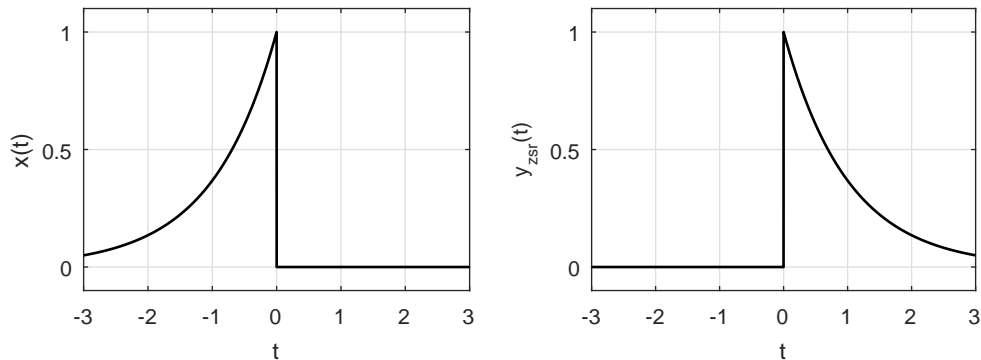

Figure S2.4-21b

Solution 2.4-22

(a) We first plot the input $x(t)$ and the impulse response $h(t)$ (shaded). To determine $y(t)$ for $t = -1$, 0, 1, 2, 3, 4, 5, and 6, we plot $x(\tau)$ and $h(t - \tau)$ (shaded) for the respective values of t; the output $y(t)$ is just the area of the product $x(\tau)h(t - \tau)$. See Fig. S2.4-22.

(b) Using Fig. S2.4-22, it is easy to show that the system response $y(t)$ to input $x(t)$ is

$$
y(t) = \begin{cases}
0 & t < -1 \\
t + 1 & -1 \le t < 0 \\
1 & 0 \le t < 5 \\
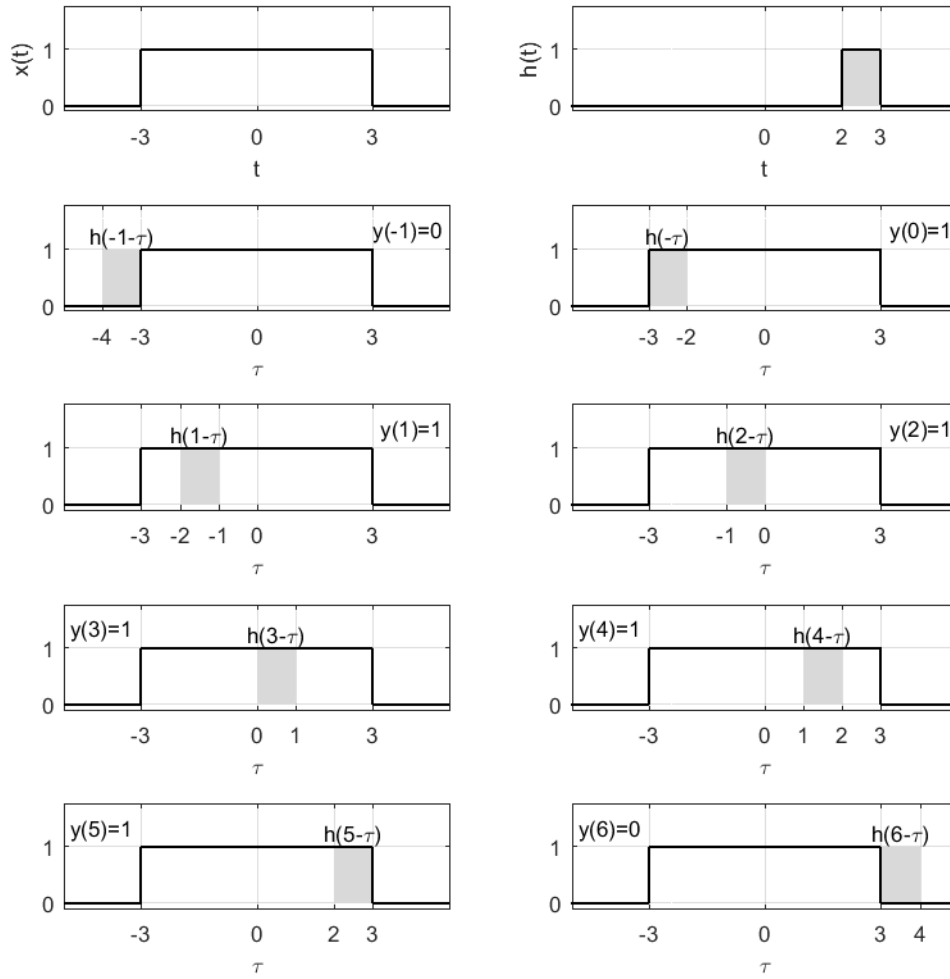6 - t & 5 \le t < 6 \\
0 & t \ge 6
\end{cases}.
$$

Figure S2.4-22

Solution 2.4-23

The output has the term $e^{-3t}u(t)$ that is not in the input. Hence, $h(t)$ should include the term $e^{-3t}u(t)$. There is also a possibility of an impulse term in $h(t)$ that will result in a term of the form $e^{-2t}u(t)$ in the output. Let us try

$$h(t) = a\delta(t) + be^{-3t}u(t).$$

This yields the output

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= 2e^{-2t}u(t) * \left[a\delta(t) + be^{-3t}u(t)\right] \\
&= 2ae^{-2t}u(t) + 2b\left[e^{-2t} - e^{-3t}\right]u(t) \\
&= (2a + 2b)e^{-2t} - 2be^{-3t}u(t).
\end{aligned}$$

Matching the coefficients of similar terms yields

$$\left.\begin{array}{r} 2a + 2b = 4 \\ -2b = 6 \end{array}\right\} \implies \begin{array}{l} a = 5 \\ b = -3 \end{array}.$$

Hence,

$$h(t) = 5\delta(t) - 3e^{-3t}u(t).$$

Solution 2.4-24

Here,

$$\frac{1}{t^2 + 1} * u(t) = \int_{-\infty}^{\infty} \frac{1}{\tau^2 + 1} u(t - \tau)\, d\tau.$$

Because $u(t - \tau) = 1$ for $\tau < t$ and is 0 for $\tau > t$, we need integrate only up to $\tau = t$.

$$\frac{1}{t^2 + 1} * u(t) = \int_{-\infty}^{t} \frac{1}{\tau^2 + 1}\, d\tau = \left.\tan^{-1}\tau\right|_{-\infty}^{t} = \tan^{-1} t + \frac{\pi}{2}.$$

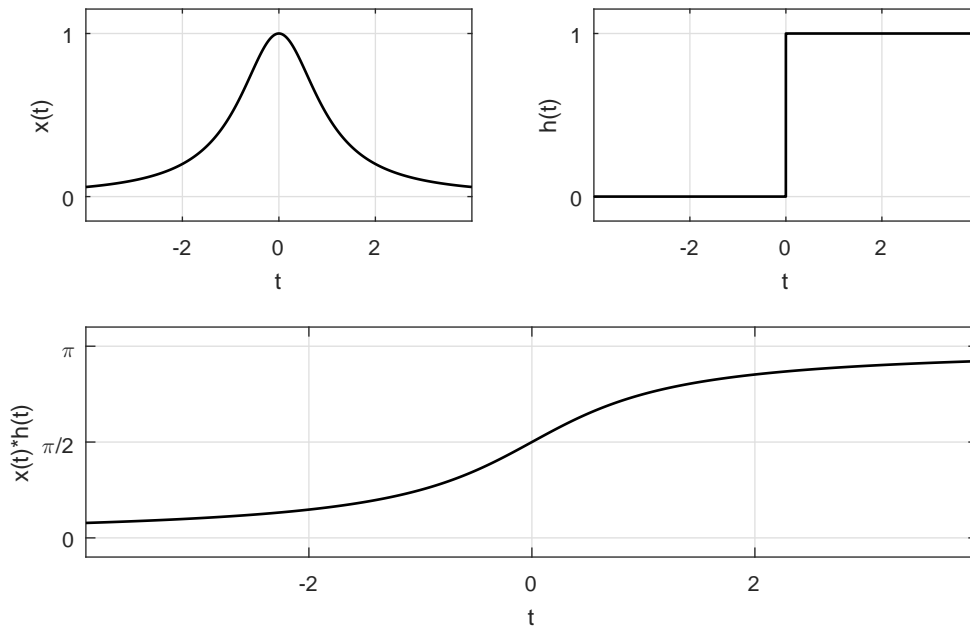Figure S2.4-24 shows $x(t)$, $h(t)$, and $x(t) * h(t)$.

Figure S2.4-24

Solution 2.4-25

To help visualize this problem, plots of $x(\tau)$ and $g(t - \tau)$ are shown in Fig. S2.4-25. There are three regions (R1 to R3) for the convolution $c(t) = x(t) * g(t)$.

$$\begin{array}{lll} \text{R1:} & t < 0 & c(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0 \\ \text{R2:} & 0 \le t < 2\pi & c(t) = \int_{0}^{t} \sin\tau\, d\tau = 1 - \cos t \\ \text{R3:} & t \ge 2\pi & c(t) = \int_{0}^{2\pi} \sin\tau\, d\tau = 0 \end{array}$$

Thus,

$$c(t) = \begin{cases} 0 & t < 0 \\ 1 - \cos t & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}.$$
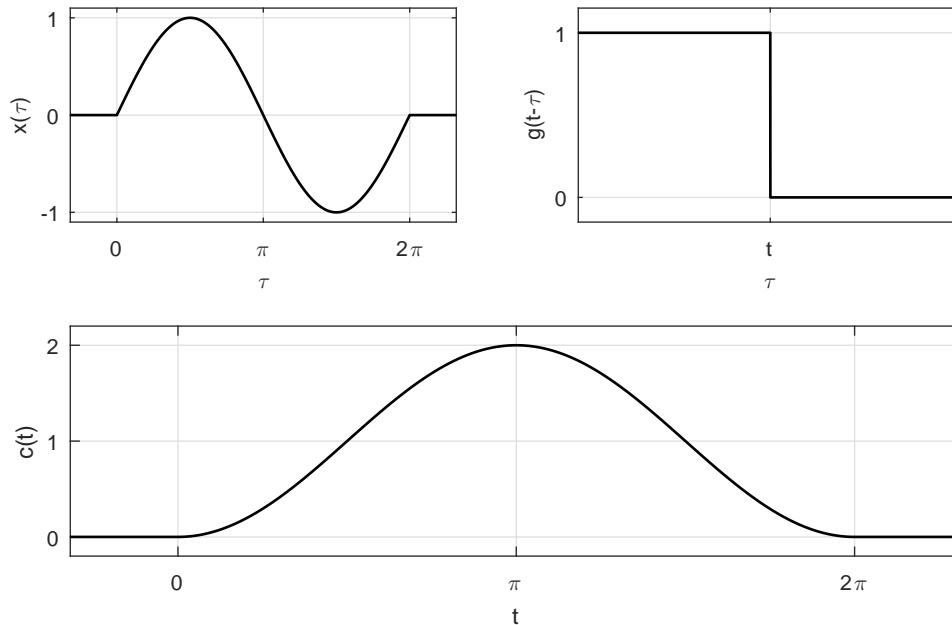
A plot of $c(t)$ is also found in Fig. S2.4-25.

Figure S2.4-25

Solution 2.4-26

To help visualize this problem, plots of $x(\tau)$ and $g(t - \tau)$ are shown in Fig. S2.4-26. There are four regions (R1 to R4) for the convolution $c(t) = x(t) * g(t)$.

$$
\begin{array}{lll}
\text{R1:} & t < 0 & c(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0 \\
\text{R2:} & 0 \leq t < 2\pi & c(t) = \int_{0}^{t} \sin \tau \, d\tau = 1 - \cos t \\
\text{R2:} & 2\pi \leq t < 4\pi & c(t) = \int_{t-2\pi}^{2\pi} \sin \tau \, d\tau = \cos t - 1 \\
\text{R3:} & t \geq 4\pi & c(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0
\end{array}
$$

Thus,

$$
c(t) = \begin{cases}
0 & t < 0 \\
1 - \cos t & 0 \leq t < 2\pi \\
\cos t - 1 & 2\pi \leq t < 4\pi \\
0 & t \geq 4\pi
\end{cases}.
$$

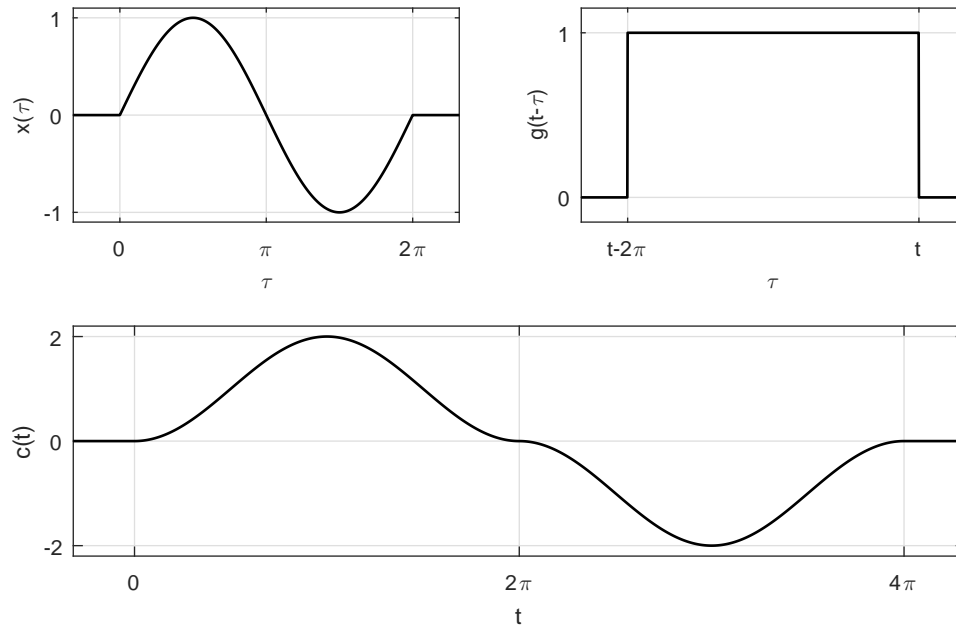A plot of $c(t)$ is also found in Fig. S2.4-26.

Figure S2.4-26

Solution 2.4-27

(a) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t - \tau)$ are shown in Fig. S2.4-27a. There are five regions (R1 to R5) for the convolution $c(t) = x_1(t) * x_2(t)$.

$$\text{R1:} \quad t < -1 \quad c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0$$

$$\text{R2:} \quad -1 \leq t < 0 \quad c(t) = \int_{4}^{t+5} AB \, d\tau = AB(t + 1)$$

$$\text{R3:} \quad 0 \leq t < 1 \quad c(t) = \int_{t+4}^{t+5} AB \, d\tau = AB$$

$$\text{R4:} \quad 1 \leq t < 2 \quad c(t) = \int_{t+4}^{6} AB \, d\tau = AB(2 - t)$$

$$\text{R5:} \quad t \geq 2 \quad c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0$$
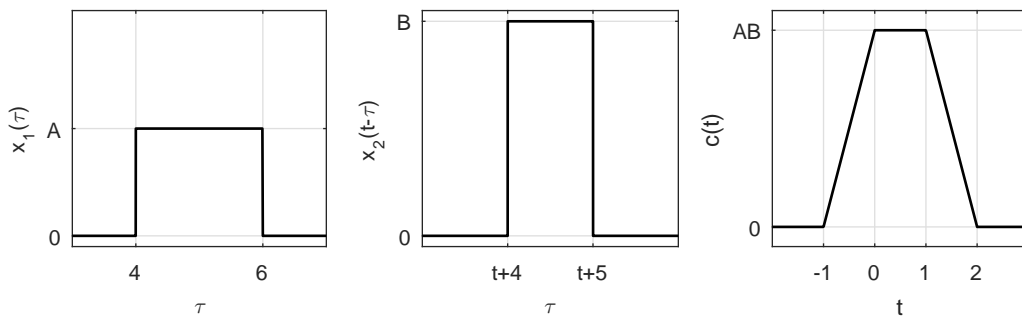
A plot of $c(t)$ is also found in Fig. S2.4-27a.

Figure S2.4-27a

(b) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t - \tau)$ are shown in Fig. S2.4-27b. There

are four regions (R1 to R4) for the convolution $c(t) = x_1(t) * x_2(t)$.

$$\begin{array}{llll}
\text{R1:} & t < -2 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0 \\
\text{R2:} & -2 \le t < 0 & c(t) = \int_{3}^{t+5} AB \, d\tau = AB(t+2) \\
\text{R3:} & 0 \le t < 2 & c(t) = \int_{t+3}^{5} AB \, d\tau = AB(2-t) \\
\text{R4:} & t \ge 2 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0
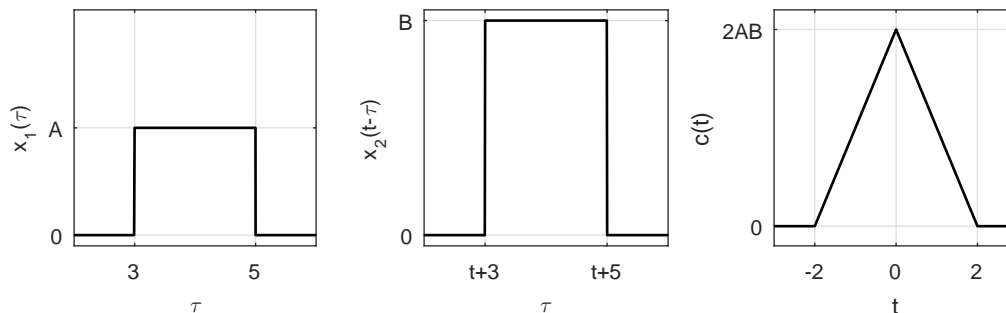\end{array}$$

A plot of $c(t)$ is also found in Fig. S2.4-27b.

Figure S2.4-27b

(c) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t-\tau)$ are shown in Fig. S2.4-27c. There are three regions (R1 to R3) for the convolution $c(t) = x_1(t) * x_2(t)$.

$$\begin{array}{llll}
\text{R1:} & t < -4 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0 \\
\text{R2:} & -4 \le t < -1 & c(t) = \int_{-2}^{t+2} 1 \, d\tau = t + 4 \\
\text{R3:} & t \ge -1 & c(t) = \int_{t-1}^{t+2} 1 \, d\tau = 3
\end{array}$$

A plot of $c(t)$ is also found in Fig. S2.4-27c.

Figure S2.4-27c

(d) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t-\tau)$ are shown in Fig. S2.4-27d. There are three regions (R1 to R3) for the convolution $c(t) = x_1(t) * x_2(t)$.

$$\begin{array}{llll}
\text{R1:} & t < -3 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0 \\
\text{R2:} & -3 \le t < 0 & c(t) = \int_{0}^{t+3} e^{-\tau} \, d\tau = 1 - e^{-(t+3)} \\
\text{R3:} & t \ge 0 & c(t) = \int_{t}^{t+3} e^{-\tau} \, d\tau = e^{-t} - e^{-(t+3)}
\end{array}$$

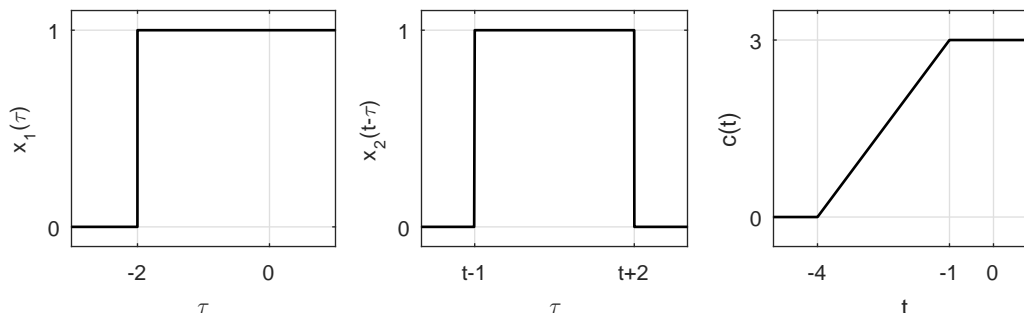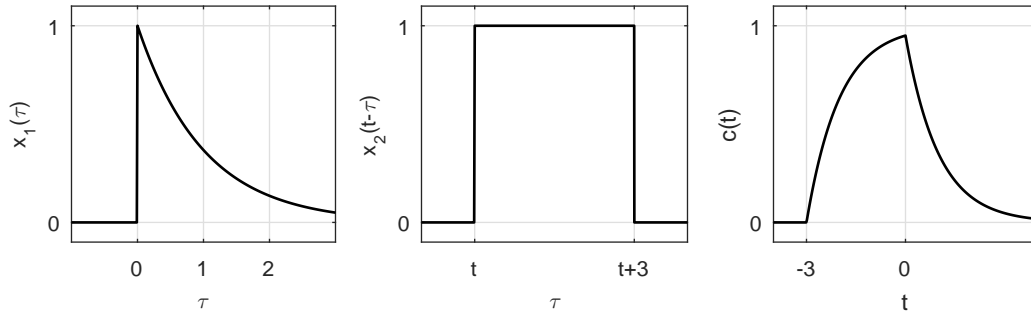A plot of $c(t)$ is also found in Fig. S2.4-27d.

Figure S2.4-27d

(e) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t-\tau)$ are shown in Fig. S2.4-27e. There are two regions (R1 and R2) for the convolution $c(t) = x_1(t) * x_2(t)$.

R1: $\quad t < 1 \quad c(t) = \int_{-\infty}^{t-1} \frac{1}{\tau^2+1}\, d\tau = \tan^{-1}(t-1) + \frac{\pi}{2}$

R2: $\quad t \geq 1 \quad c(t) = \int_{-\infty}^{0} \frac{1}{\tau^2+1}\, d\tau = \frac{\pi}{2}$

A plot of $c(t)$ is also found in Fig. S2.4-27e.

Figure S2.4-27d

(f) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t-\tau)$ are shown in Fig. S2.4-27f. There are three regions (R1 to R3) for the convolution $c(t) = x_1(t) * x_2(t)$.

R1: $\quad t < 0 \quad\quad c(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0$

R2: $\quad 0 \leq t < 3 \quad c(t) = \int_{0}^{t} e^{-\tau}\, d\tau = 1 - e^{-t}$

R3: $\quad t \geq 0 \quad\quad c(t) = \int_{t-3}^{t} e^{-\tau}\, d\tau = e^{-(t-3)} - e^{-t}$

A plot of $c(t)$ is also found in Fig. S2.4-27f.

Figure S2.4-27f

(g) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t-\tau)$ are shown in Fig. S2.4-27g. There are three regions (R1 to R3) for the convolution $c(t) = x_1(t) * x_2(t)$.

$$
\begin{array}{lll}
\text{R1:} & t < -1 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0 \\[4pt]
\text{R2:} & -1 \le t < 0 & c(t) = \int_{-1}^{t} -\tau \, d\tau = \frac{1}{2} - \frac{t^2}{2} \\[4pt]
\text{R3:} & t \ge 0 & c(t) = \int_{-1}^{0} -\tau \, d\tau = \frac{1}{2}
\end{array}
$$

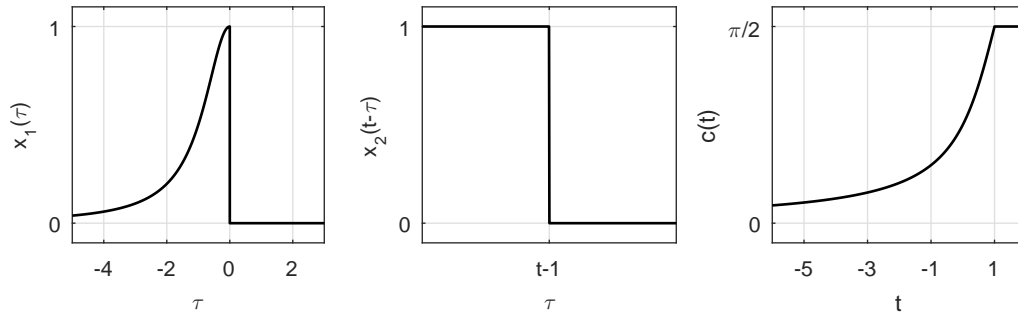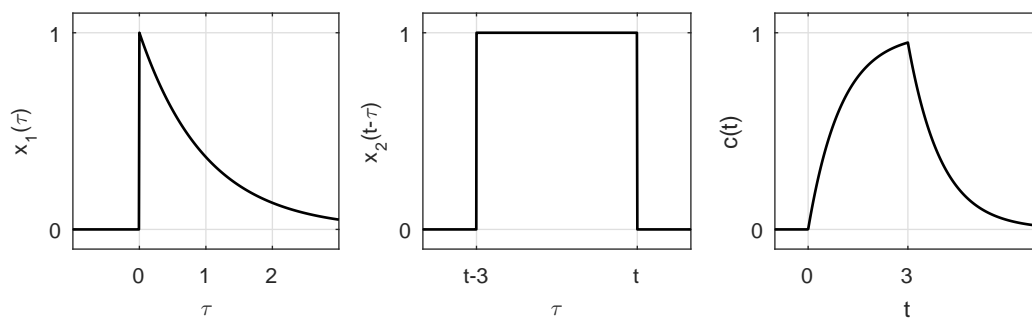A plot of $c(t)$ is also found in Fig. S2.4-27g.
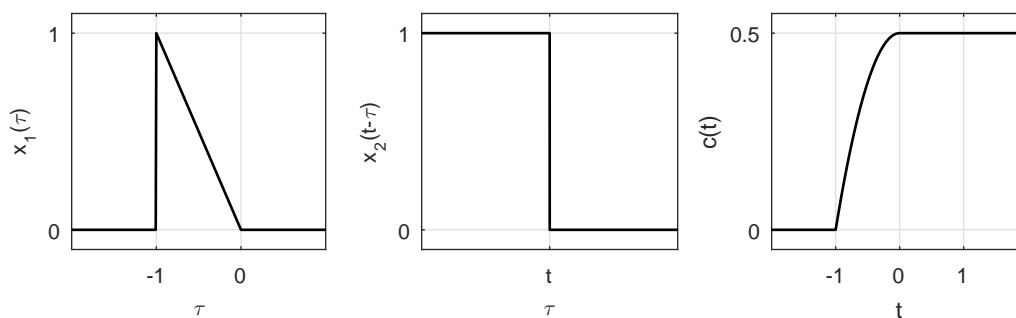
Figure S2.4-27g

(h) To help visualize this problem, plots of $x_1(\tau)$ and $x_2(t-\tau)$ are shown in Fig. S2.4-27h. There are five regions (R1 to R5) for the convolution $c(t) = x_1(t) * x_2(t)$.

$$
\begin{array}{lll}
\text{R1:} & t < -2 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0 \\[4pt]
\text{R2:} & -2 \le t < -1 & c(t) = \int_{-2}^{t} e^{\tau} e^{-2(t-\tau)} \, d\tau = \frac{e^{-2t}}{3}\left(e^{3t} - e^{-6}\right) \\[4pt]
\text{R3:} & -1 \le t < 0 & c(t) = \int_{t-1}^{t} e^{\tau} e^{-2(t-\tau)} \, d\tau = \frac{e^{-2t}}{3}\left(e^{3t} - e^{3(t-1)}\right) \\[4pt]
\text{R4:} & 0 \le t < 1 & c(t) = \int_{t-1}^{0} e^{\tau} e^{-2(t-\tau)} \, d\tau = \frac{e^{-2t}}{3}\left(1 - e^{3(t-1)}\right) \\[4pt]
\text{R5:} & t \ge 1 & c(t) = \int_{-\infty}^{\infty} 0 \, d\tau = 0
\end{array}
$$

A plot of $c(t)$ is also found in Fig. S2.4-27h.
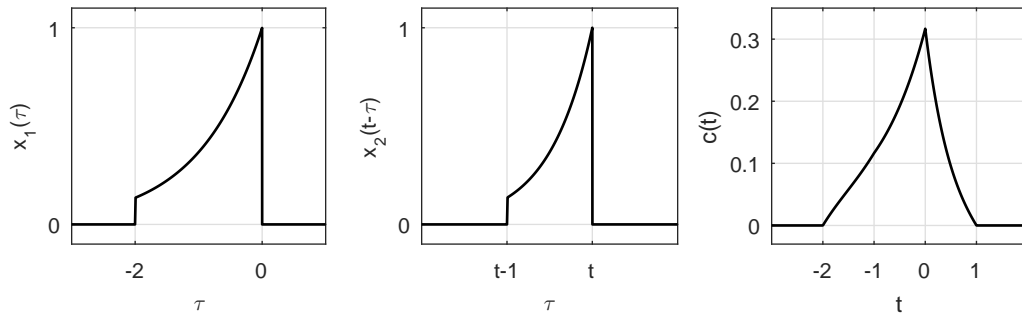
Figure S2.4-27h

Solution 2.4-28

By inspection, we find

$$\dot{x}(t) = \delta(t) - \delta(t-2)$$

and

$$\int_0^t w(\tau)d\tau = \Delta\left(\frac{t-1}{2}\right),$$

where $\Delta(t)$ is a unit-triangle function. Therefore [see Eq. (2.37)],

$$x(t) * w(t) = [\delta(t) - \delta(t-2)] * \Delta\left(\frac{t-1}{2}\right)$$

$$= \Delta\left(\frac{t-1}{2}\right) - \Delta\left(\frac{t-3}{2}\right)$$

Figure S2.4-28 shows $x(t) * w(t)$.
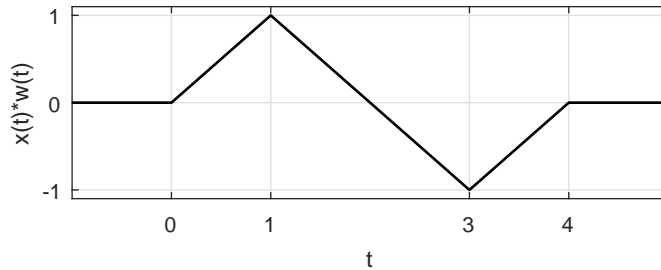
Figure S2.4-28

Solution 2.4-29

The unit impulse response of an ideal delay of T seconds is $h(t) = \delta(t-T)$. Using Eq. (2.39), we obtain

$$H(s) = \int_{-\infty}^{\infty} \delta(t-T)e^{-st}dt = e^{-sT}.$$

For an input $x(t) = e^{st}$, the output of the delay is $y(t) = e^{s(t-T)}$. Hence, according to Eq. (2.40)

$$H(s) = \frac{e^{s(t-T)}}{e^{st}} = e^{-sT}.$$

## Solution 2.4-30

To help visualize this problem, plots of $h(\tau)$ and $x(t-\tau)$ are shown in Fig. S2.4-30. There are three

regions (R1 to R3) for the convolution $y(t) = x(t) * h(t)$.

$$
\begin{array}{lll}
\text{R1:} & t < -1 & y(t) = \int_{-\infty}^{\infty} 0\, d\tau = 0 \\
\text{R2:} & -1 \le t < 0 & y(t) = \int_{-1}^{t}(\tau + 1)\, d\tau = \frac{t^2}{2} + t + \frac{1}{2} \\
\text{R3:} & t \ge 0 & y(t) = \int_{-1}^{0}(\tau + 1)\, d\tau = \frac{1}{2}
\end{array}
$$

Thus,

$$
y(t) = \begin{cases}
0 & t < -1 \\
\frac{t^2}{2} + t + \frac{1}{2} & -1 \le t < 0 \\
\frac{1}{2} & t \ge 0
\end{cases}.
$$

A plot of $y(t)$ is also found in Fig. S2.4-30g.
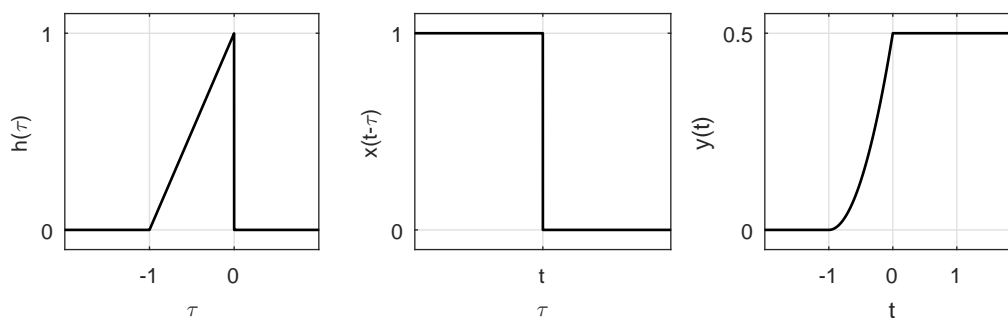
Figure S2.4-30

Solution 2.4-31

Using the graph of the system response, $h(t) = (-t/2 + 1)(u(t) - u(t - 2))$. $y(1) = \int_{-\infty}^{\infty} h_{\text{total}}(\tau)x(1 - \tau)d\tau$. Since $x(t)$ is causal, the upper limit of the integral is one. Furthermore, since $h(t)$ is causal, the total response $h_{\text{total}}(t) = h(t) * h(t)$ is also causal, which makes the lower limit of the integral zero. Over $[0, 1]$, $x(t) = u(t) = 1$. Thus, $y(1) = \int_{0}^{1} h_{\text{total}}(\tau)d\tau$. To compute $y(1)$, it is only necessary to know $h_{\text{total}}(t)$ up to $t = 1$.

Over $(0 \le t < 2)$, $h_{\text{total}}(t) = \int_{0}^{t}(-\tau/2 + 1)(-(t - \tau)/2 + 1)d\tau = \int_{0}^{t}(-\tau/2 + 1)(\tau/2 + 1 - t/2)d\tau = \int_{0}^{t}\left(-\tau^2/4 + \tau(1 - 1 + t/2)/2 + (1 - t/2)\right)d\tau = -\frac{t^3}{12} + \frac{t^3}{8} + (1 - t/2)t = \frac{t^3}{24} - \frac{t^2}{2} + t$.

Thus,

$$
y(1) = \int_{0}^{1}(\tau^3/24 - \tau^2/2 + \tau)d\tau = \left.\frac{\tau^4}{96} - \frac{\tau^3}{6} + \frac{\tau^2}{2}\right|_{\tau=0}^{1} = \frac{1}{96} - \frac{1}{6} + \frac{1}{2} = \frac{11}{32} = 0.34375.
$$

Solution 2.4-32

(a) Using KVL, $x(t) = v_L(t) + y(t)$. Also, $i_C(t) = C\frac{dy}{dt}$ and $v_L(t) = L\frac{di_L}{dt} = L\frac{di_C}{dt} = LC\frac{d^2y}{dt^2}$. Combining yields

$$
\frac{d^2y}{dt^2} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t).
$$

(b) The characteristic equation is

$$
\lambda^2 + \frac{1}{LC} = 0.
$$

The characteristic roots are

$$
\lambda_{1,2} = \frac{\pm j}{\sqrt{LC}}.
$$

(c) The form of the zero-input response is $y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. Using $\lambda_1 = -\lambda_2$, $i_c(0) = 0 = C\frac{dy}{dt}\big|_{t=0} = C(c_1\lambda_1 + c_2\lambda_2) = C(c_1\lambda_1 - c_2\lambda_1) = C\lambda_1(c_1 - c_2)$. Thus, $c_1 = c_2$. Also, $v_c(0) = 1 = y(0) = c_1 + c_2$. Combining yields $2c_1 = 2c_2 = 1$ or $c_1 = c_2 = 0.5$. The zero-input response is thus $y_0(t) = 0.5(e^{jt/\sqrt{LC}} + e^{-jt/\sqrt{LC}})$. Using Euler's identity yields

$$y_0(t) = \cos\left(\frac{t}{\sqrt{LC}}\right).$$

(d) Figure S2.4-32 shows $y_0(t)$ for $t \geq 0$. Since $y_0(t)$ is a non-decaying sinusoid, the zero-input response continues forever; the ICs never die out. Notice here that $\omega_0 = \frac{1}{\sqrt{LC}}$ and $T = \frac{2\pi}{\sqrt{LC}}$.
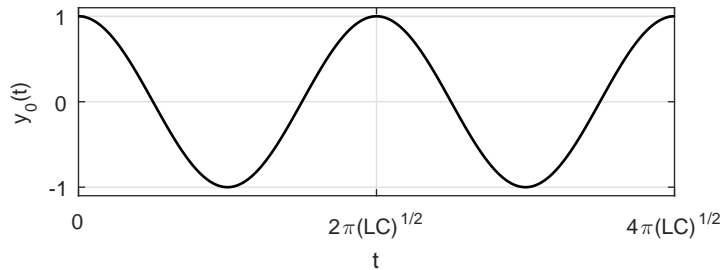
Figure S2.4-32

(e) Since $L = C = 1$, $\lambda_{1,2} = \pm j$. Let $\tilde{y}_0(t) = \tilde{c}_1 e^{jt} + \tilde{c}_2 e^{-jt}$. Using $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$, we know $\tilde{c}_1 = -\tilde{c}_2$. Combining with $\tilde{y}_0^{(1)}(0) = 1 = j\tilde{c}_1 - j\tilde{c}_2$, we know $2j\tilde{c}_1 = 1$ or $\tilde{c}_1 = -j0.5$. Thus, $\tilde{c}_2 = j0.5$ and $\tilde{y}_0(t) = \frac{e^{jt} - e^{-jt}}{2j} = \sin(t)$. From this, the impulse response is determined to be

$$h(t) = \frac{1}{LC}\sin(t)u(t) = \sin(t)u(t).$$

Next, the zero-state response is computed as

$$x(t) * h(t) = \int_0^t \sin\tau\, e^{-(t-\tau)}d\tau = \left(e^{-t}\int_0^t \text{Im}\left(e^{j\tau}e^{\tau}\right)d\tau\right)u(t)$$

$$= \left(\text{Im}\left(e^{-t}\int_0^t e^{\tau(1+j)}d\tau\right)\right)u(t) = \left(\text{Im}\left(e^{-t}\frac{e^{\tau(1+j)}}{1+j}\Big|_{\tau=0}^t\right)\right)u(t)$$

$$= \left(\text{Im}\left(e^{-t}\frac{e^{t(1+j)} - 1}{1+j}\right)\right)u(t) = \left(\text{Im}\left(\frac{e^{jt} - e^{-t}}{1+j}\right)\right)u(t)$$

$$= \left(\text{Im}\left(0.5e^{jt} - j0.5e^{jt} - 0.5(1-j)e^{-t}\right)\right)u(t)$$

$$= \left(0.5\sin(t) - 0.5\cos(t) + 0.5e^{-t}\right)u(t).$$

Summing the zero-state response and the zero-input response calculated in part (c) yields the total response $y(t) = x(t) * h(t) + y_0(t) = (0.5\sin(t) - 0.5\cos(t) + 0.5e^{-t} + \cos(t))\,u(t)$. Thus,

$$y(t) = \left(0.5\sin(t) + 0.5\cos(t) + 0.5e^{-t}\right)u(t).$$

Solution 2.4-33

(a) MATLAB is used to plot $h_1(t)$ and $h_2(t)$ (see Fig. S2.4-33).

```
>>  u = @(t) 1.0*(t>=0); t = -2.5:.001:3.5;
>>  h1 = @(t) (1-t).*(u(t)-u(t-1));
>>  h2 = @(t) t.*(u(t+2)-u(t-2));
>>  subplot(221); plot(t,h1(t)); grid on;
>>  xlabel('t'); ylabel('h_1(t)');
>>  axis([-2.5 3.5 -2.5 2.5]);
>>  subplot(222); plot(t,h2(t)); grid on;
>>  xlabel('t'); ylabel('h_2(t)');
>>  axis([-2.5 3.5 -2.5 2.5]);
```

(b) For a parallel connection, $h_p(t) = h_1(t) + h_2(t)$. MATLAB is used to plot $h_p(t)$ (see Fig. S2.4-33).

```
>>  hp = @(t) h1(t)+h2(t);
>>  subplot(223); plot(t,hp(t)); grid on;
>>  xlabel('t'); ylabel('h_p(t)');
>>  axis([-2.5 3.5 -2.5 2.5]);
```

(c) For a series connection, $h_s(t) = h_1(t) * h_2(t)$.

For $(t < -2)$, $h_s(t) = 0$.

For $(-2 \leq t < -1)$, $h_s(t) = \int_0^{t+2}(1-\tau)(t-\tau)d\tau = \int_0^{t+2}\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=0}^{t+2} = t(t+2) - (t+1)(t+2)^2/2 + (t+2)^3/3 = -t^3/6 + t^2/2 + 2t + 2/3$.

For $(-1 \leq t < 2)$, $h_s(t) = \int_0^1(1-\tau)(t-\tau)d\tau = \int_0^1\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=0}^1 = t - (t+1)/2 + 1/3 = t/2 - 1/6$.

For $(2 \leq t < 3)$, $h_s(t) = \int_{t-2}^1(1-\tau)(t-\tau)d\tau = \int_{t-2}^1\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=t-2}^1 = t/2 - 1/6 - \left(t(t-2) - (t+1)(t-2)^2/2 + (t-2)^3/3\right) = t/2 - 1/6 - \left(-t^3/6 + t^2/2 + 2t - 14/3\right) = t^3/6 - t^2/2 - 3t/2 + 9/2$.

For $(t > 3)$, $h_s(t) = 0$.

Combining all pieces yields

$$h_s(t) = \begin{cases} -t^3/6 + t^2/2 + 2t + 2/3 & -2 \leq t < -1 \\ t/2 - 1/6 & -1 \leq t < 2 \\ t^3/6 - t^2/2 - 3t/2 + 9/2 & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot $h_s(t)$ (see Fig. S2.4-33).

```
>>  hs = @(t) (-t.^3/6+t.^2/2+2*t+2/3).*(u(t+2)-u(t+1))+...
>>      (t/2-1/6).*(u(t+1)-u(t-2))+...
>>      (t.^3/6-t.^2/2-3*t/2+9/2).*(u(t-2)-u(t-3));
>>  subplot(224); plot(t,hs(t)); grid on;
>>  xlabel('t'); ylabel('h_s(t)');
>>  axis([-2.5 3.5 -1 1]);
```
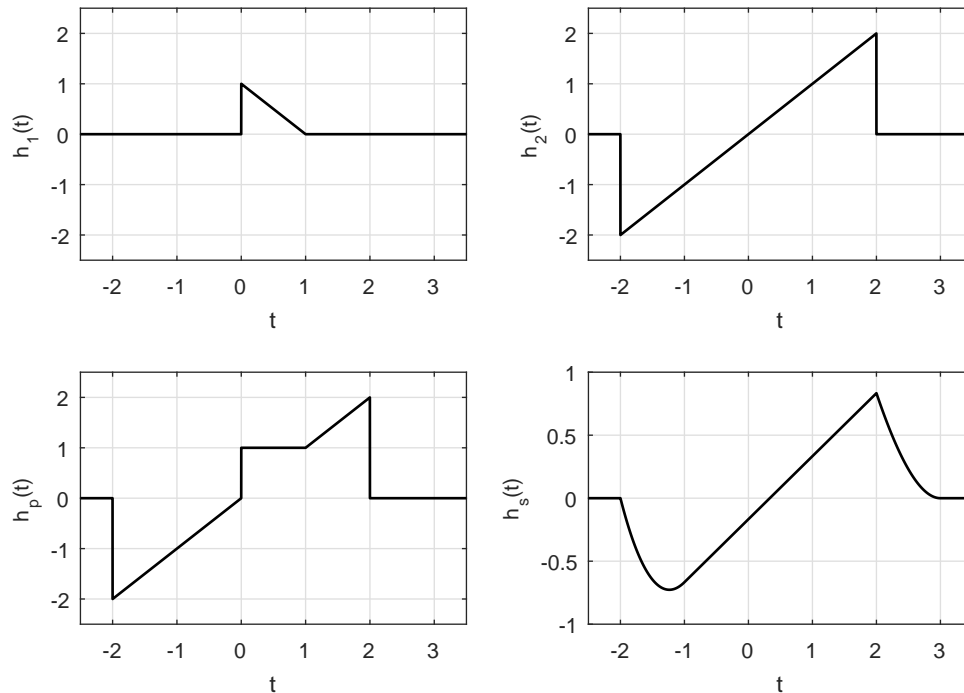
Figure S2.4-33

Solution 2.4-34

(a) Using KVL, $x(t) = RC\dot{y}(t) + y(t)$ or $\dot{y}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$. The characteristic root is $\lambda = \frac{-1}{RC}$.

The zero-input response has form $y_0(t) = c_1 e^{-t/(RC)}$. Using the IC, $y_0(0) = 2 = c_1$. Thus, $y_0(t) = 2e^{-t/(RC)}$.

The zero-state response is $x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t)$. For this first-order system, $y_n(t) = c_1 e^{-t/(RC)}$ and $y_n(0) = 1 = c_1$. Using $y_n(t) = e^{-t/(RC)}$, $b_0 = 0$, and $P(D) = \frac{1}{RC}$, the impulse response is $h(t) = \frac{1}{RC}e^{-t/(RC)}u(t)$. Thus, the zero-state response is

$$
\begin{aligned}
y_{\text{zsr}}(t) &= \left(\int_0^t \frac{1}{RC} e^{-\tau/(RC)} d\tau \right) u(t) \\
&= \left(-e^{-\tau/(RC)} \Big|_{\tau=0}^t \right) u(t) \\
&= \left(1 - e^{-t/(RC)} \right) u(t).
\end{aligned}
$$

For $t \geq 0$, the total response is the sum of the zero-input response and the zero state response,

$$
y(t) = \left(1 + e^{-t/(RC)} \right) u(t).
$$

(b) From part (a), we know the zero-input response is $y_0(t) = y_0(0)e^{-t/(RC)}$. Since the system is time-invariant, the unit step response from part (a) is shifted by one to provide the response to $x(t) = u(t-1)$. Thus, the zero-state response to $x(t) = u(t-1)$ is $\left(1 - e^{-(t-1)/(RC)}\right)u(t-1)$. Summing the two parts together and evaluating at $t = 2$ yields $y(2) = 1/2 = y_0(0)e^{-2/(RC)} + \left(1 - e^{-1/(RC)}\right)$. Solving for $y_0(0)$ yields

$$
y_0(0) = e^{1/(RC)} - 0.5e^{2/(RC)}.
$$

Solution 2.4-35

Notice, $x(2t)$ is a compressed version of $x(t)$. The convolution $y(t) = x(t) * x(2t)$ has several distinct regions.

For $t < 0$ and $t \geq 3/2$, $y(t) = 0$.

For $0 \leq t < 1/2$, $y(t) = \int_0^t 2\tau(t - \tau)d\tau = t\tau^2 - 2\tau^3/3\big|_{\tau=0}^{t} = t^3/3$.

For $1/2 \leq t < 1$, $y(t) = \int_0^{1/2} 2\tau(t - \tau)d\tau = t\tau^2 - 2\tau^3/3\big|_{\tau=0}^{1/2} = t/4 - 1/12$.

For $1 \leq t < 3/2$, $y(t) = \int_{t-1}^{1/2} 2\tau(t - \tau)d\tau = t\tau^2 - 2\tau^3/3\big|_{\tau=t-1}^{1/2} = t/4 - 1/12 - (t^3 - 2t^2 + t - 2t^3/3 + 2t^2 - 2t + 2/3) = -t^3/3 + 5t/4 - 3/4$.

Thus,

$$
y(t) = \begin{cases}
t^3/3 & 0 \leq t < 1/2 \\
t/4 - 1/12 & 1/2 \leq t < 1 \\
-t^3/3 + 5t/4 - 3/4 & 1 \leq t < 3/2 \\
0 & \text{otherwise}
\end{cases}.
$$

MATLAB is used to plot $y(t)$ (see Fig. S2.4-35).

```
>>  u = @(t) 1.0*(t>=0); t = -1:.001:2;
>>  y = @(t) (t.^3/3).*(u(t)-u(t-1/2))+...
>>      (t/4-1/12).*(u(t-1/2)-u(t-1))+...
>>      (-t.^3/3+5*t/4-3/4).*(u(t-1)-u(t-3/2));
>>  plot(t,y(t)); grid on; axis([-1 2 -.05 .25])
>>  xlabel('t'); ylabel('y(t)');
```
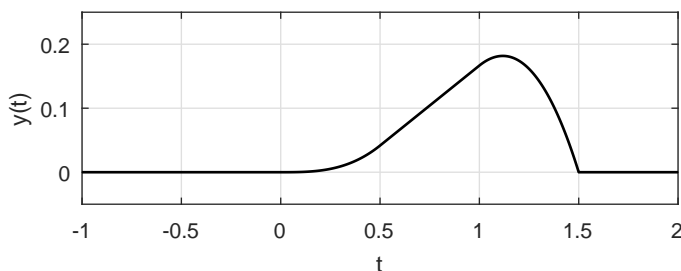

Figure S2.4-35

Solution 2.4-36

Notice, $v_R(t) = v_{L_1}(t) = v_{L_2}(t) = v(t)$.

(a) KCL at the top node gives $x(t) = y(t) + i_{L_1}(t) + i_R$. Since $v(t) = L_2\dot{y}(t)$, we know $i_R(t) = v(t)/R = \frac{L_2}{R}\dot{y}(t)$. Thus, $x(t) = y(t) + \frac{L_2}{R}\dot{y}(t) + i_{L_1}(t)$. Differentiating this expression yields $\dot{x}(t) = \dot{y}(t) + \frac{L_2}{R}\ddot{y}(t) + i_{L_1}^{(1)}(t)$. However, $i_{L_1}^{(1)}(t) = v(t)/L_1 = \frac{L_2}{L_1}\dot{y}(t)$. Thus, $\dot{x}(t) = \dot{y}(t) + \frac{L_2}{R}\ddot{y}(t) + \frac{L_2}{L_1}\dot{y}(t)$ or

$$
\ddot{y}(t) + \left(\frac{R}{L_1} + \frac{R}{L_2}\right)\dot{y}(t) = \frac{R}{L_2}\dot{x}(t).
$$

(b) The characteristic equation is $\lambda^2 + \left(\frac{R}{L_1} + \frac{R}{L_2}\right)\lambda = 0$ which yields characteristic roots of $\lambda_1 = 0$ and $\lambda_2 = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right)$.

(c) The zero-input response has form $y_0(t) = c_1 + c_2 e^{\lambda_2 t}$. Each inductor has an initial current of one amp each. Thus, $y_0(0) = 1 = c_1 + c_2$. The initial resistor current is $i_R(0) = -i_{L_1}(0) - i_{L_2}(0) = $

-2 and the initial resistor voltage is $v(0) = i_R(0)R = -2R$. Thus, $\dot{y}_0(0) = -\frac{2R}{L_2} = \lambda_2 c_2$. Solving yields $c_2 = \frac{2L_1}{L_1+L_2}$ and $c_1 = 1 - c_2 = \frac{L_2-L_1}{L_1+L_2}$. Thus,

$$y_0(t) = \frac{L_2 - L_1}{L_1 + L_2} + \frac{2L_1}{L_1 + L_2}e^{-tR/L_1 - tR/L_2}.$$

Solution 2.4-37

Since the system step response is $g(t) = e^{-t}u(t) - e^{-2t}u(t)$, the system impulse response is $h(t) = \frac{d}{dt}s(t) = -e^{-t}u(t) + \delta(t) + 2e^{-2t}u(t) - \delta(t) = (2e^{-2t} - e^{-t})u(t)$. The input $x(t) = \delta(t - \pi) - \cos(\sqrt{3})u(t)$ is just a sum of a shifted delta function and a scaled step function. Since the system is LTI, the output is quickly computed using $h(t)$ and $g(t)$. That is,

$$y(t) = h(t - \pi) - \cos(\sqrt{3})g(t)$$
$$= (2e^{-2(t-\pi)} - e^{-(t-\pi)})u(t - \pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})u(t).$$

Solution 2.4-38

Since $x(t)$ is $(T = 2)$-periodic, the convolution $y(t) = x(t) * h(t)$ is also $(T = 2)$-periodic. Thus, it is sufficient to evaluate $y(t)$ over any interval of length two.

For $0 \le t < 1/2$, $y(t) = \int_0^t \tau d\tau + \int_{t+1}^{3/2} \tau d\tau = \frac{\tau^2}{2}\Big|_{\tau=0}^{t} + \frac{\tau^2}{2}\Big|_{\tau=t+1}^{3/2} = t^2/2 + 9/8 - (t^2/2 + t + 1/2) = -t + 5/8$.

For $1/2 \le t < 1$, $y(t) = \int_0^t \tau d\tau = t^2/2$.

For $1 \le t < 3/2$, $y(t) = \int_{t-1}^t \tau d\tau = \frac{\tau^2}{2}\Big|_{\tau=t-1}^{t} = t^2/2 - (t^2/2 - t + 1/2) = t - 1/2$.

For $3/2 \le t < 2$, $y(t) = \int_{t-1}^{3/2} \tau d\tau = \frac{\tau^2}{2}\Big|_{\tau=t-1}^{3/2} = 9/8 - (t^2/2 - t + 1/2) = -t^2/2 + t + 5/8$.

Combining,

$$y(t) = \begin{cases} -t + 5/8 & 0 \le t < 1/2 \\ t^2/2 & 1/2 \le t < 1 \\ t - 1/2 & 1 \le t < 3/2 \\ -t^2/2 + t + 5/8 & 3/2 \le t < 2 \\ y(t + 2) & \forall t \end{cases}.$$

Figure S2.4-38 shows the resulting signal $y(t)$ over $-3 \le t \le 3$. This interval includes three periods of the $(T = 2)$-periodic function $y(t)$.

Figure S2.4-38

```
>>  t = -3:.001:3;
>>  y = @(t) (-mod(t,2)+5/8).*((mod(t,2)>=0)&(mod(t,2)<1/2))+...
>>     (mod(t,2)).^2/2.*((mod(t,2)>=1/2)&(mod(t,2)<1))+...
>>     (mod(t,2)-1/2).*((mod(t,2)>=1)&(mod(t,2)<3/2))+...
>>     (-(mod(t,2)).^2/2+mod(t,2)+5/8).*((mod(t,2)>=3/2)&(mod(t,2)<2));
>>  plot(t,y(t)); grid on; axis([-3 3 -.05 1.05])
>>  xlabel('t'); ylabel('y(t)');
```
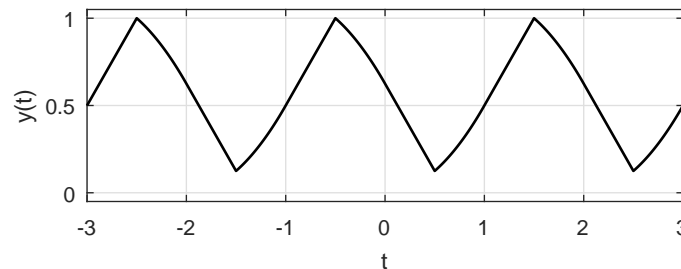
Solution 2.4-39

(a) Using KVL, $x(t) = i(t)R + v_{C_1}(t) + y(t) = RC_2\dot{y}(t) + v_{C_1}(t) + y(t)$. Differentiating yields $\dot{x}(t) = RC_2\ddot{y}(t) + \dot{v}_{C_1}(t) + \dot{y}(t) = RC_2\ddot{y}(t) + \frac{1}{C_1}i(t) + \dot{y}(t) = RC_2\ddot{y}(t) + \frac{C_2}{C_1}\dot{y}(t) + \dot{y}(t)$. Thus,

$$\ddot{y}(t) + \left(\frac{1}{RC_1} + \frac{1}{RC_2}\right)\dot{y}(t) = \frac{1}{RC_2}\dot{x}(t).$$

(b) Since $R = 1$, $C_1 = 1$, and $C_2 = 2$, the differential equation becomes $\ddot{y}(t) + 3/2\dot{y}(t) = 1/2\dot{x}(t)$.

The characteristic equation is $\lambda^2 + 3/2\lambda = 0$, and the characteristic roots are $\lambda_1 = 0$ and $\lambda_2 = -3/2$. Thus, the form of the zero-input response is $y_0(t) = c_1 + c_2e^{-3t/2}$. From the initial conditions, we see that $y(0) = V_{C_2} = 1 = c_1 + c_2$. The initial voltage across the resistor is $v_R(0) = -(V_{C_1} + V_{C_2}) = -2 - 1 = -3$ which yields $i_R(0) = -3/R = -3$. Also, $i_R(0) = -3 = i_{C_2}(0) = C_2\dot{y}(0) = 2\dot{y}(0)$. Thus, $\dot{y}(0) = -3/2 = -3c_2/2$. Solving yields $c_2 = 1$ and $c_1 = 0$. Thus,

$$y_0(t) = e^{-3t/2}.$$

The zero-state response is $x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t)$. For this second-order system, $y_n(t) = c_1 + c_2e^{-3t/2}$, $y_n(0) = 0 = c_1 + c_2$ and $\dot{y}_n(0) = 1 = -3c_2/2$. Thus, $c_2 = -2/3$ and $c_1 = 2/3$. Using $b_0 = 0$, and $P(D) = 0.5D$, the impulse response is $h(t) = 0.5D[y_n(t)]u(t) = 0.5D[2/3 - 2/3e^{-3t/2}]u(t) = 0.5[-2/3(-3/2)e^{-3t/2}]u(t) = 0.5e^{-3t/2}u(t)$. Using $x(t) = 4te^{-3t/2}u(t)$, the zero-state response is $\left(\int_0^t(4\tau e^{-3\tau/2})(0.5e^{-3(t-\tau)/2})d\tau\right)u(t) = \left(2e^{-3t/2}\int_0^t \tau d\tau\right)u(t) = \left(2e^{-3t/2}t^2/2\right)u(t)$. Thus,

$$x(t) * h(t) = t^2e^{-3t/2}u(t).$$

Since the input is driving a natural mode, resonance is expected; thus, the t^2 term seems sensible.

For $(t \geq 0)$, the total response is the sum of the zero-input response and the zero-state response.

$$y(t) = y_0(t) + x(t) * h(t) = \left(e^{-3t/2} + t^2e^{-3t/2}\right)u(t).$$

Solution 2.4-40

(a) No; the system is not causal because $h(t) \neq 0$ for all $t < 0$.

(b) The zero-state response is just the convolution of the input with the impulse response. There are two regions for the convolution. For $t < 2$,

$$y_{\text{zsr}}(t) = \int_{t-2}^0 3e^\tau \, d\tau + \int_0^\infty 3e^{-\tau} \, d\tau = 3e^\tau|_{t-2}^0 + -3e^{-\tau}|_0^\infty$$
$$= 3\left[1 - e^{t-2} + 0 - (-1)\right] = 6 - 3e^{t-2}.$$

For $t \geq 2$,

$$y_{\text{zsr}}(t) = \int_{t+2}^\infty 3e^{-\tau} \, d\tau = -3e^{-\tau}|_{t-2}^\infty$$
$$= 0 - \left(-3e^{-(t-2)}\right) = 3e^{-(t-2)}.$$

Thus,

$$y_{\text{zsr}}(t) = \begin{cases} 6 - 3e^{t-2} & t < 2 \\ 3e^{2-t} & t \geq 2 \end{cases}.$$

Solution 2.4-41

Since $h(t)$ is only provided for over $(0 \leq t < 0.5)$, it is not possible to determine with certainty whether or not the system is causal or stable. However, when looking at $h(t)$ the waveform appears to have a DC offset. This apparent DC offset can be very troubling if $h(t)$ is truly an impulse response function. If a DC offset is present, the system is neither causal nor stable. Imagine, a non-causal, unstable heart! Something is probably wrong.

One simple explanation is that a blood-filled heart always has some ventricular pressure. Unless removed, this relaxed-state pressure would likely appear as a DC offset to any measurements. It would likely be most appropriate to subtract this offset when trying to measure the impulse response function.

Another problem is that the impulse response function is most appropriate in the study of linear, time-invariant systems. It is quite unlikely that the heart is either linear or time-invariant. Even if the impulse response could be reliably measured at a particular time, it might not provide much useful information.

Solution 2.4-42

(a) We obtain $h(t)$ by directly substituting $\delta(t)$ for $x(t)$:

$$h_{\mathrm{i}}(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \Rightarrow h_{\mathrm{i}}(t) = u(t).$$

(b) For two systems in parallel, their impulse responses add. Thus,

$$h_{\mathrm{p}}(t) = h_{\mathrm{i}}(t) + h_{\mathrm{i}}(t) = 2u(t).$$

(c) For two systems in series, the overall impulse response is the convolution of the two component impulse responses. Thus,

$$h_{\mathrm{s}} = h_{\mathrm{i}}(t) * h_{\mathrm{i}}(t) = \left(\int_{0}^{t} d\tau \right) u(t) \Rightarrow h_{\mathrm{s}}(t) = tu(t).$$

Solution 2.4-43

(a) $x(t) * x(-t) = \int_{-\infty}^{\infty} x(\tau)x(-(t-\tau))d\tau = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau = r_{xx}(t).$

(b) Since $r_{xx}(t)$ is an even function, we only need to compute $r_{xx}(t)$ for either $t \geq 0$ or $t \leq 0$. In either case, the autocorrelation function is computed by convolving the original signal with its reflection.

For $t < -2$, $r_{xx}(t) = 0$.

For $-2 \leq t < -1$, $r_{xx}(t) = \int_{0}^{t+2} \tau d\tau = 0.5\tau^2 \big|_{\tau=0}^{t+2} = t^2/2 + 2t + 2.$

For $-1 \leq t < 0$, $r_{xx}(t) = \int_{0}^{t+1} \tau(\tau-t)d\tau + \int_{t+1}^{1} \tau d\tau + \int_{1}^{t+2} d\tau = \left(\frac{\tau^3}{3} - t\frac{\tau^2}{2} \Big|_{\tau=0}^{t+1} \right) + \frac{\tau^2}{2} \Big|_{\tau=t+1}^{1} +$

$\tau \big|_{\tau=1}^{t+2} = \frac{t^3+3t^2+3t+1}{3} - \frac{t^3+2t^2+t}{2} + \frac{1}{2} - \frac{t^2+2t+1}{2} + (t+2-1) = -t^3/6 - t^2/2 + t/2 + 4/3.$

Combining and using $r_{xx}(t) = r_{xx}(-t)$ yields

$$r_{xx}(t) = \begin{cases} t^2/2 + 2t + 2 & -2 \leq t < -1 \\ -t^3/6 - t^2/2 + t/2 + 4/3 & -1 \leq t < 0 \\ t^3/6 - t^2/2 - t/2 + 4/3 & 0 \leq t < 1 \\ t^2/2 - 2t + 2 & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

MATLAB is used to plot the result (see Fig. S2.4-43).

```
>>  LSS3eSMMATLABFigFormat(4,1.5,3);
>>  u = @(t) 1.0*(t>=0); t = -2.5:.001:2.5;
>>  rxx = @(t) (t.^2/2+2*t+2).*(u(t+2)-u(t+1))+...
>>      (-t.^3/6-t.^2/2+t/2+4/3).*(u(t+1)-u(t))+...
>>      (t.^3/6-t.^2/2-t/2+4/3).*(u(t)-u(t-1))+...
>>      (t.^2/2-2*t+2).*(u(t-1)-u(t-2));
>>  plot(t,rxx(t)); grid on; axis([-2.5 2.5 -.05 1.5]);
>>  xlabel('t'); ylabel('r_{xx}(t)'); set(gca,'ytick',0:1/3:4/3);
>>  set(gca,'yticklabel',{'0','1/3','2/3','1','4/3'});
```
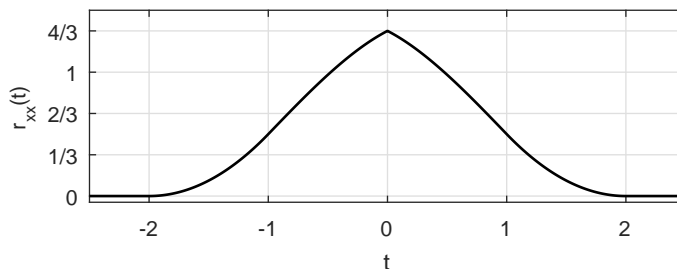

Figure S2.4-43

Solution 2.4-44

(a) KCL at the negative terminal of the op-amp yields $\frac{x(t)-0}{R} + C\dot{y}(t) = 0$. Thus,

$$\dot{y}(t) = -\frac{1}{RC}x(t).$$

(b) The zero-state response is $y(t) = x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a first order system with $\lambda = 0$, thus $\tilde{y}_0(t) = \tilde{c}_1 e^{\lambda t} = \tilde{c}_1$. Since $\tilde{y}_0(0) = 1 = \tilde{c}_1$, $b_0 = 0$, and $P(D) = -\frac{1}{RC}$, the impulse response is $h(t) = -\frac{1}{RC}u(t)$. Thus,

$$y(t) = \left(\int_0^t -\frac{1}{RC}d\tau\right)u(t) = -\frac{t}{RC}u(t).$$

Notice, $|y(t)|$ ramps toward infinity as time increases. Intuitively, this makes sense; a DC input to an integrator should output an unbounded ramp function.

Solution 2.4-45

The system response to $u(t)$ is $g(t)$ and the response to step $u(t-\tau)$ is $g(t-\tau)$. The input $x(t)$ is made up of step components. The step component at τ has a height $\triangle f$ which can be expressed as

$$\triangle f = \frac{\triangle f}{\triangle \tau}\triangle\tau = \dot{x}(\tau)\triangle\tau.$$

The step component at $n\triangle\tau$ has a height $\dot{x}(n\triangle\tau)\triangle\tau$ and it can be expressed as $[\dot{x}(n\triangle\tau)\triangle\tau]u(t-n\triangle\tau)$. Its response $\triangle y(t)$ is

$$\triangle y(t) = [\dot{x}(n\triangle\tau)\triangle\tau]g(t-n\triangle\tau).$$

The total response due to all components is

$$y(t) = \lim_{\triangle\tau\to 0}\sum_{n=-\infty}^{\infty}\dot{x}(n\triangle\tau)g(t-n\triangle\tau)\triangle\tau$$

$$= \int_{-\infty}^{\infty}\dot{x}(\tau)g(t-\tau)\,d\tau = \dot{x}(\tau)*g(\tau)$$

Solution 2.4-46

Consider the input $x(t) = e^{j\omega_0 t}$. Letting $s = j\omega_0$ in Eq. (2.38), the system response is found as

$$y(t) = H(j\omega_0)e^{j\omega_0 t}.$$

Using Eq. (2.31), the system response to input $\hat{x}(t) = \cos\omega_0 t = \text{Re}[e^{j\omega_0 t}]$ is $\hat{y}(t)$, where

$$\begin{aligned}
\hat{y}(t) &= \text{Re}[H(j\omega_0)e^{j\omega_0 t}] \\
&= \text{Re}\left\{|H(j\omega_0)|e^{j[\omega_0 t + \angle H(j\omega_0)]}\right\} \\
&= |H(j\omega_0)|\cos[\omega_0 t + \angle H(j\omega_0)].
\end{aligned}$$

Where $H(j\omega)$ is $H(s)|_{s=j\omega}$ in Eq. (2.39). Hence,

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\, d\tau.$$

Solution 2.4-47

An element of length $\triangle\tau$ at point $n\triangle\tau$ has a charge (Fig. S2.4-47). A point x is at a distance $x - n\triangle\tau$ from this charge. The electric field at point x due to the charge $Q(n\triangle\tau)\triangle\tau$ is

$$\triangle E = \frac{Q(n\triangle\tau)\triangle\tau}{4\pi\epsilon(x - n\triangle\tau)^2}.$$

The total field due to the charge along the entire length is

$$\begin{aligned}
E(x) &= \lim_{\triangle\tau \to 0} \sum_{n=-\infty}^{\infty} \frac{Q(n\triangle\tau)\triangle\tau}{4\pi\epsilon(x - n\triangle\tau)^2} \\
&= \int_{-\infty}^{\infty} \frac{Q(\tau)}{4\pi\epsilon(x - \tau)^2}\, d\tau = Q(x) * \frac{1}{4\pi\epsilon x}.
\end{aligned}$$
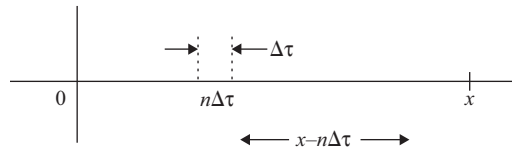
Figure S2.4-47

Solution 2.4-48

(a) Yes, the system is causal since $h(t) = 0$ for $t < 0$.

(b) To compute the zero-state response $y_1(t)$, the convolution of two rectangular pulses is required: a pulse of amplitude j and width two and a pulse of amplitude one and a width of one. The convolution involves several regions.

For $t < 0$, $y_1(t) = 0$.

For $0 \leq t < 1$, $y_1(t) = \int_0^t j\, dt = jt$.

For $1 \leq t < 2$, $y_1(t) = \int_{t-1}^t j\, dt = j(t - (t - 1)) = j$.

For $2 \leq t < 3$, $y_1(t) = \int_{t-1}^2 j\, dt = j(2 - (t - 1)) = j(3 - t)$.

For $t \geq 3$, $y_1(t) = 0$.

Thus,

$$
y_1(t) = \begin{cases} jt & 0 \leq t < 1 \\ j & 1 \leq t < 2 \\ j\,(3 - t) & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}.
$$

(c) To compute $y_2(t)$, first note that $x_2(t) = 2x_1(t - 1) + x_1(t - 2)$. Using the system properties of linearity and time-invariance, the output $y_2(t)$ is given by

$$
y_2(t) = 2y_1(t - 1) + y_1(t - 2).
$$

Solution 2.5-1

(a) Here, $\lambda^2 + 8\lambda + 12 = (\lambda + 2)(\lambda + 6)$. Both roots are in LHP. The system is BIBO stable and also asymptotically stable.

(b) In this case, $\lambda(\lambda^2 + 3\lambda + 2) = \lambda(\lambda + 1)(\lambda + 2)$. The characteristic roots are 0, -1, -2. One root is on the imaginary axis and none are in the RHP. The system is BIBO unstable and marginally stable.

(c) The characteristic polynomial is $\lambda^2(\lambda^2 + 2) = \lambda^2(\lambda + j\sqrt{2})(\lambda - j\sqrt{2})$. The characteristic roots are 0 (repeated twice) and $\pm j\sqrt{2}$. Because there are repeated roots on imaginary axis, the system is BIBO unstable and asymptotically unstable.

(d) Here, $(\lambda + 1)(\lambda^2 - 6\lambda + 5) = (\lambda + 1)(\lambda - 1)(\lambda - 5)$. The roots are -1, 1 and 5. There are two roots in the RHP. The system is BIBO unstable and asymptotically unstable.

Solution 2.5-2

(a) The characteristic polynomial is $(\lambda + 1)(\lambda^2 + 2\lambda + 5)^2 = (\lambda + 1)(\lambda + 1 - j2)^2(\lambda + 1 + j2)^2$. The characteristic roots -1, $-1 \pm j2$ (repeated twice) are all in the LHP. The system is BIBO stable and asymptotically stable.

(b) In this case, $(\lambda + 1)(\lambda^2 + 9) = (\lambda + 1)(\lambda + j3)(\lambda - j3)$. The roots are -1, $\pm j3$. There are two (simple) roots on the imaginary axis and no roots in the RHP. The system is BIBO unstable and marginally stable.

(c) Here, $(\lambda + 1)(\lambda^2 + 9)^2 = (\lambda + 1)(\lambda + j3)^2(\lambda - j3)^2$. The roots are -1 and $\pm j3$ repeated twice. Since there is a repeated root on the imaginary axis, the system is BIBO unstable and asymptotically unstable.

(d) The characteristic polynomial is $(\lambda^2 + 1)(\lambda^2 + 4)(\lambda^2 + 9) = (\lambda + j1)(\lambda - j1)(\lambda + j2)(\lambda - j2)(\lambda + j3)(\lambda - j3)$. The roots are $\pm j1$, $\pm j2$ and $\pm j3$. All roots are simple and on the imaginary axis. No roots are in the RHP. The system is BIBO unstable and marginally stable.

Solution 2.5-3

Notice that $h(t) = e^t \left(\frac{2}{3} \cos(\frac{3}{2}t) + \frac{1}{3} \sin(\pi t) \right) u(123 - t)$ is bounded by the envelope e^t. Thus, $|h(t)| < e^t u(123 - t)$ and

$$
\int_{-\infty}^{\infty} |h(t)|\, dt < \int_{-\infty}^{\infty} e^t u(123 - t)\, dt = \int_{-\infty}^{123} e^t\, dt = e^{123} < \infty.
$$

Since $h(t)$ is absolutely integrable, the system is BIBO stable.

Solution 2.5-4

(a) To be causal, a system's impulse response must equal 0 for all $t < 0$. Thus,

$$\text{an LTIC system with } h(t) = \tfrac{1}{t}u(t - T) \text{ is causal for } T \geq 0.$$

(b) To be BIBO stable, a system's impulse response must be absolutely integrable. In the current case,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{T}^{\infty} \frac{1}{t}\, dt = \ln(\infty) - \ln(T) \nless \infty.$$

Since there is no value T for which $h(t)$ is absolutely integrable,

$$\text{there is no value } T \text{ for which the system is BIBO stable.}$$

Solution 2.5-5

It is better to chose a system that is guaranteed internally stable rather than a system that is only guaranteed externally stable. The reason is simple: internal stability automatically guarantees external stability, but external stability does not always guarantee internal stability. That is, by choosing a system that is guaranteed internally stable, external stability is likewise guaranteed; the converse is not necessarily true.

Solution 2.5-6

In this problem, we assume the system is first order and that the system mode is visible in the impulse response $h(t)$.

(a) Because $u(t) = e^{0t}u(t)$, the characteristic root is 0.

(b) The root lies on the imaginary axis, and the system is marginally stable.

(c) Since $\int_{0}^{\infty} h(t)\, dt = \infty$, the system is BIBO unstable.

(d) The integral of $\delta(t)$ is $u(t)$. Likewise, the system response to $\delta(t)$ is $u(t)$. Clearly, the system is an ideal integrator, which has numerous uses.

Solution 2.5-7

Assume that a system exists that violates Eq. (2.45)) and yet produces a bounded output for every bounded input. The response at $t = t_1$ is

$$y(t_1) = \int_{0}^{\infty} h(\tau)x(t_1 - \tau)\, d\tau.$$

Consider a bounded input $x(t)$ such that at some instant t_1,

$$x(t_1 - \tau) = \begin{cases} 1 & \text{if } h(\tau) > 0 \\ -1 & \text{if } h(\tau) < 0 \end{cases}.$$

In this case

$$h(\tau)x(t_1 - \tau) = |h(\tau)|$$

and

$$y(t_1) = \int_{0}^{\infty} |h(\tau)|\, d\tau = \infty.$$

This violates the assumption.

Solution 2.5-8

(a) For this convolution, there are several regions. For $(t < -2)$ and $(t \geq 4)$, $y(t) = 0$.

For $(-2 \leq t < 0)$, $y(t) = \int_0^{t+2} \tau d\tau = (t+2)^2/2 = t^2/2 + 2t + 2$.

For $(0 \leq t < 2)$, $y(t) = \int_0^2 \tau d\tau = 2$.

For $(2 \leq t < 4)$, $y(t) = \int_{t-2}^2 \tau d\tau = 2^2/2 - (t-2)^2/2 = 2 - t^2/2 - 2t - 2 = -t^2/2 + 2t$.

Combining yields

$$y(t) = \begin{cases} t^2/2 + 2t + 2 & -2 \leq t < 0 \\ 2 & 0 \leq t < 2 \\ -t^2/2 + 2t & 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot the result (see Fig. S2.5-8).

```
>>  u = @(t) 1.0*(t>=0); t = -3:.001:5;
>>  y = @(t) (t.^2/2+2*t+2).*(u(t+2)-u(t))+2.*(u(t)-u(t-2))+...
>>     (-t.^2/2+2*t).*(u(t-2)-u(t-4));
>>  plot(t,y(t)); grid on; axis([-3 5 -.25 2.25]);
>>  xlabel('t'); ylabel('y(t)');
```
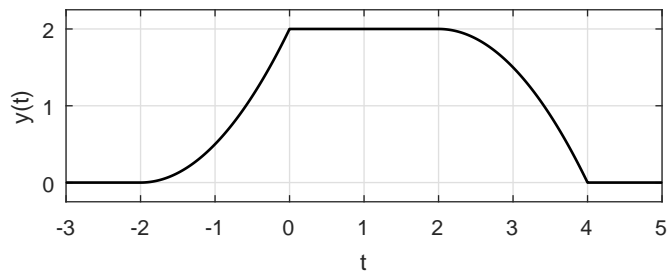

Figure S2.5-8

(b) Yes, the system is stable since $\int |h(t)| = 4 < \infty$.

No, the system is not causal since $h(t) \neq 0$ for all $t < 0$.

Solution 2.5-9

The system is stable since $h(t)$ is absolutely integrable. That is, $\int_{-\infty}^{\infty} h(t)dt = \int_0^1 1dt = 1 < \infty$. The system is causal since $h(t) = 0$ for all $t < 0$.

Solution 2.5-10

Expanding $h(t) = \sum_{i=0}^{\infty} (0.5)^i \delta(t-i)$ yields

$$h(t) = \left(\delta(t) + 0.5\delta(t-1) + 0.25\delta(t-2) + 0.125\delta(t-3) + \cdots \right).$$

(a) Yes, the system is causal since $h(t) = 0$ for all $t < 0$.

(b) Yes, the system is stable since the impulse response is absolutely integrable. That is, $\int_{-\infty}^{\infty} \sum_{i=0}^{\infty} (0.5)^i \delta(t-i)dt = \sum_{i=0}^{\infty} (0.5)^i \int_{-\infty}^{\infty} \delta(t-i) = \sum_{i=0}^{\infty} (0.5)^i = \frac{1-0}{1-0.5} = 2 < \infty$.

Solution 2.6-1

(a) The time-constant (rise-time) of the system is $T_h = 10^{-5}$. The rate of pulse communication $< \frac{1}{T_h} = 10^5$ pulses/sec. The channel cannot transmit million pulses/second.

(b) The bandwidth of the channel is

$$B = \frac{1}{T_h} = 10^5 \, \text{Hz}$$

The channel can transmit audio signal of bandwidth 15 kHz readily.

Solution 2.6-2

The system described by $(D^2 + 2D + 13/4)\{y(t)\} = x(t)$ has characteristic equation $\lambda^2 + 2\lambda + 13/4 = 0$ and characteristic roots

$$\lambda = \frac{-2 \pm \sqrt{4 - 13}}{2} = -1 \pm j\frac{3}{2}.$$

The input $x(t) = \cos(\omega t) = 0.5e^{j\omega t} + 0.5e^{-j\omega t}$ will produce a strong response if the input frequency ω is chosen as close as possible to the characteristic roots of the system. This can be accomplished by choosing ω to match the imaginary portion of the characteristic roots. That is,

$$\cos(\omega t) \text{ will produce a strong response when } \omega = \pm\frac{3}{2} \text{ rad/s.}$$

Solution 2.6-3

For rectangular impulse response $\hat{h}(t)$ to be an appropriate approximation of $h(t)$, $\hat{h}(t)$ should have the same peak amplitude and area of $h(t)$. To have the same peak amplitude requires that $A = 2$. The area of $h(t)$ is

$$\int_0^2 (4t - 2t^2) \, dt = (2t^2 - 2t^3/3)|_0^2 = 8 - \frac{16}{3} = \frac{8}{3}.$$

The area of $\hat{h}(t)$ is

$$AT_h = \frac{8}{3} \quad \Rightarrow \quad T_h = \frac{4}{3}.$$

Thus,

$$\hat{h}(t) \text{ appropriately approximates } h(t) \text{ when } A = 2 \text{ and } T_h = \frac{4}{3}.$$

Solution 2.6-4

$$T_h = \frac{1}{B} = \frac{1}{10^4} = 10^{-4} = 0.1 \, \text{ms}$$

The received pulse width $= (0.5 + 0.1) = 0.6$ ms. Each pulse takes up 0.6 ms interval. The maximum pulse rate (to avoid interference between successive pulses) is

$$\frac{1}{0.6 \times 10^{-3}} \simeq 1667 \, \text{pulses/sec.}$$

Solution 2.6-5

Using Eq. (2.47),

(a) The system rise time is $T_r = T_h = -\frac{1}{\lambda} = 10^{-4}$.

(b) The system bandwidth is $f_c = 1/T_h = 1/T_r = 10^4$.

(c) The pulse transmission rate is $f_c = 10^4$ pulses/sec.

Solution 2.6-6

The impulse response $h(t)$ of a 6 MHz LP system can be approximated as a rectangle of width $T_h = \frac{1000}{6}$ ns. Applying a rectangular data pulse input $x(t)$ that is $\frac{500}{6}$ ns wide produces a (roughly) trapezoidal output $y(t) = h(t) * x(t)$ that is, using the width property of convolution, $\frac{1500}{6} = 250$ ns wide. To avoid overlap, output pulses should be spaced no closer than 250 ns apart. Thus, a suitable transmission rate for this system is

$$F_{\text{rate}} = \frac{1}{250(10^{-9})} = 4(10^6) \text{ pulses/sec} .$$

Solution 2.6-7

A 5 kHz LP system has a time constant of $T_h = \frac{1}{5000} = 0.2$ ms. Further, the impulse response $h(t)$ of an LP system is approximately rectangular, with a width equal to T_h. To be noncausal, the impulse response cannot be zero for all $t < 0$. Taken together, a possible impulse response of a 5 kHz noncausal LP system is

$$h_1(t) = A[u(t + 0.0002) - u(t)].$$

A plot of $h_1(t)$ is shown in Fig. S2.6-7. Many other possible solutions exist, such as the impulse response $h_2(t)$, also shown in Fig. S2.6-7.
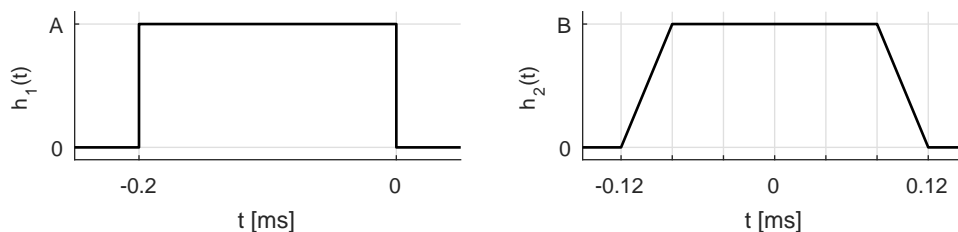

Figure S2.6-7

Solution 2.6-8

System 1 is LP with a cutoff frequency of $f_c = 1$ Hz; this is a really slow system. System 2 is allpass (first delta) with echoes (next 2 deltas) and is obviously a much faster system. Since a fast system is desired in the transmission of high-speed digital data,

system 2 (allpass w/ echoes) is more appropriate than system 1 (slow lowpass) .

Solution 2.6-9

(a) For a causal system with finite duration $h(t)$, the rise time is exactly equal to the time when the signal is last non-zero. That is,

$$T_r = 4 \text{ seconds.}$$

(b) The impulse response function $h(t)$ is consistent with a channel that has the following three characteristics: 1) a channel with delay from input to output (for example, signal propagation delay), 2) a channel with low-pass character (pulse dispersion that results in a $\delta(t)$ input spreading into a square pulse), and 3) a channel with two signal paths (for example, a primary signal path and an echo path).

For systems with predominantly low-pass character, digital information can be transmitted without significant interference at a rate of $\mathcal{F}_c = \frac{1}{T_r} = 1/4$. However, this estimate is too conservative for the present system. Notice that $h(t) = 0$ for $0 \le t < 1$, corresponding to a transmission delay in the primary signal path. The remaining portion of $h(t)$ has a width

of three, so it is therefore practical to transmit at rates of $\mathcal{F}_c = 1/3$. By clever interleaving of data, it is possible to transmit at rates of $\mathcal{F}_c = 1/2$. Consider transmitting the binary sequence $\{b_0, b_1, b_2, b_3, \dots\}$ using a $(t = 1)$-spaced delta train weighted by the pulse sequence $\{b_0, b_1, 0, 0, b_2, b_3, 0, 0, \dots\}$. The output is the series of non-overlapping unit-duration pulses given by $\{b_0, b_1, b_0, b_1, b_2, b_3, b_2, b_3, \dots\}$. The effective transmission rate is 0.5 bits per unit time.

(c) The resulting convolution $y(t) = x(t) * h(t)$ has many regions.

For $t < 1$, $y(t) = 0$.

For $1 \le t < 2$, $y(t) = \int_1^t (-1)dt = 1 - t$.

For $2 \le t < 3$, $y(t) = \int_1^2 (-1)dt = -1$.

For $3 \le t < 4$, $y(t) = \int_{t-2}^2 (-1)dt + \int_3^t (-1)dt = (t-2) - 2 + 3 - t = -1$.

For $4 \le t < 5$, $y(t) = \int_3^4 (-1)dt = -1$.

For $5 \le t < 6$, $y(t) = \int_{t-2}^4 (-1)dt = (t-2) - 4 = t - 6$.

For $6 \le t$, $y(t) = 0$.

Thus,

$$y(t) = \begin{cases} 1 - t & 1 \le t < 2 \\ -1 & 2 \le t < 5 \\ t - 6 & 5 \le t < 6 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot the result (see Fig. S2.6-9).

```
>>  u = @(t) 1.0*(t>=0); t = 0:.001:10;
>>  y = @(t)   (1-t).*(u(t-1)-u(t-2))-1*(u(t-2)-u(t-5))+...
>>     (t-6).*(u(t-5)-u(t-6));
>>  plot(t,y(t)); grid on; axis([0 10 -1.2 0.2]);
>>  xlabel('t'); ylabel('y(t)');
```
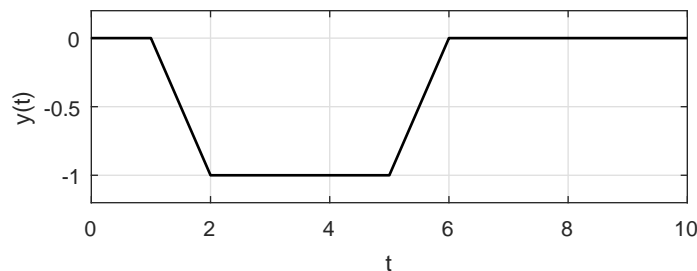

Figure S2.6-9

Solution 2.6-10

(a) We use MATLAB to accurately plot $h(t)$, the result of which is shown in Fig. S2.6-10.

```
>>  u = @(t) 1.0*(t>=0); h = @(t) -t.*exp(-t).*u(t);
>>  t = -1:.001:8; plot(t,h(t)); grid on;
>>  set(gca,'ytick',[-1/exp(1) 0],'yticklabel',{'-1/e','0'});
>>  axis([-1 8 -.5 0.1]); xlabel('t'); ylabel('h(t)');
```
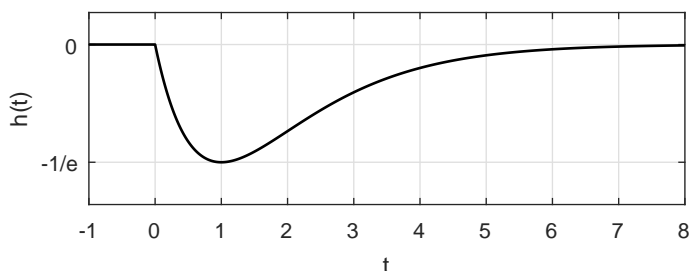
Figure S2.6-10

(b) A rectangular approximation $\hat{h}(t)$ to impulse response $h(t)$ needs the same area and peak height. First, we compute the area of $h(t)$.

$$\text{area}_h = \int_{-\infty}^{\infty} h(t)\, dt = \int_{0}^{\infty} -te^{-t}\, dt.$$

Using integration by parts ($\int u\, dv = uv - \int v\, du$ with $u = t$, $du = dt$, $v = e^{-t}$ and $dv = -e^{-t}\, dt$), we find that

$$\text{area}_h = -te^{-t}\Big|_{0}^{\infty} - \int_{0}^{\infty} e^{-t}\, dt = 0 - 0 + e^{-t}\Big|_{0}^{\infty} = 0 - 1 = -1$$

The peak of $h(t)$ occurs when $\frac{d}{dt}h(t) = te^{-t} - e^{-t} = (t-1)e^{-t} = 0$, or at $t = 1$. The peak height of $h(t)$ is thus $h(1) = -e^{-1}$, as confirmed in Fig. S2.6-10.

The time constant T_h, which is also the width of $\hat{h}(t)$, is thus

$$T_h = \frac{\text{area}_h}{\text{peak}_h} = \frac{-1}{-e^{-1}} = e.$$

Thus,

$$\hat{h}(t) \text{ is a rectangle of width } T_h = e \text{ and height } -e^{-1}.$$

That is,

$$\hat{h}(t) = -e^{-1}[u(t) - u(t - e)].$$

The approximate cutoff frequency f_c of this system is

$$f_c = \frac{1}{T_h} = \frac{1}{e} = 0.3679 \text{ Hz .}$$

Solution 2.6-11

(a) A rectangular approximation $\hat{h}(t)$ to impulse response $h(t)$ needs the same area and peak height. First, we compute the area of $h(t)$.

$$\text{area}_h = \int_{-\infty}^{\infty} h(t)\, dt = \int_{0}^{1} 1\, dt + \int_{1}^{3} \frac{3 - t}{2}\, dt$$

$$= 1 + \left(\frac{3t}{2} - \frac{t^2}{4}\Big|_{1}^{3} \right) = 1 + \frac{9}{4} - \frac{5}{4} = 2.$$

By inspection of Fig. P2.4-8, the peak of $h(t)$ is just 1. The width of $\hat{h}(t)$, which is also the time constant T_h, is thus

$$T_h = \frac{\text{area}_h}{\text{peak}_h} = \frac{2}{1} = 2.$$

Thus,
$$\hat{h}(t) \text{ is a rectangle of width } T_h = 2 \text{ and height } 1.$$

That is,
$$\hat{h}(t) = u(t) - u(t-2).$$

(b) From part (a), the time constant is
$$T_h = 2.$$

(c) The approximate cutoff frequency f_c of our LP system with a time-constant $T_h = 2$ is $f_c = \frac{1}{T_h} = 0.5$ Hz. Expressed in radians, we see that
$$\omega_c = 2\pi f_c = \pi \text{ rad/s}.$$

(d) Since $\omega_0 \ll \omega_c$, our lowpass system will pass a sinusoidal input with a simple gain of $\int h(t)\, dt = 2$. Thus,
$$x(t) = \sin(\omega_0 t + \pi/3) \quad \Longrightarrow \quad y(t) = 2\sin(\omega_0 t + \pi/3).$$

Technically there is also a small phase shift, but we can safely ignore it since $\omega_0 \ll \omega_c$.

Solution 2.6-12

We approximate the impulse response of the first system as a rectangle of duration 4 μs and some height A. Similarly, we approximate the impulse response of the second system as a rectangle of duration 2 μs and some height B. When connected in series, the resulting impulse response is the convolution of these two rectangular signals, which produces a trapezoidal waveform of duration 6 μs and height $2AB(10^{-6})$. All three impulse responses are shown in Fig. S2.6-12. The time constant T_{series} is the area divided by the peak height,

$$T_{\text{series}} = \frac{2AB(10^{-6})(6-2)(10^{-6})}{2AB(10^{-6})} = 4 \ \mu\text{s} = T_1.$$

It is not surprising that the series system has the same time constant as the first and slowest system in the cascade; the speed of a chain of systems is limited by the slowest system in that chain!

Figure S2.6-12

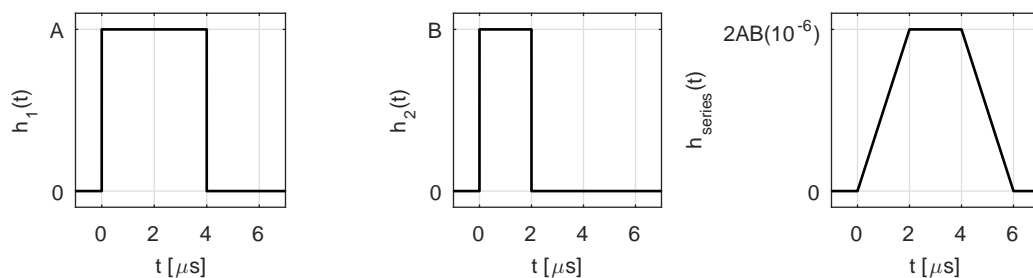### Solution 2.7-1

(a) Here, the characteristic equation is $\lambda^4 - 16 = 0$. Rearranging, we obtain $\lambda^4 = 16e^{j2\pi k}$. Solving, we see that $\lambda = 2e^{j\pi k/2}$. Using $k = 0$, 1, 2, and 3, we obtain the four unique characteristic roots:
$$\lambda_1 = 2, \ \lambda_2 = 2j, \ \lambda_3 = -2, \text{ and } \lambda_4 = -2j \ .$$

This result is easily confirmed with MATLAB.

```
>> lambda = roots([1 0 0 0 -16])
   lambda = -2.0000 + 0.0000i
            0.0000 + 2.0000i
            0.0000 - 2.0000i
            2.0000 + 0.0000i
```

(b) From Eq. (2.17), computing $h(t)$ requires a signal $\tilde{y}_n(t) = \sum_{k=1}^{4} c_k e^{\lambda_k t}$. To begin, we express the needed system of equations with a matrix representation.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 2j & -2 & -2j \\
4 & -4 & 4 & -4 \\
8 & -8j & -8 & 8j
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}.
$$

Next, we use MATLAB to compute the coefficients c_k.

```
>> lambda = [2, 2j, -2, -2j];
>> A = [lambda.^0; lambda.^1; lambda.^2; lambda.^3];
>> c = A\[0;0;0;1]
   c =  0.0313 + 0.0000i
        0.0000 + 0.0313i
       -0.0313 + 0.0000i
        0.0000 - 0.0313i
```

Since $0.0313 = \frac{1}{32}$, we see that

$$
c_1 = \tfrac{1}{32},\ c_2 = \tfrac{j}{32},\ c_3 = -\tfrac{1}{32},\ \text{and}\ c_4 = -\tfrac{j}{32}.
$$

Solution 2.7-2

(a) Here we use MATLAB to plot $x(t)$, $h_1(t)$, and $h_2(t)$ over the interval $-2.5 \le t \le 3.5$. Figure S2.7-2a shows the results.

```
>> u = @(t) 1.0*(t>=0); x = @(t) 2*u(t+2/3)-2*u(t);
>> h1 = @(t) mod(t,1); h2 = @(t) h1(t).*(u(t-1)-u(t-2));
>> t = -2.5:.001:3.5;
>> subplot(131); plot(t,x(t)); grid on;
>> axis([-2.5 3.5 -.1 2.1]); xlabel('t'); ylabel('x(t)');
>> set(gca,'xtick',[-2/3 0],'xticklabel',{'-2/3','0'});
>> subplot(132); plot(t,h1(t)); grid on;
>> axis([-2.5 3.5 -.1 2.1]); xlabel('t'); ylabel('h_1(t)');
>> set(gca,'xtick',[-2:3]);
>> subplot(133); plot(t,h2(t)); grid on;
>> axis([-2.5 3.5 -.1 2.1]); xlabel('t'); ylabel('h_2(t)');
>> set(gca,'xtick',[-2:3]);
```

(b) To help visualize $y_2(t) = x(t) * h_2(t)$, plots of $h_2(\tau)$ and $x(t - \tau)$ are shown in Fig. S2.7-2b. There are five regions (R1 to R5) for this convolution .

$$
\begin{array}{lll}
\text{R1:} & t < \tfrac{1}{3} & y_2(t) = \int 0\,d\tau = 0 \\[4pt]
\text{R2:} & \tfrac{1}{3} \le t < 1 & y_2(t) = \int_1^{t+\frac{2}{3}} 2(\tau - 1)\,d\tau = \tau^2 - 2\tau\big|_1^{t+\frac{2}{3}} = t^2 - \tfrac{2}{3}t + \tfrac{1}{9} \\[4pt]
\text{R3:} & 1 \le t < \tfrac{4}{3} & y_2(t) = \int_t^{t+\frac{2}{3}} 2(\tau - 1)\,d\tau = \tau^2 - 2\tau\big|_t^{t+\frac{2}{3}} = \tfrac{4}{3}t - \tfrac{8}{9} \\[4pt]
\text{R4:} & \tfrac{4}{3} \le t < 2 & y_2(t) = \int_t^2 2(\tau - 1)\,d\tau = \tau^2 - 2\tau\big|_t^2 = -t^2 + 2t \\[4pt]
\text{R5:} & t \ge 2 & y_2(t) = \int 0\,d\tau = 0
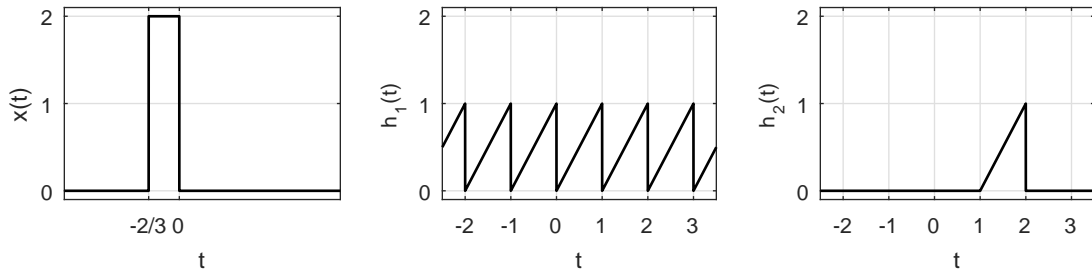\end{array}
$$

Figure S2.7-2a

Thus,

$$
y_2(t) = \begin{cases}
0 & t < \frac{1}{3} \\
t^2 - \frac{2}{3}t + \frac{1}{9} & \frac{1}{3} \leq t < 1 \\
\frac{4}{3}t - \frac{8}{9} & 1 \leq t < \frac{4}{3} \\
-t^2 + 2t & \frac{4}{3} \leq t < 2 \\
0 & t \geq 2
\end{cases}.
$$

A plot of $y_2(t)$ is also found in Fig. S2.7-2b.
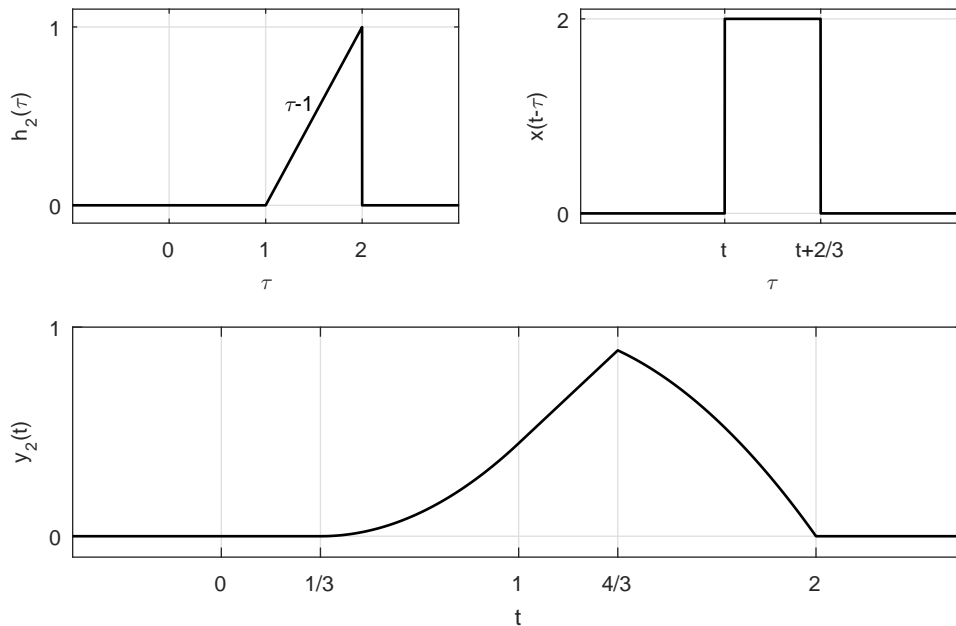
Figure S2.7-2b

(c) Since $h_1(t)$ is 1-periodic, $y_1(t) = x(t) * h_1(t)$ is also 1-periodic. Thus, we only need to determine $y_1(t)$ over a 1 second interval. By flipping and shifting $x(t)$, there are only two regions that need to be computed.

R1: $0 \leq t < \frac{1}{3}$ $y_1(t) = \int_t^{t+\frac{2}{3}} 2\tau \, d\tau = \tau^2 \big|_t^{t+\frac{2}{3}} = \frac{4}{3}t + \frac{4}{9}$

R2: $\frac{1}{3} \leq t < 1$ $y_1(t) = \int_t^1 2\tau \, d\tau + \int_1^{t+\frac{2}{3}} 2(\tau - 1) \, d\tau = \tau^2 \big|_t^1 + \tau^2 - 2\tau \big|_1^{t+\frac{2}{3}} = -\frac{2}{3}t + \frac{10}{9}$

Applying a periodic extension, the final result is

$$
y_1(t) = \begin{cases} \frac{4}{3}t + \frac{4}{9} & 0 \le t < \frac{1}{3} \\ -\frac{2}{3}t + \frac{10}{9} & \frac{1}{3} \le t < 1 \\ y_1(t+1) & \forall t \end{cases} .
$$

A plot of $y_1(t)$ is shown at the top of Fig. S2.7-2c. This result is graphically verified (see bottom of Fig. S2.7-2c) by modifying program CH2MP4.m in Sec. 2.7.4 with updated definitions of $x(t)$ and $h(t)$, finer resolution time vector, and cosmetic changes to axis limits.
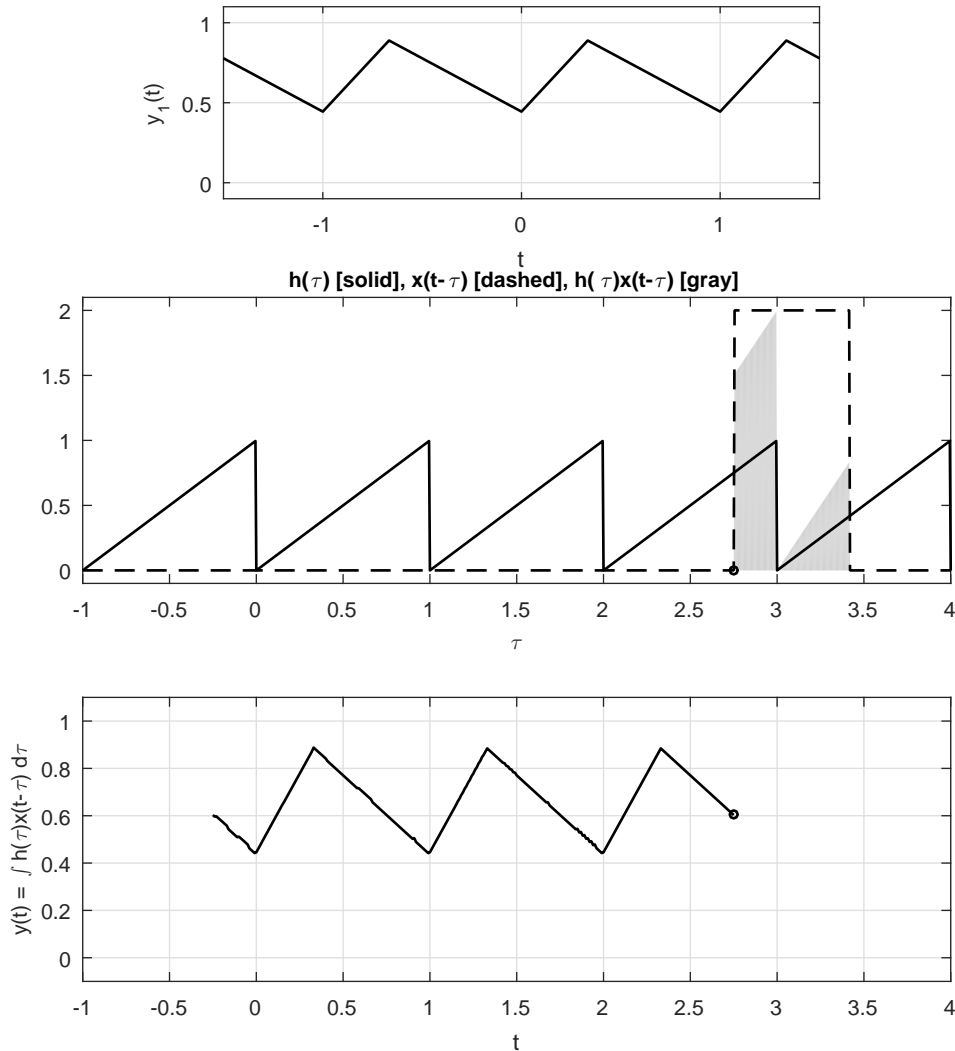

Figure S2.7-2c

Solution 2.7-3

(a) KCL at the negative terminal of the op-amp yields $\frac{x(t)-0}{R_{in}} + \frac{y(t)-0}{R_f} + i_c(t) = 0$. Also, $i_c(t) = C\dot{y}(t)$. Thus, $\frac{x(t)}{R_{in}} + \frac{y(t)}{R_f} + C\dot{y}(t) = 0$ or

$$
\dot{y}(t) + \frac{1}{CR_f}y(t) = \frac{-1}{CR_{in}}x(t).
$$

The characteristic equation is $\lambda + \frac{1}{CR_f} = 0$, and the characteristic root is

$$\lambda = \frac{-1}{CR_f}.$$

(b) The zero-state response is $y(t) = x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a first order system with $\lambda = \frac{-1}{CR_f}$, thus $\tilde{y}_0(t) = \tilde{c}_1 e^{\lambda t}$. Since $\tilde{y}_0(0) = 1 = \tilde{c}_1$, $b_0 = 0$, and $P(D) = \frac{-1}{CR_{in}}$, the impulse response is $h(t) = \frac{-1}{CR_{in}}e^{-t/(CR_f)}u(t)$. Thus,

$$y(t) = \left(\int_0^t \frac{-1}{CR_{in}}e^{-\tau/(CR_f)}d\tau\right)u(t) = \left(\frac{R_f}{R_{in}}e^{-\tau/(CR_f)}\Big|_{\tau=0}^t\right)u(t) = \frac{R_f}{R_{in}}(e^{-t/(CR_f)} - 1)u(t).$$

$$y(t) = \frac{R_f}{R_{in}}(e^{-t/(CR_f)} - 1)u(t).$$

Notice, $y(t)$ approaches $\frac{-R_f}{R_{in}}$ as time increases; unlike a true integrator, the "lossy" integrator provides a bounded output in response to a DC input.

(c) For this system, the characteristic root is only affected by C and R_f. Using 10% resistors, the resistor R_f is generally expected to lie in the range $(0.9R_f, 1.1R_f)$. Using 25% capacitors, the capacitor C is generally expected to lie in the range $(.25C, 1.25C)$. Since $\lambda = \frac{-1}{CR_f}$, the characteristic root is expected to lie in the range $(\lambda/[(0.9)(0.75)], \lambda/[(1.1)(1.25)])$. Thus,

The characteristic root is expected within the interval $(1.48\lambda, 0.73\lambda)$.

Solution 2.7-4

Identify the output of the first op-amp as $v(t)$.

(a) KCL at the negative terminal of the first op-amp yields $\frac{x(t)}{R_1} + C_1\dot{v}(t) = 0$ or $\frac{1}{R_1 C_1}x(t) = -\dot{v}(t)$. KCL at the negative terminal of the second op-amp yields $\frac{v(t)}{R_2} + \frac{y(t)}{R_3} + C_2\dot{y}(t) = 0$ or $v(t) = -\frac{R_2}{R_3}y(t) - R_2 C_2\dot{y}(t)$. Substituting this expression for $v(t)$ into the first expression yields $\frac{1}{R_1 C_1}x(t) = \frac{R_2}{R_3}\dot{y}(t) + R_2 C_2\ddot{y}(t)$. Thus,

$$\ddot{y}(t) + \frac{1}{R_3 C_2}\dot{y}(t) = \frac{1}{R_1 R_2 C_1 C_2}x(t).$$

The characteristic equation is $\lambda^2 + \frac{1}{R_3 C_2}\lambda = 0$ and the characteristic roots are

$$\lambda_1 = 0 \text{ and } \lambda_2 = -\frac{1}{R_3 C_2}.$$

Substituting $C_1 = C_2 = 10\mu\text{F}$, $R_1 = R_2 = 100k\Omega$, and $R_3 = 50k\Omega$ yields

$$\ddot{y}(t) + 2\dot{y}(t) = x(t), \lambda_1 = 0, \text{ and } \lambda_2 = -2.$$

Since one root lies on the ω-axis, the circuit is not BIBO stable. In particular, a DC input results in an unbounded output.

(b) The zero-input response has form $y_0(t) = c_1 + c_2 e^{-2t}$. Each op-amp has an initial output of one volt. Thus, $y_0(0) = 1 = c_1 + c_2$. KCL at the negative terminal of the second op-amp yields $\frac{1}{R_2} + \frac{1}{R_3} + C_2\dot{y}_0(0) = 0$ or $\dot{y}_0(0) = -\frac{1}{R_2 C_2} - \frac{1}{R_3 C_2} = -1 - 2 = -3$. Thus, $\dot{y}_0(0) = -3 = -2c_2$. Thus, $c_2 = 3/2$ and $c_1 = 1 - 3/2 = -1/2$ and

$$y_0(t) = -1/2 + 3/2e^{-2t}.$$

(c) The zero-state response is $y(t) = x(t) * h(t)$, where $h(t) = b_0 \delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a second order system with $\lambda_1 = 0$ and $\lambda_2 = -2$, so $\tilde{y}_0(t) = \tilde{c}_1 + \tilde{c}_2 e^{-2t}$. Solving $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$ and $\tilde{y}_0^{(1)}(t) = 1 = -2\tilde{c}_2$ yields $\tilde{c}_2 = -1/2$ and $\tilde{c}_1 = 1/2$. Since $b_0 = 0$ and $P(D) = 1$, $h(t) = \left(1/2 - e^{-2t}/2\right)u(t)$.

Next, $y(t) = x(t) * h(t) = \left(\int_0^t (1/2 - e^{-2\tau}/2)d\tau\right)u(t) = \left(\tau/2 + e^{-2\tau}/4\Big|_{\tau=0}^t\right)u(t) = \left(t/2 + e^{-2t}/4 - 1/4\right)u(t)$.

$$y(t) = \left(t/2 + e^{-2t}/4 - 1/4\right)u(t).$$

As expected, the DC nature of the unit step input results in an unbounded output.

(d) For this system, $\lambda_1 = 0$ is not affected by the components and λ_2 is only affected by C_2 and R_3. Using 10% resistors, the resistor R_3 is generally expected to lie in the range $(0.9R_3, 1.1R_3)$. Using 25% capacitors, the capacitor C_2 is generally expected to lie in the range $(.25C_2, 1.25C_2)$. Since $\lambda_2 = -\frac{1}{R_3 C_2}$, the characteristic root is expected to lie in the range $(\lambda_2/[(0.9)(0.75)], \lambda_2/[(1.1)(1.25)])$. Thus,

$$\lambda_1 \text{ is unaffected and } \lambda_2 \text{ is expected to lie within } (-2.9630, -1.4545).$$

Solution 2.7-5

A plot of $y_1(t)$ is shown at the top of Fig. S2.7-5c. Here, we compute $y(t) = x(t) * h(t)$ graphically by modifying program `CH2MP4.m` in Sec. 2.7.4. As shown in Fig. S2.7-5, the final result has a total of 7 regions.

```
% CH2MP4.m : Chapter 2, MATLAB Program 4
% Script M-file graphically demonstrates the convolution process.

figure(1) % Create figure window and make visible on screen
u = @(t) 1.0*(t>=0);
x = @(t) 3*(u(t)-u(t-1))+2*(u(t-2)-u(t-3));
h = @(t) (4-t).*(u(t)-u(t-2));
dtau = 0.005; tau = -3:dtau:6;
ti = 0; tvec = -0.5:.02:5.5;
y = NaN*zeros(1,length(tvec)); % Pre-allocate memory
for t = tvec,
    ti = ti+1; % Time index
    xh = x(t-tau).*h(tau); lxh = length(xh);
    y(ti) = sum(xh.*dtau); % Trapezoidal approximation of convolution integral
    subplot(2,1,1),plot(tau,h(tau),'k-',tau,x(t-tau),'k--',t,0,'ok');
    axis([tau(1) tau(end) -1 12]);
    patch([tau(1:end-1);tau(1:end-1);tau(2:end);tau(2:end)],...
        [zeros(1,lxh-1);xh(1:end-1);xh(2:end);zeros(1,lxh-1)],...
        [.8 .8 .8],'edgecolor','none');
    xlabel('\tau'); title('h(\tau) [solid], x(t-\tau) [dashed], h(\tau)x(t-\tau) [gray]');
    c = get(gca,'children'); set(gca,'children',[c(2);c(3);c(4);c(1)]);
    subplot(2,1,2),plot(tvec,y,'k',tvec(ti),y(ti),'ok');
    xlabel('t'); ylabel('y(t) = \int h(\tau)x(t-\tau) d\tau');
    axis([tau(1) tau(end) -1 12]); grid;
    drawnow;
end
```
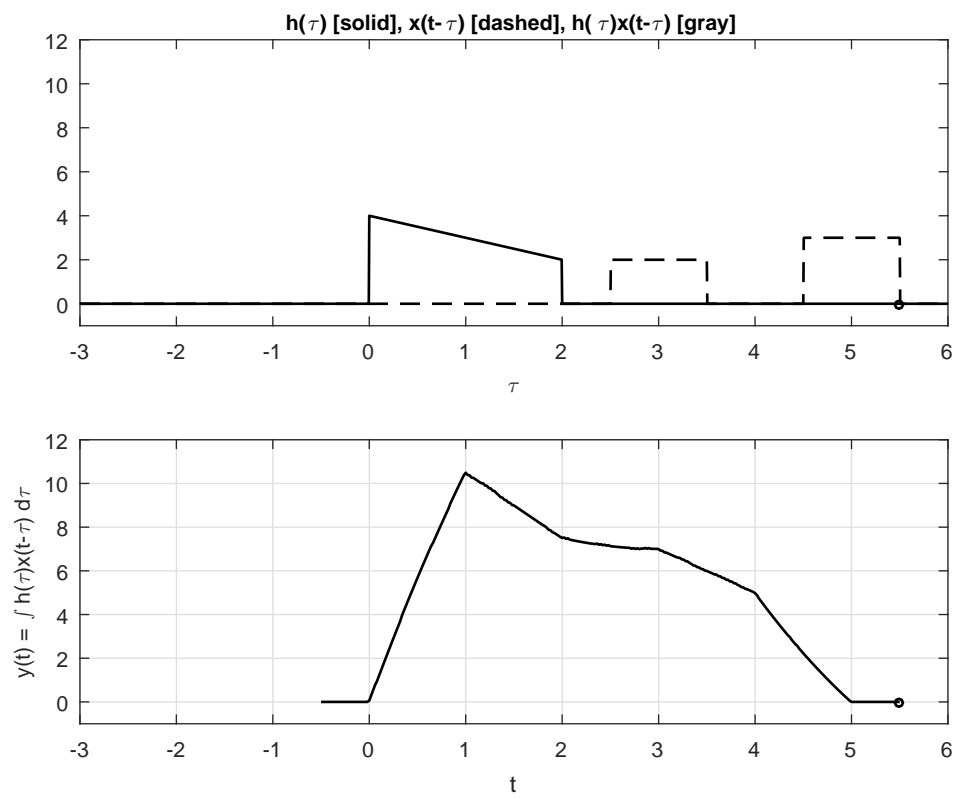
Figure S2.7-5

Chapter 3 Solutions

Solution 3.1-1

(a) $E_x = (3)^2 + 2(2)^2 + 2(1)^2 = 19$

(b) $E_x = (3)^2 + 2(2)^2 + 2(1)^2 = 19$

(c) $E_x = 2(3)^2 + 2(6)^2 + 2(9)^2 = 252$

(d) $E_x = 2(2)^2 + 2(4)^2 = 40$

Solution 3.1-2

(a)
$$P_x = \frac{1}{N_0 + 1} \sum_{n=-3}^{3} |x[n]|^2 = \frac{1}{6} \left[2(1)^2 + 2(2)^2 + 3^2 \right] = \frac{19}{6}$$

(b)
$$P_x = \frac{1}{12} \left[2(1)^2 + 2(2)^2 + 2(3)^2 \right] = \frac{7}{3}$$

(c)
$$P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} a^n = \frac{1}{N_0} \left[\frac{a^{N_0} - 1}{a - 1} \right] = \frac{a^{N_0} - 1}{N_0(a - 1)}$$

Solution 3.1-3

First, we show that the power of a signal $\mathcal{D}e^{j(2\pi/N_0)n}$ is $|\mathcal{D}|^2$.

$$P = \frac{1}{N_0} \sum_{n=0}^{N_0-1} \left| \mathcal{D}e^{jr\frac{2\pi}{N_0}n} \right|^2 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |\mathcal{D}|^2 = |\mathcal{D}|^2.$$

Next, we show that the power of a signal $x[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr(2\pi/N_0)n}$ is $P_x = \sum_{r=0}^{N_0-1} |\mathcal{D}_r|^2$

$$P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \left| \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \right|^2$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \sum_{m=0}^{N_0-1} \mathcal{D}_m^* e^{-jm\Omega_0 n}$$

Interchanging the order of summation yields

$$
P_x = \frac{1}{N_0} \sum_{r=0}^{N_0-1} \sum_{m=0}^{N_0-1} \mathcal{D}_r \mathcal{D}_m^* \left[\sum_{n=0}^{N_0-1} e^{j(r-m)\Omega_0 n} \right]
$$

The summation within square brackets is N_0 when $r = m$ and 0 otherwise. Hence,

$$
P_x = \sum_{r=0}^{N_0-1} \mathcal{D}_r \mathcal{D}_r^* = \sum_{r=0}^{N_0-1} |\mathcal{D}_r|^2 .
$$

Solution 3.1-4

(a) Here,

$$
x[n] = 0.8^n u[n] = \underbrace{\frac{1}{2} \left\{ 0.8^n u[n] + 0.8^{-n} u[-n] \right\}}_{x_e[n]} + \underbrace{\frac{1}{2} \left\{ 0.8^n u[n] - 0.8^{-n} u[-n] \right\}}_{x_o[n]}
$$

and

$$
E_x = \sum_{n=0}^{\infty} (0.8)^{2n} = \sum_{n=0}^{\infty} 0.64^n = \frac{1}{1 - 0.64} = 2.78.
$$

(b) To find the energy of the even component $x_e[n]$, we observe that both terms $0.8^n u[n]$ and $0.8^{-n} u[-n]$ are nonzero at $n = 0$. Hence, the two terms are not disjoint, and the energy E_{x_e} is not the sum of the energies of the two terms. For this reason, we rearrange $x_e[n]$ as

$$
x_e[n] = \delta[n] + \frac{1}{2} \left\{ 0.8^n u[n-1] + 0.8^{-n} u[-n-1] \right\} .
$$

All the above three terms are disjoint. Hence E_{x_e} is the sum of energies of the three terms. Thus,

$$
E_{x_e} = 1 + \frac{1}{4} \sum_{n=1}^{\infty} 0.64^n + \frac{1}{4} \sum_{n=-\infty}^{-1} 0.64^{-n}
$$

$$
= 1 + \frac{1}{2} \sum_{n=1}^{\infty} 0.64^n = 1 + \frac{1}{2} \left[\frac{0.64}{1 - 0.64} \right] = 1.89.
$$

The energy of $x_o[n]$ is

$$
E_{x_o} = \frac{1}{4} \left[\sum_{n=1}^{\infty} 0.64^n + \sum_{n=-\infty}^{-1} 0.64^{-n} \right] = \frac{1}{2} \sum_{n=1}^{\infty} 0.64^n = 0.89.
$$

Hence

$$
E_{x_e} + E_{x_o} = 1.89 + 0.89 = 2.78 = E_x
$$

(c) Let us first consider a causal signal $x[n]$, which can be expressed as

$$
x[n] = x[0]\delta[n] + \underbrace{\frac{1}{2} \left\{ x[n]u[n-1] + x[-n]u[-n-1] \right\}}_{x_e[n]} + \underbrace{\frac{1}{2} \left\{ x[n]u[n-1] - x[-n]u[-n-1] \right\}}_{x_o[n]} .
$$

The energy of the even component $x_e[n]$ is

$$E_{x_e} = x^2[0] + \frac{1}{4}\sum_{n=1}^{\infty}|x[n]|^2 + \frac{1}{4}\sum_{n=-1}^{-\infty}|x[-n]|^2 = x^2[0] + \frac{1}{2}\sum_{n=1}^{\infty}|x[n]|^2.$$

Similarly,

$$E_{x_o} = \frac{1}{2}\sum_{n=1}^{\infty}|x[n]|^2.$$

Hence,

$$E_{x_e} + E_{x_o} = x^2[0] + \sum_{n=1}^{\infty}|x[n]|^2 = E_x.$$

Similarly, we can show that this result applies to anticausal signals also.

A general signal is made up of causal and anticausal components, which are disjoint. Hence the energy of a signal is the sum of energies of the causal and anticausal components. Moreover the energy of each causal and anticausal component is equal to the sum of the respective even and odd components. Also the sum of the even components of the causal and anticausal signals equals the even component of $x[n]$. The same is true of odd components. Hence, it follows that the energy of $x[n]$ is the sum of energies of even and odd components of $x[n]$.

Solution 3.1-5

In this problem, $x[n] = x[n]u[n]$ is a causal energy signal. By definition, we know that

$$x_e[n] = \frac{x[n] + x[-n]}{2} = \frac{x[n]u[n] + x[-n]u[-n]}{2}$$

and

$$x_o[n] = \frac{x[n] - x[-n]}{2} = \frac{x[n]u[n] - x[-n]u[-n]}{2}.$$

(a) The energy of a DT signal $x[n]$ is computed as

$$E_x = \sum_{n=-\infty}^{\infty}|x[n]u[n]|^2 = \sum_{n=0}^{\infty}|x[n]|^2.$$

Computing the energy of $x_e[n]$, we obtain

$$
\begin{aligned}
E_{x_e} &= \sum_{n=-\infty}^{\infty}|x_e[n]|^2 \\
&= \sum_{n=-\infty}^{\infty}\left|\frac{x[n]u[n] + x[-n]u[-n]}{2}\right|^2 \\
&= \frac{1}{4}\sum_{n=-\infty}^{\infty}|x[n]|^2 u[n] + 2|x[0]|^2\delta[n] + |x[-n]|^2 u[-n] \\
&= \frac{1}{4}\sum_{n=0}^{\infty}|x[n]|^2 + \sum_{n=-\infty}^{\infty}\frac{1}{2}|x[0]|^2\delta[n] + \frac{1}{4}\sum_{n=-\infty}^{0}|x[-n]|^2 \\
&= \frac{1}{4}E_x + \frac{1}{2}|x[0]|^2 + \frac{1}{4}E_x = \frac{1}{2}E_x + \frac{1}{2}|x[0]|^2.
\end{aligned}
$$

Similarly,

$$E_{x_o} = \sum_{n=-\infty}^{\infty} |x_o[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| \frac{x[n]u[n] - x[-n]u[-n]}{2} \right|^2$$

$$= \frac{1}{4} \sum_{n=-\infty}^{\infty} |x[n]|^2 u[n] - 2|x[0]|^2 \delta[n] + |x[-n]|^2 u[-n]$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} |x[n]|^2 - \sum_{n=-\infty}^{\infty} \frac{1}{2} |x[0]|^2 \delta[n] + \frac{1}{4} \sum_{n=-\infty}^{0} |x[-n]|^2$$

$$= \frac{1}{4} E_x - \frac{1}{2} |x[0]|^2 + \frac{1}{4} E_x = \frac{1}{2} E_x - \frac{1}{2} |x[0]|^2.$$

Adding E_{x_e} and E_{x_o} we obtain

$$E_{x_e} + E_{x_o} = \frac{1}{2} E_x + \frac{1}{2} |x[0]|^2 + \frac{1}{2} E_x - \frac{1}{2} |x[0]|^2 = E_x.$$

Clearly, $E_{x_e} = E_{x_o} = 0.5 E_x$ holds true if $x[n]$ is causal and $x[0] = 0$.

(b) Next, we compute the cross energy of x_e and x_o as

$$E_{x_e, x_o} = \sum_{n=-\infty}^{\infty} x_e[n] x_o^*[n]$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{x[n]u[n] + x[-n]u[-n]}{2} \right) \left(\frac{x[n]u[n] - x[-n]u[-n]}{2} \right)^*$$

$$= \frac{1}{4} \sum_{n=-\infty}^{\infty} |x[n]|^2 u[n] + |x[0]|^2 \delta[n] - |x[0]|^2 \delta[n] - |x[-n]|^2 u[-n]$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} |x[n]|^2 - \frac{1}{4} \sum_{n=-\infty}^{0} |x[-n]|^2$$

$$= \frac{1}{4} (E_x - E_x) = 0.$$

Solution 3.1-6

In this problem,

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n & n \geq 0 \\ A^n & n < 0 \end{cases},$$

where A is either 0.5, 1, or 2. For $A \neq 1$, the energy of $x[n]$ is

$$E_x = \sum_{-\infty}^{-1} A^{2n} + \sum_{0}^{\infty} \left(\frac{1}{3}\right)^{2n} = \frac{A^{-2(\infty)} - 1}{1 - A^2} + \frac{1 - 0}{1 - \frac{1}{9}}.$$

For $A \neq 1$, the power of $x[n]$ is

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \left(\sum_{-N}^{-1} A^{2n} + \sum_{0}^{N} (\tfrac{1}{3})^{2n} \right)$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left(\frac{A^{-2(\infty)} - 1}{1 - A^2} + \frac{1 - 0}{1 - \frac{1}{9}} \right)$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left(\frac{A^{-2(\infty)} - 1}{1 - A^2} \right).$$

(a) For the case $A = \frac{1}{2}$, we see that

$$E_x = \frac{2^{2(\infty)} - 1}{1 - \frac{1}{4}} + \frac{9}{8} = \infty$$

and

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \left(\frac{2^{2(\infty)} - 1}{1 - \frac{1}{4}} \right) = \infty.$$

When $A = \frac{1}{2}$, $x[n]$ is neither an energy signal nor a power signal ($E_x = P_x = \infty$).

(b) For the case $A = 1$, we see that

$$E_x = \sum_{-\infty}^{-1} 1 + \sum_{0}^{\infty} (\tfrac{1}{3})^{2n} = \infty$$

and

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \left(\sum_{-N}^{-1} 1 + \sum_{0}^{N} (\tfrac{1}{3})^{2n} \right)$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left(N + \frac{9}{8} \right) = \frac{1}{2}.$$

When $A = 1$, $x[n]$ a power signal ($E_x = \infty$ and $P_x = \frac{1}{2}$).

(c) For the case $A = 2$, we see that

$$E_x = \frac{2^{-2(\infty)} - 1}{1 - 2^2} + \frac{9}{8} = \frac{1}{3} + \frac{9}{8} = \frac{35}{24}.$$

Since energy is finite,

$$P_x = 0.$$

When $A = 2$, $x[n]$ is an energy signal ($E_x = \frac{35}{24}$ and $P_x = 0$).

Solution 3.1-7

To begin, we notice that

$$x[n] = \mathrm{Re} \left\{ 3e^{j\pi n/4} \right\} = 3\cos(\pi n/4).$$

We see that $x[n]$ is 8-periodic and, therefore, a power signal.

$$P_x = \frac{1}{8} \sum_{n=0}^{7} |x[n]|^2 = \frac{1}{8} \left[2(3)^2 + 4(3/\sqrt{(2)})^2 \right] = \frac{1}{8}[18 + 18] = \frac{9}{2}.$$

We can verify this power calculation in MATLAB.

```
>>  x = @(n) 3*cos(pi*n/4); n = 0:7; Px = sum((x(n)).^2)/8
    Px = 4.5000
```

Since $0 < P_x < \infty$, we know that $E_x = \infty$. Thus,

$$E_x = \infty \quad \text{and} \quad P_x = \frac{9}{2}.$$

Solution 3.2-1

(a) The energy E_a of $x[-n]$ is given by

$$E_a = \sum_{n=-\infty}^{\infty} |x[-n]|^2.$$

Setting $n = -m$, we obtain

$$E_a = \sum_{m=\infty}^{-\infty} |x[m]|^2 = \sum_{m=-\infty}^{\infty} |x[m]|^2 = E_x.$$

(b) The energy E_b of $x[n-m]$ is given by

$$E_b = \sum_{n=-\infty}^{\infty} |x[n-m]|^2 = \sum_{r=-\infty}^{\infty} |x[r]|^2 = E_x.$$

(c) The energy E_c of $x[m-n]$ is given by

$$E_c = \sum_{n=-\infty}^{\infty} |x[m-n]|^2 = \sum_{r=\infty}^{-\infty} |x[r]|^2 = \sum_{r=-\infty}^{\infty} |x[r]|^2 = E_x.$$

(d) The energy E_d of $Kx[n]$ is given by

$$E_d = \sum_{n=-\infty}^{\infty} |Kx[n]|^2 = K^2 \sum_{n=-\infty}^{\infty} |x[n]|^2 = K^2 E_x$$

Solution 3.2-2

We are given that a periodic signal $x[n]$ has power P_x.

(a) The power P_a of $-x[n]$ is given by

$$P_a = \frac{1}{N} \sum_{N} |-x[n]|^2 = \frac{1}{N} \sum_{N} |x[n]|^2 = P_x.$$

(b) The power P_b of $x[-n]$ is given by

$$P_b = \frac{1}{N} \sum_{N} |x[-n]|^2.$$

Setting $n' = -n$, we obtain

$$P_b = \frac{1}{N} \sum_{N} |x[n']|^2 = P_x.$$

(c) The power P_c of $x[n - m]$ is given by

$$P_c = \frac{1}{N} \sum_N |x[n - m]|^2.$$

Setting $n' = n - m$, we obtain

$$P_c = \frac{1}{N} \sum_N |x[n']|^2 = P_x.$$

(d) The power P_d of $cx[n]$ is given by

$$P_d = \frac{1}{N} \sum_N |cx[n]|^2 = |c|^2 \frac{1}{N} \sum_N |x[n]|^2 = |c|^2 P_x.$$

(e) The power P_e of $x[m - n]$ is given by

$$P_e = \frac{1}{N} \sum_N |x[m - n]|^2.$$

Setting $n' = m - n$, we obtain

$$P_e = \frac{1}{N} \sum_N |x[n']|^2 = P_x.$$

These results show that time-shift or time-inversion operations do not affect the power of a signal. The same is the case with a sign change. Multiplication of a signal by a constant c causes $|c|^2$-fold increase in power.

Solution 3.2-3

This problem considers a signal $x[n]$ that is defined as $[-1, 2, -3, 4, -5, 4, -3, \overset{\downarrow}{2}, -1]$.

(a) We use a simple table to construct $y[n] = x[-3n + 2]$.

n	$y[n]$
0	$x[-3(0) + 2] = x[2] = 0$
1	$x[-3(1) + 2] = x[-1] = -3$
2	$x[-3(2) + 2] = x[-4] = 4$
3	$x[-3(3) + 2] = x[-7] = -1$
4	$x[-3(4) + 2] = x[-10] = 0$

Thus,

$$y[n] = [\overset{\downarrow}{0}, -3, 4, -1].$$

(b) For $z[n] = x[n/2 - 3]$, we see that $z[n] = x[-7]$ (the left-most nonzero value of $x[n]$) when $n/2 - 3 = -7$ or $n = -8$. We obtain the remaining values of $z[n]$ by interlacing each value of $x[n]$ with a single zero (upsample by 2). Thus,

$$z[n] = [-1, 0, 2, 0, -3, 0, 4, 0, \overset{\downarrow}{-5}, 0, 4, 0, -3, 0, 2, 0, -1].$$

Solution 3.2-4

This problem considers a signal $x[n]$ that is defined as $[-1, \overset{\downarrow}{2}, -3, 4, -5, 4, -3, 2, -1]$.

(a) Since $x[n]$ is finite-duration, it is an energy signal and $P_x = 0$. The energy E_x is just a summation of the values of $|x[n]|^2$.

$$E_x = (1 + 4 + 9 + 16 + 25 + 16 + 9 + 4 + 1) = 85.$$

(b) We use a simple table to construct $y[n] = x[2(n + 2)]$.

n	$y[n]$
-3	$x[2(-3 + 2)] = x[-2] = 0$
-2	$x[2(-2 + 2)] = x[0] = 2$
-1	$x[2(-1 + 2)] = x[2] = 4$
0	$x[2(0 + 2)] = x[4] = 4$
1	$x[2(1 + 2)] = x[6] = 2$
2	$x[2(2 + 2)] = x[8] = 0$

Thus,

$$y[n] = [2, 4, \overset{\downarrow}{4}, 2].$$

(c) For $z[n] = x[-\frac{n-6}{3}]$, we see that $z[n] = x[-1]$ (the left-most nonzero value of $x[n]$) when $-\frac{n-6}{3} = -1$ or $n = 9$. We obtain the remaining values of $z[n]$ by reflecting $x[n]$ and interlacing each value with two zeros (upsample by 3). Thus,

$$z[n] = [-1, 0, 0, 2, 0, 0, -3, 0, 0, 4, 0, 0, -5, 0, 0, \overset{\downarrow}{4}, 0, 0, -3, 0, 0, 2, 0, 0, -1].$$

Solution 3.2-5

This problem considers a 4-periodic signal $w[n]$ defined as $[\cdots, 1, 2, 3, 4, \overset{\downarrow}{1}, 2, 3, 4, 1, 2, 3, 4, \cdots]$.

(a) For $x[n] = w[2n]$, $x[n]$ is just $w[n]$ downsampled by 2. Thus,

$$x[n] = [\cdots 1, 3, \overset{\downarrow}{1}, 3, 1, 3, \cdots].$$

Clearly, $x[n]$ is 2-periodic and thus a power signal with $P_x = \frac{1}{2}(1^2 + 3^2) = \frac{10}{2} = 5$. Thus,

$$E_x = \infty \quad \text{and} \quad P_x = 5.$$

(b) For $y[n] = w[2 - \frac{n}{3}]$, $y[n]$ is just $w[n]$ reflected, upsampled by 3, and then shifted. To help get oriented, we see that $y[0] = w[2 - \frac{0}{3}] = w[2] = 3$. Thus,

$$y[n] = [\cdots 4, 0, 0, \overset{\downarrow}{3}, 0, 0, 2, 0, 0, 1, 0, 0, 4, 0, 0, 3, 0, 0, 2, 0, 0, 1, 0, 0, \cdots].$$

Clearly, $y[n]$ is 12-periodic and thus a power signal with $P_y = \frac{1}{12}(1^2 + 2^2 + 3^2 + 4^2) = \frac{30}{12} = \frac{5}{2}$. Thus,

$$E_x = \infty \quad \text{and} \quad P_x = \frac{5}{2}.$$

Solution 3.2-6

(a) Since $x[n]$ is finite duration, it is an energy signal and $P_x = 0$. Further, $E_x = \sum_{n=0}^{5} |x[n]|^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$. Thus,

$$E_x = 91 \quad \text{and} \quad P_x = 0.$$

(b) Since $y[n]$ is reflected with every other term of $x[n]$ missing, we know that $N_1 = -2$. Since $y[n]$ has 2 zeros inserted between each value from $x[n]$, we know that $N_2 = 3$. We want $x[4]$ to go to $y[0] = x[-2(0)/3 + N_3]$, so we know that $N_3 = 4$. Thus,

$$N_1 = -2, \ N_2 = 3, \text{ and } N_3 = 4.$$

For obvious reasons, it is also possible to have

$$N_1 = 2, \ N_2 = -3, \text{ and } N_3 = 4.$$

(c) Noting that $y[n]$ has a duration of 7, a 6-periodic replication of $y[n]$ has overlapping values and, over one period, is given by $\tilde{y}[n] = [\cdots, (5\overset{\downarrow}{+}1), 0, 0, 3, 0, 0, (5+1), 0, 0, 3, 0, 0, \cdots]$. The power of $\tilde{y}[n]$ is computed as

$$P_{\tilde{y}} = \frac{1}{6} \sum_{n=0}^{5} |\tilde{y}[n]|^2 = \frac{1}{6}(6^2 + 3^2) = \frac{45}{6} = \frac{15}{2}.$$

Thus,

$$E_{\tilde{y}} = \infty \quad \text{and} \quad P_{\tilde{y}} = \frac{15}{2}.$$

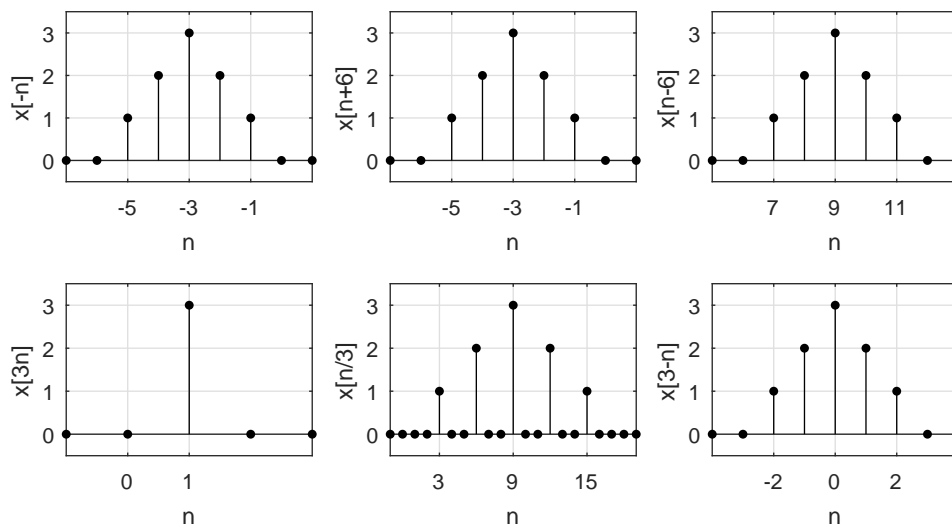## Solution 3.2-7

Figure S3.2-7 shows the desired signals.

Figure S3.2-7

Solution 3.2-8

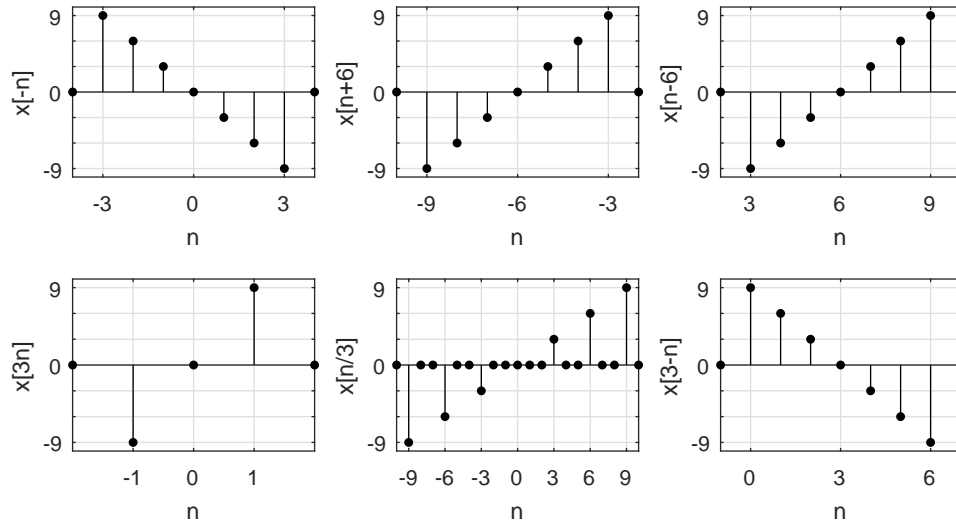Figure S3.2-8 shows the desired signals.

Figure S3.2-8

Solution 3.2-9

This problem considers a DT signal $x[n]$ whose nonzero values are given as $x[n] = [1, -3, 2, 2, 3, -2,$
-1, 1, 2, -3, $\overset{\downarrow}{3}$, 3, -2, 1, -3, 2, 3, -1]. Over $-5 \le n \le 4$, the desired signals $y[n] = x[-1-2n]$ and
$z[n] = x[-2+n/3]$ are easily determined by direct substitution.

n	-5	-4	-3	-2	-1	0	1	2	3	4
y[n]	x[9]	x[7]	x[5]	x[3]	x[1]	x[-1]	x[-3]	x[-5]	x[-7]	x[-9]
z[n]	0	0	x[-3]	0	0	x[-2]	0	0	x[-1]	0

As Fig. S3.2-9 makes clear, $y[n]$ is a reflected, downsampled, and shifted version of $x[n]$ while $z[n]$
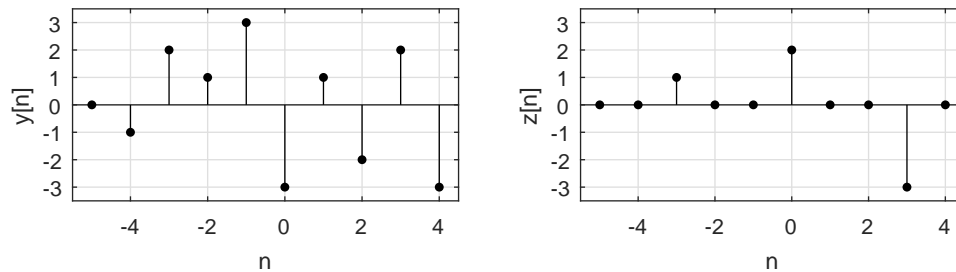is an upsampled and shifted version of $x[n]$.

Figure S3.2-9

Solution 3.2-10

In this problem, we determine and locate the two largest nonzero values of signals that are functions
of $x[n] = (\frac{1}{2})^n u[n]$. The largest nonzero values of $x[n]$ are $x[0] = 1$, $x[1] = \frac{1}{2}$, $x[2] = \frac{1}{4}$, and so forth.

(a) For $y_a[n] = x[2n]$, we see that

n	−1	0	1	2	3	4
$y_a[n]$	x[−2]	x[0]	x[2]	x[4]	x[6]	x[8]

Thus, the two largest nonzero values are

$$y_a[0] = x[0] = 1 \quad \text{and} \quad y_a[1] = x[2] = \frac{1}{4}.$$

(b) For $y_b[n] = x[n/3]$ we see that

n	-1	0	1	2	3	4
$y_b[n]$	0	$x[0]$	0	0	$x[1]$	0

Thus, the two largest nonzero values are

$$y_b[0] = x[0] = 1 \quad \text{and} \quad y_b[3] = x[1] = \frac{1}{2}.$$

(c) For $y_c[n] = x[3n+1]$ we see that

n	-1	0	1	2	3	4
$y_c[n]$	$x[-2]$	$x[1]$	$x[4]$	$x[7]$	$x[10]$	$x[13]$

Thus, the two largest nonzero values are

$$y_c[0] = x[1] = \frac{1}{2} \quad \text{and} \quad y_c[1] = x[4] = \frac{1}{16}.$$

(d) For $y_d[n] = x[-2n+5]$ we see that

n	-1	0	1	2	3	4
$y_d[n]$	$x[7]$	$x[5]$	$x[3]$	$x[1]$	$x[-1]$	$x[-3]$

Thus, the two largest nonzero values are

$$y_d[2] = x[1] = \frac{1}{2} \quad \text{and} \quad y_d[1] = x[3] = \frac{1}{8}.$$

(e) For $y_e[n] = x[-(n+8)/2]$ we see that

n	-12	-11	-10	-9	-8	-7	-6	-5
$y_e[n]$	$x[2]$	0	$x[1]$	0	$x[0]$	0	$x[-1]$	0

Thus, the two largest nonzero values are

$$y_e[-8] = x[0] = 1 \quad \text{and} \quad y_e[-10] = x[1] = \frac{1}{2}.$$

Solution 3.3-1

Figure S3.3-1 shows the desired signals.

(a) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} (1)^{2n} = \lim_{N\to\infty} \frac{2N+1}{2N+1} = 1$

(b) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} (-1)^{2n} = \lim_{N\to\infty} \frac{2N+1}{2N+1} = 1$

(c) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} (1)^2 = \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}$

(d) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} (-1)^2 = \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}$

(e) Here, $N_0 = \frac{2\pi}{\frac{\pi}{3}} = 6$. Thus,

$$P_x = \frac{1}{6} \sum_{0}^{5} (\cos[\frac{\pi}{3}n + \frac{\pi}{6}])^2 = \frac{1}{6}\left[\left(\frac{\sqrt{3}}{2}\right)^2 + 0^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right] = \frac{1}{2}.$$
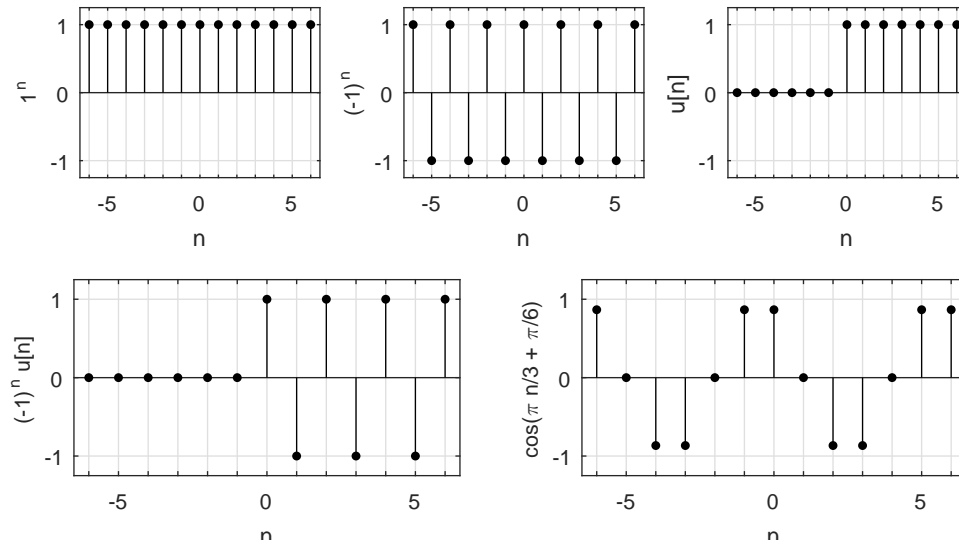
Figure S3.3-1

Solution 3.3-2

(a) Since, $u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$, we know that $u[n-2] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \cdots$. Thus, $u[n] - u[n-2] = (\delta[n] + \delta[n-1] + \delta[n-2] + \cdots) - (\delta[n-2] + \delta[n-3] + \delta[n-4] + \cdots) = \delta[n] + \delta[n-1]$.

(b) Because $\sin\frac{\pi n}{3} = 0$ for $n = 0$, we see that $2^{n-1}\sin\left(\frac{\pi n}{3}\right)u[n] = \frac{1}{2}2^n\sin\left(\frac{\pi n}{3}\right)u[n-1]$.

(c) Because $n(n-1) = 0$ for $n = 0$ and $n = 1$, we see that $n(n-1)\gamma^n u[n] = n(n-1)\gamma^n u[n-2]$.

(d) Because $\sin\frac{\pi n}{2} = 0$ for even n and $u[n] + (-1)^n u[n] = 0$ for odd n, we see that $(u[n] + (-1)^n u[n])\sin\left(\frac{\pi n}{2}\right) = 0$ for all n.

(e) Because $\cos\frac{\pi n}{2} = 0$ for odd n and $u[n] + (-1)^{n+1} = 0$ for even n, we see that $(u[n] + (-1)^{n+1}u[n])\cos\left(\frac{\pi n}{2}\right) = 0$ for all n.

Solution 3.3-3

Figure S3.3-3 shows the signals
$x_a[n] = u[n-2] - u[n-6]$
$x_b[n] = n\{u[n] - u[n-7]\}$
$x_c[n] = (n-2)\{u[n-2] - u[n-6]\}$
$x_d[n] = (-n+8)\{u[n-6] - u[n-9]\}$, and
$x_e[n] = (n-2)\{u[n-2] - u[n-6]\} + (-n+8)\{u[n-6] - u[n-9]\} = x_c[n] + x_d[n]$.

Figure S3.3-3

Solution 3.3-4
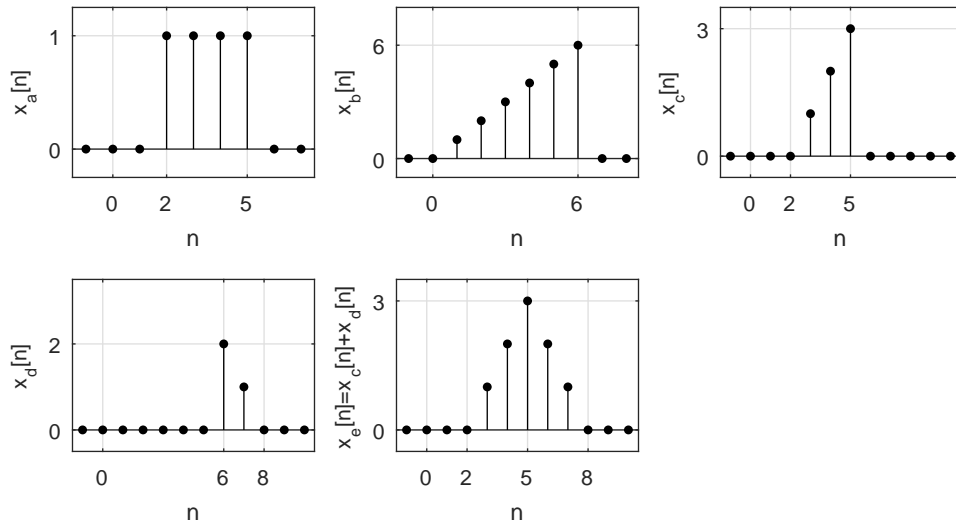
(a) $x[n] = (n+3)\left(u[n+3] - u[n]\right) + (-n+3)\left(u[n] - u[n-4]\right)$

(b) $x[n] = n\left(u[n] - u[n-4]\right) + (-n+6)\left(u[n-4] - u[n-7]\right)$

(c) $x[n] = 3n\left(u[n+3] - u[n-4]\right)$

(d) $x[n] = -2n\left(u[n+2] - u[n]\right) + 2n\left(u[n] - u[n-3]\right)$

In all four cases, $x[n]$ may be represented by other (slightly different) expressions. For instance, in case (a), we may also use $x[n] = (n+3)\left(u[n+3] - u[n-1]\right) + (-n+3)\left(u[n-1] - u[n-4]\right)$. Moreover because $x[n] = 0$ at $n = \pm 3$, $u[n+3]$ and $u[n-4]$ may be replaced with $u[n+2]$ and $u[n-3]$, respectively. Similar observations apply to the other cases also. What is important is that the expression evaluates to the correct waveform shape.

Solution 3.3-5

As eigensignals of LTID system, everlasting exponentials z^n pass through LTID systems modified only in gain and phase – no shape distortion occurs. As shown in Sec. 3.8.2 [see Eq. (3.38)], this idea is expressed mathematically as

$$z^n \xrightarrow{H} z^n H(z),$$

where $H(z)$ is the LTID system's transfer function.

Solution 3.3-6

The Kronecker delta function $\delta[n]$ serves a similar role in the study of DT systems as the Dirac delta function $\delta(t)$ does for CT systems. The two functions have similarities and differences.

The Kronecker delta function is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, we see that the Kronecker delta in nonzero only at time instant $n = 0$. Further, we see that a Kronecker delta has an accumulated sum (area) of 1, $\sum_{n=-\infty}^{\infty} \delta[n] = 1$. The Kronecker delta

function can be generated in practical DT systems, and it is used as an input to determine a DT system's impulse response $h[n]$.

The Dirac delta function is often defined with the sifting property as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau.$$

This definition requires that the Dirac delta function be nonzero only at time instant $t = 0$ and that the area of $\delta(t)$ is 1. The Dirac delta function is used as an input to determine a CT system's impulse response $h(t)$. These properties are very similar to the above properties of the Kronecker delta function. The Dirac delta function, however, has unbounded amplitude at $t = 0$, which makes it impossible to generate $\delta(t)$ in real-world situations.

Solution 3.3-7

(a) $e^{\lambda_a n} = e^{-0.5n} = (0.6065)^n = (\gamma_a)^n$

(b) $e^{\lambda_b n} = e^{0.5n} = (1.6487)^n = (\gamma_b)^n$

(c) $e^{\lambda_c n} = e^{-j\pi n} = (e^{-j\pi})^n = (-1)^n = (\gamma_c)^n$

(d) $e^{\lambda_d n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n = (\gamma_d)^n$

Figure S3.3-7 shows locations of λ and γ in each case. By plotting these signals, we observe that when $|\gamma| < 1$ (λ in LHP), the signal decays exponentially. When $|\gamma| > 1$ (λ in RHP), the signal grows exponentially. When $|\gamma| = 1$ (λ on imaginary axis), the signal amplitude is constant.
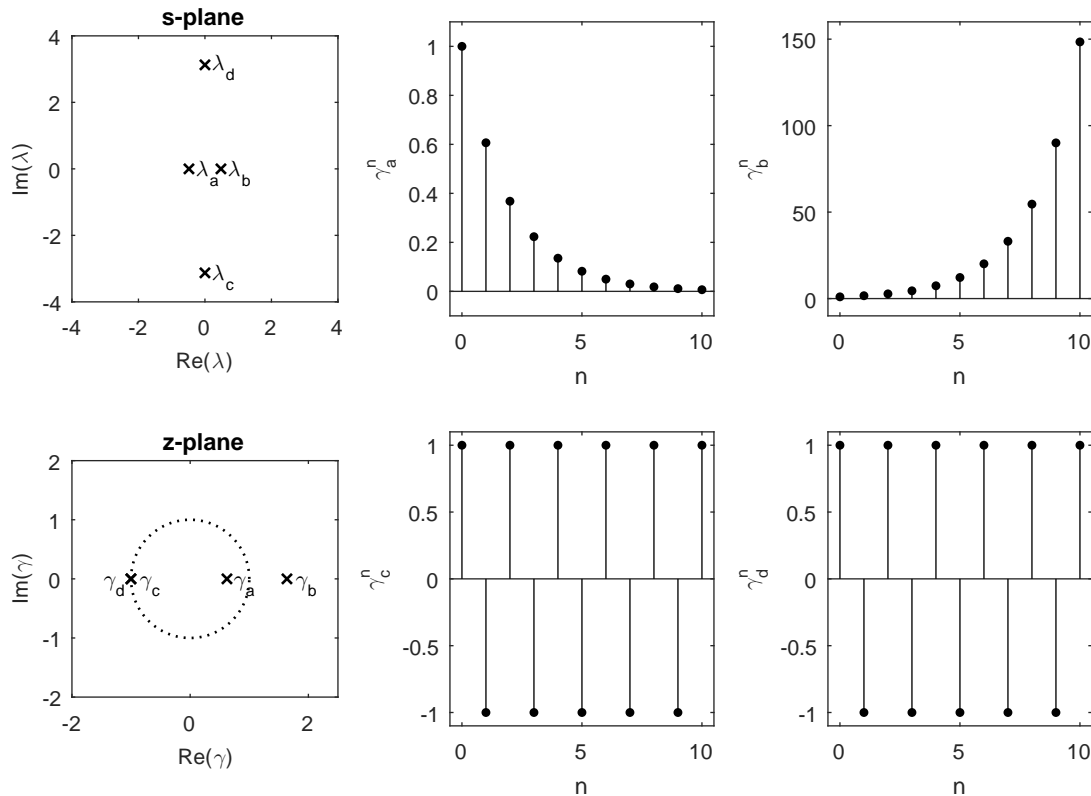

Figure S3.3-7

Solution 3.3-8

(a) $e^{-(1+j\pi)n} = (e^{-1}e^{-j\pi})^n = \left(-\frac{1}{e}\right)^n$

(b) $e^{-(1-j\pi)n} = (e^{-1}e^{j\pi})^n = \left(-\frac{1}{e}\right)^n$

(c) $e^{(1+j\pi)n} = (e\,e^{j\pi})^n = (-e)^n$

(d) $e^{(1-j\pi)n} = (e\,e^{-j\pi})^n = (-e)^n$

(e) $e^{-(1+j\frac{\pi}{3})n} = (e^{-1})^n\,e^{-j\frac{\pi}{3}n} = \left(\frac{1}{e}\right)^n[\cos\frac{\pi}{3}n - j\sin\frac{\pi}{3}n]$

(f) $e^{(1-j\frac{\pi}{3})n} = (e^1)^n\,e^{-j\frac{\pi}{3}n} = e^n[\cos\frac{\pi}{3}n - j\sin\frac{\pi}{3}n]$

Solution 3.3-9

In general, any signal can be decomposed into a sum of even and odd components as

$$x[n] = \underbrace{\frac{1}{2}\{x[n] + x[-n]\}}_{x_e[n]} + \underbrace{\frac{1}{2}\{x[n] - x[-n]\}}_{x_o[n]}.$$

(a)

$$u[n] = \underbrace{\frac{1}{2}\{u[n] + u[-n]\}}_{x_e[n]} + \underbrace{\frac{1}{2}\{u[n] - u[-n]\}}_{x_o[n]}$$
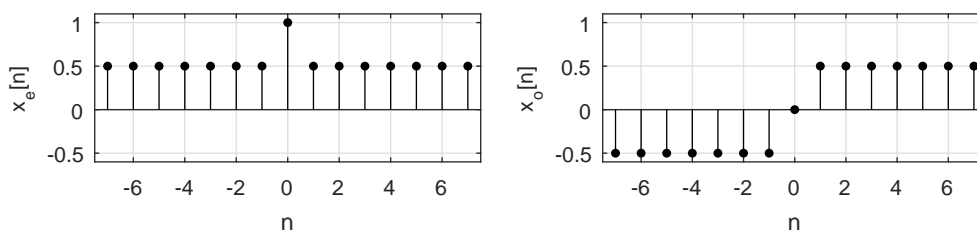

Figure S3.3-9a

(b)

$$nu[n] = \underbrace{\frac{1}{2}\{nu[n] - nu[-n]\}}_{x_e[n]} + \underbrace{\frac{1}{2}\{nu[n] + nu[-n]\}}_{x_o[n]}$$
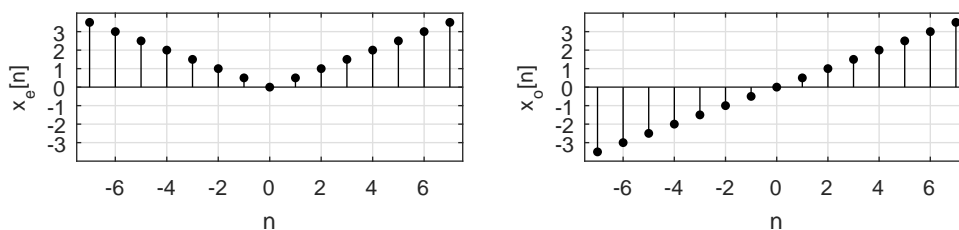

Figure S3.3-9b

(c) $\sin\left(\frac{\pi n}{4}\right)$ is an odd function, so its even component is zero.
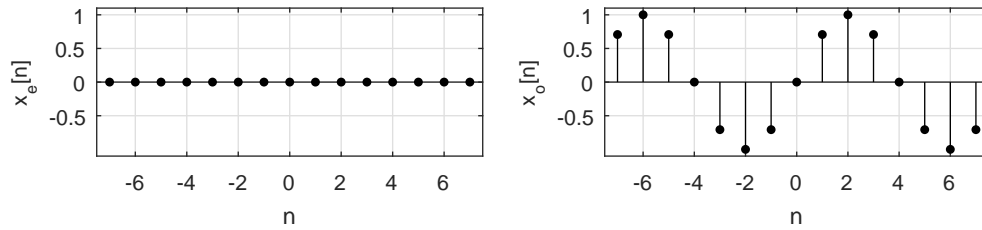
Figure S3.3-9c

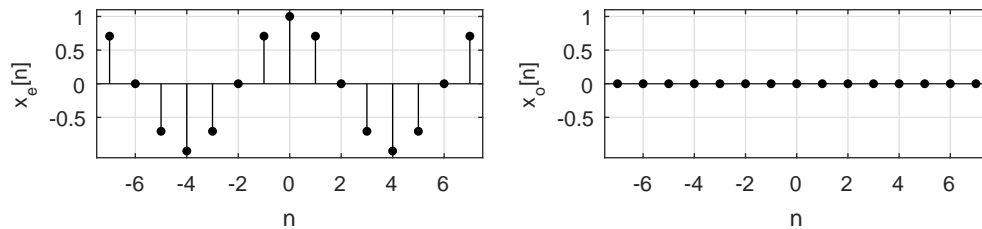**(d)** $\cos\left(\frac{\pi n}{4}\right)$ is an even function, so its odd component is zero.

Figure S3.3-9d

Solution 3.4-1

(a) Because $y[n] = y[n-1] + x[n]$, the standard-form difference equation is

$$y[n] - y[n-1] = x[n]$$

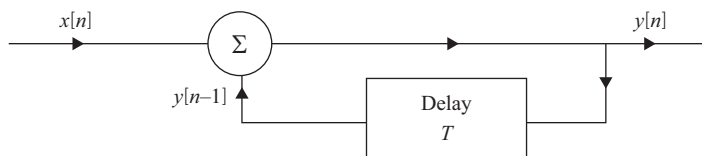**(b)** Realization of this equation is shown in Fig. S3.4-1.

Figure S3.4-1

Solution 3.4-2

The net growth rate of the native population is $3.3 - 1.3 = 2\%$ per year. Assuming the immigrants enter at a uniform rate throughout the year, their birth and death rate will be $(3.3/2)\%$ and $(1.3/2)\%$, respectively of the immigrants at the end of the year. The population $p[n]$ at the beginning of the kth year is $p[n-1]$ plus the net increase in the native population plus $i[n-1]$, the immigrants entering during $(n-1)$st year plus the net increase in the immigrant population for the year $(n-1)$.

$$p[n] = p[n-1] + \frac{3.3 - 1.3}{100}p[n-1] + i[n-1] + \frac{3.3 - 1.3}{2 \times 100}i[n-1]$$
$$= 1.02p[n-1] + 1.01i[n-1]$$

Thus,

$$p[n] - 1.02p[n-1] = 1.01i[n-1] \quad \text{or} \quad p[n+1] - 1.02p[n] = 1.01i[n].$$

Solution 3.4-3

(a)
$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]$$

(b) This system can be realized according to Fig. S3.4-3.

D represents unit delay

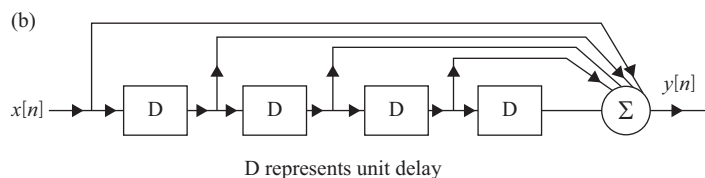Figure S3.4-3

Solution 3.4-4

The input $x[n] = u[n]$, which has a constant value of unity for all $n \geq 0$. Also $y[n] - y[n-1] = Tu[n]$. Hence the difference between two successive output values is always constant of value T. Clearly $y[n]$ must be a ramp with a possible constant component. Thus,

$$y[n] = (nT + c)u[n].$$

To find the value of unknown constant c, we let $n = 0$ and obtain

$$y[0] = c.$$

But from the input equation $y[n] - y[n-1] = Tu[n]$, we find $y[0] = T$ [remember that $y[-1] = 0$]. Hence,

$$y[n] = (n+1)Tu[n] \simeq nTu[n] \qquad \text{for} \quad T \to 0.$$

Solution 3.4-5

The differential equation is

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y(t) = x(t).$$

We use the notation $y[n]$ to represent $y(nT)$, $x[n]$ to represent $x(nT), \cdots$ etc. and assume that T is small enough so that the assumption $T \to 0$ may be made. We have

$$y(t) = y[n]$$

$$\frac{dy}{dt} \simeq \frac{y[n] - y[n-1]}{T}$$

$$\frac{d^2y}{dt^2} \simeq \frac{\frac{y[n]-y[n-1]}{T} - \frac{y[n-1]-y[n-2]}{T}}{T} = \frac{y[n] - 2y[n-1] + y[n-2]}{T^2}.$$

Substituting approximate difference expressions in the differential equation yields

$$\frac{y[n] - 2y[n-1] + y[n-2]}{T^2} + a_1\frac{y[n] - y[n-1]}{T} + a_0 y[n] = x[n].$$

Simplifying, we obtain

$$(1 + a_1 T + a_0 T^2)y[n] - (2 + a_1 T)y[n-1] + y[n-2] = T^2 x[n].$$

Solution 3.4-6

(a) We can determine $h[n]$ by direct substitution.

n	≤ -2	-1	0	1	2	3	≥ 4
$h[n]$	$x[\geq 3] = 0$	$x[1] = 1$	$x[-1] = 3$	$x[-3] = 5$	$x[-5] = 3$	$x[-7] = 1$	$x[\leq -9] = 0$

Thus,

$$h[n] = [1, \overset{\downarrow}{3}, 5, 3, 1].$$

(b) Since $h[n]$ is finite duration, the desired difference equation follows immediately as

$$y[n] = x[n+1] + 3x[n] + 5x[n-1] + 3x[n-2] + x[n-3].$$

(c) Since the system is described by a constant-coefficient linear difference equation, the system is necessarily linear and time-invariant. To show linearity, we assume $x_1[n] \longrightarrow y_1[n]$ and $x_2[n] \longrightarrow y_2[n]$. Next, determine the output to $x[n] = ax_1[n] + bx_2[n]$:

$$\begin{aligned}
y[n] &= x[n+1] + 3x[n] + 5x[n-1] + 3x[n-2] + x[n-3] \\
&= ax_1[n+1] + bx_2[n+1] + 3ax_1[n] + 3bx_2[n] + 5ax_1[n-1] + 5bx_2[n-1] + \\
&\quad 3ax_1[n-2] + 3bx_2[n-2] + ax_1[n-3] + bx_2[n-3] \\
&= a\left(x_1[n+1] + 3x_1[n] + 5x_1[n-1] + 3x_1[n-2] + x_1[n-3]\right) + \\
&\quad b\left(x_2[n+1] + 3x_2[n] + 5x_2[n-1] + 3x_2[n-2] + x_2[n-3]\right) \\
&= ay_1[n] + by_2[n].
\end{aligned}$$

Since a linear combination of inputs generates the corresponding linear combination of outputs, the system is linear.

To show time invariance, we note that delaying any input by some amount N causes a corresponding delay in the output. That is, if $x[n] \longrightarrow y[n]$, we see that $x_\mathrm{d}[n] = x[n - N]$ produces output

$$\begin{aligned}
y_\mathrm{d}[n] &= x_\mathrm{d}[n+1] + 3x_\mathrm{d}[n] + 5x_\mathrm{d}[n-1] + 3x_\mathrm{d}[n-2] + x_\mathrm{d}[n-3] \\
&= x[n - N + 1] + 3x[n - N] + 5x[n - N - 1] + 3x[n - N - 2] + x[n - N - 3] \\
&= y[n - N].
\end{aligned}$$

(d) Yes. The system is BIBO stable since $h[n]$ is absolutely summable,

$$\sum_{n=-\infty}^{\infty} |h[n]| = 1 + 3 + 5 + 3 + 1 = 13 < \infty.$$

(e) No. The system is not memoryless since $h[n] \neq 0$ for all $n \neq 0$.

(f) No. The system is not causal since $h[n] \neq 0$ for all $n < 0$. In particular, we see that $h[-1] = 1$.

Solution 3.4-7

(a) Since $h[n]$ is an involves upsampling $u[n]$ by a factor of 3, we expect $h[n]$ to possess a comb-like appearance where the unit values of $u[n]$ are interleaved with pairs of zeros. The first nonzero value occurs when $-(5 - n)/3 = 0$ or $n = 5$. Figure S3.4-7 shows the start of $h[n]$, which forever repeats repeat 1, 0, 0 starting at $n = 5$.
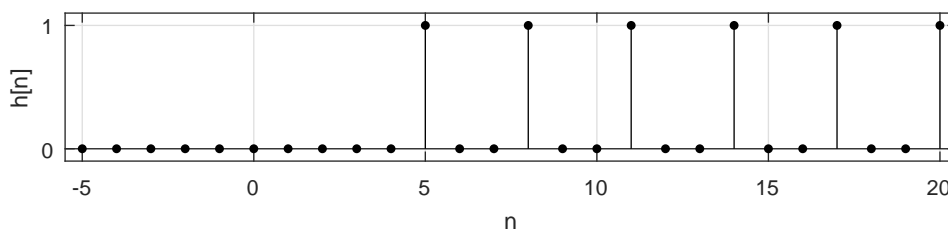
Figure S3.4-7

(b) To determine BIBO stability, we see whether $h[n]$ is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=5,8,11,\cdots} 1 = \infty \not< \infty.$$

Since $h[n]$ is not absolutely summable,

the system is not BIBO stable.

(c) No. The system is not memoryless since $h[n] \neq 0$ for all $n \neq 0$.

(d) Yes. The system is causal since $h[n] = 0$ for all $n < 0$.

Solution 3.4-8

The node equation at the kth node is $\quad i_1 + i_2 + i_3 = 0$, or

$$\frac{v[n-1] - v[n]}{R} + \frac{v[n+1] - v[n]}{R} - \frac{v[n]}{aR} = 0.$$

Therefore

$$a(v[n-1] + v[n+1] - 2v[n]) - v[n] = 0$$

or

$$v[n+1] - \left(2 + \frac{1}{a}\right)v[n] + v[n-1] = 0.$$

That is,

$$v[n+2] - \left(2 + \frac{1}{a}\right)v[n+1] + v[n] = 0.$$

Solution 3.4-9

(a) True; all finite power signals have infinite energy, and therefore cannot be energy signals. Energy signals and power signals are mutually exclusive.

(b) False; a signal with infinite energy need not be a power signal. For example, the signal $x[n] = 2^n u[n]$ has infinite energy and infinite power. Thus, it is neither an energy signal nor a power signal.

(c) True; the system is causal. Even though the input is scaled by $(n+1)$, the current output only depends on the current input. Another way to see this is to rewrite the expression as $y[n] = nx[n] + x[n]$.

(d) False; the system is not causal. The current output depends on a future input value. To help see this, substitute $n' = n - 1$ to yield $y[n'] = x[n' + 1]$; the output at time n' requires the future input value at time $n' + 1$.

(e) False; a signal $x[n]$ with energy E does not guarantee that signal $x[an]$ has energy $\frac{E}{|a|}$. Although this statement is true for continuous-time signals, it is not true for discrete-time signals. Remember, the discrete operation $x[an]$ results in a loss of information and thus a likely loss in energy. For example, consider $x[n] = \delta[n-1]$, which has energy $E = 1$. The signal $y[n] = x[2n] = 0$ has zero energy, not $E/2 = 1/2$.

Solution 3.4-10

Notice, $y_1[n] = -\delta[n] + \delta[n-1] + 2\delta[n-2]$. Furthermore, $x_2[n] = x_1[n-1] - 2x_1[n-2]$. Since the system is LTI,

$$y_2[n] = y_1[n-1] - 2y_1[n-2].$$

MATLAB is used to plot the result.

```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0);
>>  y1 = @(n) -delta(n)+delta(n-1)+2*delta(n-2);
>>  y2 = @(n) y1(n-1)-2*y1(n-2); n = -2:8;
>>  stem(n,y2(n),'k.'); axis([-2.5 8.5 -4.5 3.5]);
>>  grid on; xlabel('n'); ylabel('y_2[n]');
```
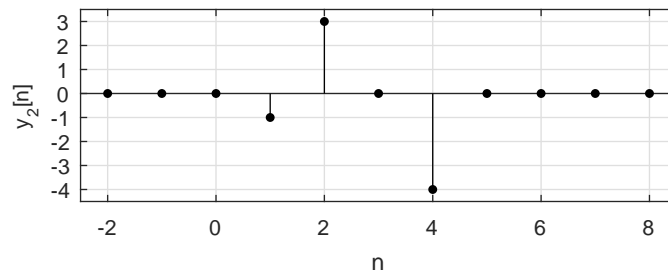

Figure S3.4-10

Solution 3.4-11

Using the sifting property, this system operation is rewritten as $y[n] = 0.5\,(x[n] + x[-n])$.

(a) This system extracts the even portion of the input.

(b) Yes, the system is BIBO stable. If the input is bounded, then the output is necessarily bounded. That is, if $|x[n]| \le M_x < \infty$, then $|y[n]| = |0.5\,(x[n] + x[-n])| \le 0.5\,(|x[n]| + |x[-n]|) \le M_x < \infty$.

(c) Yes, the system is linear. Let $y_1[n] = 0.5(x_1[n] + x_1[-n])$ and $y_2[n] = 0.5(x_2[n] + x_2[-n])$. Applying $ax_1[n] + bx_2[n]$ to the system yields $y[n] = 0.5\,(ax_1[n] + bx_2[n] + (ax_1[-n] + bx_2[-n])) = 0.5a(x_1[n] + x_1[-n]) + 0.5b(x_2[n] + x_2[-n]) = ay_1[n] + by_2[n]$.

(d) No, the system is not memoryless. For example, at time $n = 1$ the output $y[1] = 0.5(x[1] + x[-1])$ depends on a past value of the input, $x[-1]$.

(e) No, the system is not causal. For example, at time $n = -1$ the output $y[-1] = 0.5(x[-1] + x[1])$ depends on a future value of the input, $x[1]$.

(f) No, the system is not time-invariant. For example, let the input be $x_1[n] = u[n+10] - u[n-11]$. Since this input is already even, the output is just the input, $y_1[n] = x_1[n]$. Shifting by a non-zero integer N, the signal $x_2[n] = x_1[n-N]$ is not even, and the output is $y_2[n] \ne y_1[n-N] = x_1[n-N]$. Thus, the system cannot be time-invariant.

Solution 3.4-12

It is convenient to substitute $n' = n+1$ and rewrite the system expression as $y[n'] = x[n'-1]/x[n']$.

(a) No, the system is not BIBO stable. Input values of zero can result in unbounded outputs. For example, at $n' = 0$ the bounded input $x[n'] = \delta[n']$ yields an unbounded output $y[1] = 1/0 = \infty$.

(b) No, the system is not memoryless. The current output relies on a past input. For example, at $n' = 0$, the output $y[n']$ requires both the current input $x[n']$ and a stored past input $x[n'-1]$.

(c) Yes, the system is causal. The current output $y[n']$ does not depend on any future value of the input.

Solution 3.4-13

The operation $y(t) = x(2t)$ is a one-to-one mapping, where no information is lost. Any one-to-one mapping is invertible. In this case, $x(t)$ is recovered by taking $y(t/2)$.

Since every other sample of $x[n]$ is removed in the operation $y[n] = x[2n]$, one half of $x[n]$ is lost and the process is not invertible. Thought of another way, the operation $y[n] = x[2n]$ is not a one-to-one mapping; many different signals $x[n]$ map to the same signal $y[n]$, which makes inversion impossible.

Solution 3.4-14

Notice that $y_1[n]$ is obtained by multiplying $x[n]$ by the repeating sequence $\left\{\sin(\frac{\pi}{2}n+1)\right\} = \{\ldots, 0.5403, -0.8415, -0.5403, 0.8415, \ldots\}$. In this particular case, $x[n]$ is recovered by simply multiplying $y[n]$ by the inverse repeating sequence $\left\{\frac{1}{\sin(\frac{\pi}{2}n+1)}\right\} = \{\ldots, 1.8508, -1.1884, -1.8508, 1.1884, \ldots\}$.

In the case second, however, $y_2[n]$ is obtained by multiplying $x[n]$ by the repeating sequence $\left\{\sin(\frac{\pi}{2}(n+1))\right\} = \{\ldots, 0, -1, 0, 1, \ldots\}$. Since the sequence includes zeros, information is lost and the original sequence cannot be recovered.

Solution 3.4-15

Using the definition of the ramp function, the system expression is rewritten as $y[n] = nx[n]u[n]$.

(a) No, the system is not BIBO stable. For example, if the input is a unit step $x[n] = u[n]$, then the output is a ramp function $y[n] = r[n]$, which grows unbounded with time.

(b) Yes, the system is linear. Let $y_1[n] = nx_1[n]u[n]$ and $y_2[n] = nx_2[n]u[n]$. Applying $ax_1[n] + bx_2[n]$ to the system yields $y[n] = n\left(ax_1[n] + bx_2[n]\right)u[n] = anx_1[n]u[n] + bnx_2[n]u[n] = ay_1[n] + by_2[n]$.

(c) Yes, the system is memoryless. The current output only depends on the current input multiplied by a known (time-varying) scale factor.

(d) Yes, the system is causal. All memoryless systems are causal. The output does not depend on future values of the input or output.

(e) No, the system is not time-invariant. For example, applying $x_1[n] = \delta[n]$ yields the output $y_1[n] = n\delta[n]u[n] = 0$. Applying $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1]u[n] = \delta[n-1]$. Note, $x_2[n] = x_1[n-1]$ but $y_2[n] \neq y_1[n-1]$. Shifting the input does not produce a corresponding shift in the output.

Solution 3.4-16

(a) Position measurements $x[n]$ are in meters. The difference $x[n] - x[n-1]$ is the change in position, in meters, per frame of film. Since the camera operates at 60 frames per second, dimensional analysis requires $k = \frac{60}{\text{seconds}}$.

(b) Since $v[n] = k(x[n] - x[n-1])$ is an estimate of the velocity $v(t) = \frac{d}{dt}x(t)$, it is sensible to use $a[n] = k(v[n] - v[n-1])$ as an estimate of the acceleration $a(t) = \frac{d}{dt}v(t)$. Combining estimates yields $a[n] = k(k(x[n] - x[n-1]) - k(x[n-1] - x[n-2]))$ or

$$a[n] = k^2\left(x[n] - 2x[n-1] + x[n-2]\right) = 3600\left(x[n] - 2x[n-1] + x[n-2]\right).$$

This estimate of acceleration has two primary advantages. First, it is simple to calculate. Second, it is a causal, stable, LTI system and therefore enjoys the properties of such systems.

There are several shortcomings of the estimate as well. Of particular significance, the estimate $a[n]$ lags the actual acceleration $a(t)$. One way to see this is that the estimate $a[n]$ depends only on current and past values. A more balanced estimate of $a(t)$ is a shifted version $a_2[n] = a[n+1] = k^2\left(x[n+1] - 2x[n] + x[n-1]\right)$. While this may fix the problem of lag, the new system is no longer causal. Both cases estimate derivatives using first-order differences; there are more sophisticated (and more complex) methods to more accurately estimate derivatives.

By substituting $\delta[n]$ for $x[n]$, the impulse response is

$$h[n] = k^2\left(\delta[n] - 2\delta[n-1] + \delta[n-2]\right).$$

Solution 3.5-1

(a) By inspection, the standard advance operator form is

$$(E + 1)y[n] = \frac{1}{2}x[n].$$

(b) Substituting $\delta[n]$ for $x[n]$ and $h[n]$ for $y[n]$, we see that

$$h[n] = -h[n-1] + \frac{1}{2}\delta[n-1].$$

Since the system is causal, we know $h[n] = 0$ for all $n < 0$. We use recursion to determine the first five ($0 \le n \le 4$) values of $h[n]$.

$$h[0] = -h[-1] + \frac{1}{2}\delta[-1] = 0$$

$$h[1] = -h[0] + \frac{1}{2}\delta[0] = \frac{1}{2}$$

$$h[2] = -h[1] + \frac{1}{2}\delta[1] = -\frac{1}{2}$$

$$h[3] = -h[2] + \frac{1}{2}\delta[2] = \frac{1}{2}$$

$$h[4] = -h[3] + \frac{1}{2}\delta[3] = -\frac{1}{2}$$

Taken altogether, we see that

$$h[n] = -\frac{1}{2}(-1)^n u[n-1].$$

(c) Written in delay form for recursion, the system output is expressed as

$$y[n] = -y[n-1] + \frac{1}{2}x[n-1].$$

We use recursion to determine zero-state response $y_{zsr}[n]$ to input $x[n] = 2u[n]$ for $0 \le n \le 4$.

$$y_{zsr}[0] = -y_{zsr}[-1] + \frac{1}{2}2u[-1] = 0$$

$$y_{zsr}[1] = -y_{zsr}[0] + \frac{1}{2}2u[0] = 1$$

$$y_{zsr}[2] = -y_{zsr}[1] + \frac{1}{2}2u[1] = 0$$

$$y_{zsr}[3] = -y_{zsr}[2] + \frac{1}{2}2u[2] = 1$$

$$y_{zsr}[4] = -y_{zsr}[3] + \frac{1}{2}2u[3] = 0$$

(d) Setting $x[n] = 0$ to the recursion expression in part (c), the system (zero-input) response is expressed as

$$y[n] = -y[n-1].$$

We use recursion and initial condition $y_{zir}[-1] = 1$ to determine zero-input response $y_{zir}[n]$ for $0 \le n \le 4$.

$$y_{zir}[0] = -y_{zir}[-1] = -1$$
$$y_{zir}[1] = -y_{zir}[0] = 1$$
$$y_{zir}[2] = -y_{zir}[1] = -1$$
$$y_{zir}[3] = -y_{zir}[2] = 1$$
$$y_{zir}[4] = -y_{zir}[3] = -1$$

Solution 3.5-2

(a) Here,
$$y[n+1] = 0.5y[n]$$

Setting $n = -1$ and substituting $y[-1] = 10$ yield

$$y[0] = 0.5(10) = 5.$$

Setting $n = 0$ and substituting $y[0] = 5$ yield

$$y[1] = 0.5(5) = 2.5.$$

Setting $n = 1$ and substituting $y[1] = 2.5$ yield

$$y[2] = 0.5(2.5) = 1.25.$$

(b) In this case,
$$y[n+1] = -2y[n] + x[n+1].$$

Setting $n = -1$ and substituting $y[-1] = 0$ and $x[0] = 1$ yield

$$y[0] = 0 + 1 = 1.$$

Setting $n = 0$ and substituting $y[0] = 1$ and $x[1] = \frac{1}{e}$ yield

$$y[1] = -2(1) + \frac{1}{e} = -2 + \frac{1}{e} = -1.632$$

Setting $n = 1$ and substituting $y[1] = -2 + \frac{1}{e}$ and $x[2] = \frac{1}{e^2}$ yield

$$y[2] = -2(-2 + \frac{1}{e}) + \frac{1}{e^2} = 4 - \frac{2}{e} + \frac{1}{e^2} = 3.399.$$

Solution 3.5-3

Here,

$$y[n] = 0.6y[n-1] + 0.16[n-2].$$

Setting $n = 0$ and substituting $y[-1] = -25$ and $y[-2] = 0$ yield

$$y[0] = 0.6(-25) + 0.16(0) = -15.$$

Setting $n = 1$ and substituting $y[-1] = 0$ and $y[0] = -15$ yield

$$y[1] = 0.6(-15) + 0.16(-25) = -13.$$

Setting $n = 2$ and substituting $y[1] = -13$ and $y[0] = -15$ yield

$$y[2] = 0.6(-13) + 0.16(-15) = -10.2.$$

Solution 3.5-4

This equation can be expressed as

$$y[n+2] = -\frac{1}{4}y[n+1] - \frac{1}{16}y[n] + x[n+2].$$

Setting $n = -2$ and substituting $y[-1] = y[-2] = 0$ and $x[0] = 100$ yield

$$y[0] = -\frac{1}{4}(0) - \frac{1}{16}(0) + 100 = 100.$$

Setting $n = -1$ and substituting $y[-1] = 0$, $y[0] = 100$, and $x[1] = 100$ yield

$$y[1] = -\frac{1}{4}(100) - \frac{1}{16}(0) + 100 = 75.$$

Setting $n = 0$ and substituting $y[0] = 100$, $y[1] = 75$, and $x[2] = 100$ yield

$$y[2] = -\frac{1}{4}(75) - \frac{1}{16}(100) + 100 = 75.$$

Solution 3.5-5

Here,

$$y[n+2] = -3y[n+1] - 2y[n] + x[n+2] + 3x[n+1] + 3x[n].$$

Setting $n = -2$ and substituting $y[-1] = 3$, $y[-2] = 2$, $x[-1] = 0$, $x[-2] = 0$, and $x[0] = 1$ yield

$$y[0] = -3(3) - 2(2) + 1 + 3(0) + 3(0) = -12.$$

Setting $n = -1$ and substituting $y[0] = -12$, $y[-1] = 3$, $x[-1] = 0$, $x[0] = 1$, and $x[1] = 3$ yield

$$y[1] = -3(-12) - 2(3) + 3 + 3(1) + 3(0) = 36.$$

Proceeding along same lines, we obtain

$$y[2] = -3(36) - 2(-12) + 9 + 3(3) + 3(1) = -63.$$

Solution 3.5-6

Expressed in a form for recursion, the difference equation is

$$y[n] = -2y[n-1] - y[n-2] + 2x[n] - x[n-1].$$

Setting $n = 0$ and substituting $y[-1] = 2$, $y[-2] = 3$, $x[0] = 1$, and $x[-1] = 0$ yield

$$y[0] = -2(2) - 3 + 2(1) - 0 = -5.$$

Setting $n = 1$ and substituting $y[0] = -5$, $y[-1] = 2$, $x[0] = 1$, and $x[1] = \frac{1}{3}$ yield

$$y[1] = -2(-5) - (2) + 2(\frac{1}{3}) - 1 = 7.667.$$

Setting $n = 2$ and substituting $y[1] = 7.667$, $y[0] = -5$, $x[1] = \frac{1}{3}$, and $x[2] = \frac{1}{9}$ yield

$$y[2] = -2(7.667) - (-5) + 2(\frac{1}{9}) - \frac{1}{3} = -10.444.$$

Solution 3.6-1

In this case, the characteristic equation is

$$Q(\gamma) = \gamma^2 + \tfrac{1}{6}\gamma - \tfrac{1}{6} = (\gamma + \tfrac{1}{2})(\gamma - \tfrac{1}{3}) = 0.$$

Thus, the characteristic roots are

$$\gamma_1 = -\frac{1}{2} \quad \text{and} \quad \gamma_2 = \frac{1}{3}.$$

The form of the zero-input response is

$$y_0[n] = c_1(-\tfrac{1}{2})^n + c_2(\tfrac{1}{3})^n.$$

Using the initial conditions, we see that

$$\left. \begin{array}{l} y_0[-1] = 3 = -2c_1 + 3c_2 \\ y_0[-2] = -1 = 4c_1 + 9c_2 \end{array} \right\} \implies \begin{array}{l} c_1 = -1 \\ c_2 = \frac{1}{3} \end{array}.$$

Thus,

$$y_0[n] = -(-\tfrac{1}{2})^n + \tfrac{1}{3}(\tfrac{1}{3})^n = (\tfrac{1}{3})^{n+1} - (-\tfrac{1}{2})^n.$$

Solution 3.6-2

Here,

$$(E^2 + 3E + 2)y[n] = 0.$$

The characteristic equation is $\gamma^2 + 3\gamma + 2 = (\gamma + 1)(\gamma + 2) = 0$. Therefore,

$$y[n] = c_1(-1)^n + c_2(-2)^n.$$

Setting $n = -1$ and -2 and substituting initial conditions yields

$$\left. \begin{array}{l} 0 = -c_1 - \frac{1}{2}c_2 \\ 1 = c_1 + \frac{1}{4}c_2 \end{array} \right\} \implies \begin{array}{l} c_1 = 2 \\ c_2 = -4 \end{array}.$$

Thus,

$$y[n] = 2(-1)^n - 4(-2)^n \qquad n \geq 0.$$

Solution 3.6-3

Here,

$$(E^2 + 2E + 1)y[n] = 0.$$

The characteristic equation is $\gamma^2 + 2\gamma + 1 = (\gamma + 1)^2 = 0$. Therefore,

$$y[n] = (c_1 + c_2 n)(-1)^n.$$

Setting $n = -1$ and -2 and substituting initial conditions yields

$$\left. \begin{array}{l} 1 = -c_1 + c_2 \\ 1 = c_1 - 2c_2 \end{array} \right\} \implies \begin{array}{l} c_1 = -3 \\ c_2 = -2 \end{array}.$$

Thus,

$$y[n] = -(3 + 2n)(-1)^n \qquad n \geq 0.$$

Solution 3.6-4

For this second-order system,

$$(E^2 - 2E + 2)y[n] = 0.$$

The characteristic equation is $\gamma^2 - 2\gamma + 2 = (\gamma - 1 - j1)(\gamma - 1 + j1) = 0$. The characteristics roots are $1 \pm j1 = \sqrt{2}e^{\pm j\pi/4}$. Therefore,

$$y[n] = c(\sqrt{2})^n \cos(\frac{\pi}{4}n + \theta).$$

Setting $n = -1$ and -2 and substituting initial conditions yields

$$1 = \frac{c}{\sqrt{2}}\cos(-\frac{\pi}{4} + \theta) = \frac{c}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$0 = \frac{c}{2}\cos(-\frac{\pi}{2} + \theta) = \frac{c}{2}\sin\theta$$

Solution of these two simultaneous equations yields

$$\left. \begin{array}{l} c\cos\theta = 2 \\ c\sin\theta = 0 \end{array} \right\} \implies \begin{array}{l} c = 2 \\ \theta = 0 \end{array}.$$

Thus,

$$y[n] = 2(\sqrt{2})^n \cos(\frac{\pi}{4}n) \qquad n \geq 0.$$

Solution 3.6-5

The equation can be expressed in terms of advance operation notation as

$$E^N y[n] = \left(b_0 E^N + b_1 E^{N-1} + \cdots + b_N\right) x[n].$$

The characteristic equation is

$$\gamma^N = 0.$$

Hence, all the N characteristic roots are zero. Therefore, the zero-input component is zero and the total response in given by the zero-state component.

Solution 3.6-6

(a) By definition, any element in the Fibonacci sequence is the sum of the previous two. Thus, $f[n] = f[n-1] + f[n-2]$. Written in standard form, this yields

$$f[n] - f[n-1] - f[n-2] = 0.$$

This is a somewhat unusual system in the fact that it has no input. In the lingo of signals and systems, $f[n]$ is a zero-input response that is completely driven by the auxiliary conditions.

(b) The characteristic equation is $\gamma^2 - \gamma - 1 = 0$. This yields two characteristic roots

$$\gamma_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \gamma_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Since one characteristic root is in the right-half plane, the system is not stable.

(c) To determine a particular Fibonacci number, it is convenient to determine a closed form expression for $f[n]$. Since $f[n]$ is a zero-input response, it has form $f[n] = c_1\gamma_1^n + c_2\gamma_2^n$. The auxiliary equations yield $f[1] = 0 = c_1\gamma_1 + c_2\gamma_2$ and $f[2] = 1 = c_1\gamma_1^2 c_2\gamma_2^2$. Solving yields $c_1 = \frac{-\gamma_2}{\gamma_1\gamma_2(\gamma_2-\gamma_1)} = \frac{\sqrt{5}-1}{2\sqrt{5}} \approx 0.2764$ and $c_2 = \frac{\gamma_1}{\gamma_1\gamma_2(\gamma_2-\gamma_1)} = \frac{\sqrt{5}+1}{2\sqrt{5}} \approx 0.7236$.

MATLAB is used to solve for the requested values of $f[n]$.

```
>>  gamma1 = (1+sqrt(5))/2; gamma2 = (1-sqrt(5))/2;
>>  c1 = -gamma2/(gamma1*gamma2*(gamma2-gamma1));
>>  c2 = gamma1/(gamma1*gamma2*(gamma2-gamma1));
>>  f = @(n) c1*gamma1.^(n)+c2*gamma2.^(n);
>>  f(50)
    ans = 7.7787e+009
>>  f(1000)
    ans = 2.6864e+208
```

Thus,

$$f[50] = \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{50} + \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{50} \approx 7.7787(10^9)$$

and

$$f[1000] = \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{1000} + \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{1000} \approx 2.6864(10^{208}).$$

Solution 3.6-7

For this problem,

$$v(n+2) - 2.5v(n+1) + v(n) = 0 \quad \text{or} \quad (E^2 - 2.5E + 1)v[n] = 0.$$

The auxiliary conditions are $v(0) = 100$ and $v(N) = 0$. The characteristic equation is $\gamma^2 - 2.5\gamma + 1 = (\gamma - 0.5)(\gamma - 2) = 0$. Therefore,

$$v(n) = c_1(0.5)^n + c_2(2)^n.$$

Setting $n = 0$ and N, and substituting $v(0) = 100$ and $v(N) = 0$ yield

$$\left.\begin{array}{r} 100 = c_1 + c_2 \\ 0 = c_1(0.5)^N + c_2(2)^N \end{array}\right\} \implies \begin{array}{l} c_1 = \frac{100(2)^N}{2^N - (0.5)^N} \\ c_2 = \frac{100(0.5)^N}{(0.5)^N - 2^N} \end{array}.$$

Thus,

$$v[n] = \frac{100}{2^N - (0.5)^N}[2^N(0.5)^n - (0.5)^N(2)^n] \qquad n = 0, 1, \cdots, N.$$

Solution 3.6-8

Since we are looking for the zero-input response, the term $\sqrt{3}x[n-8]$ is irrelevant and the equation becomes $y_0[n] + y_0[n-1] + 0.25y_0[n-2] = 0$. The characteristic equation for this second-order system is $\gamma^2 + \gamma + 0.25 = 0$. This yields a repeated root at $\gamma = -0.5$, and the zero-input response has form $y_0[n] = c_1(-0.5)^n + c_2 n(-0.5)^n$. The pair of equations $y_0[-1] = 1 = -2c_1 + 2c_2$ and $y_0[1] = 1 = -c_1/2 - c_2/2$ are solved using MATLAB.

```
>>   c = [-2 2;-1/2 -1/2]\[1;1]
     c = -1.2500
         -0.7500
```

Thus,

$$y_0[n] = -1.25(-0.5)^n - 0.75n(-0.5)^n.$$

Solution 3.6-9

Here, we consider just one of the infinite possible solutions to this problem. A third-order system needs three characteristic roots. For a marginally stable LTID system, there must be at least one unrepeated root on the unit circle with any remaining roots inside the unit circle. For a stable LTIC system, all roots need to be in the left half-plane. These conditions can be met, for example, by selecting the roots -1, $-\frac{1}{2}$ and $-\frac{1}{4}$. Using these values,

$$Q(x) = (x+1)(x+\tfrac{1}{2})(x+\tfrac{1}{4}) = (x+1)(x^2 + \tfrac{3}{4}x + \tfrac{1}{8}) = x^3 + \tfrac{3}{4}x^2 3 + \tfrac{1}{8}x + x^2 + \tfrac{3}{4}x + \tfrac{1}{8}.$$

Simplifying into standard form, we see that a possible solution is

$$Q(x) = x^3 + \frac{7}{4}x^2 + \frac{7}{8}x + \frac{1}{8}.$$

Solution 3.7-1

(a) Here,
$$(E+2)y[n] = x[n].$$

The characteristic equation is $\gamma + 2 = 0$, and the characteristic root is -2. Also $a_1 = 2$ and $b_1 = 1$. Therefore,
$$h[n] = \frac{1}{2}\delta[n] + c(-2)^n.$$

We need one value of $h[n]$ to determine c. This is determined by iterative solution of

$$(E+2)h[n] = \delta[n] \quad \text{or} \quad h[n+1] + 2h[n] = \delta[n].$$

Setting $n = -1$ and substituting $h[-1] = \delta[-1] = 0$ yield

$$h[0] = 0.$$

Setting $n = 0$ and using $h[0] = 0$ yield

$$0 = \frac{1}{2} + c \quad \Longrightarrow \quad c = -\frac{1}{2}.$$

Therefore,

$$h[n] = \frac{1}{2}\delta[n] - \frac{1}{2}(-2)^n u[n].$$

(b) Here, $b_1 = 0$, $a_1 = 2$, and the characteristic root is -2. Therefore,

$$h[n] = c(-2)^n.$$

We need one value of $h[n]$ to determine c. This is done by solving iteratively

$$h[n+1] + 2h[n] = \delta[n+1].$$

Setting $n = -1$ and substituting $h[-1] = 0$ and $\delta[0] = 1$ yield

$$h[0] = 1.$$

Setting $n = 0$ and using $h[0] = 0$ yield

$$1 = c.$$

Consequently,

$$h[n] = (-2)^n u[n].$$

Solution 3.7-2

(a) The characteristic equation is $\gamma^2 + 1 = 0$, and the characteristic roots are $\gamma_1 = j$ and $\gamma_2 = -j$. The form of the impulse response is

$$h[n] = \frac{\frac{1}{2}}{1}\delta[n] + \left[c_1(j)^n + c_2(-j)^n\right] u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = -h[n-2] + \delta[n-1] + \frac{1}{2}\delta[n-2].$$

Thus, $h[0] = -h[-2] + \delta[-1] + \frac{1}{2}\delta[-2] = 0$ and $h[1] = -h[-1] + \delta[0] + \frac{1}{2}\delta[-1] = 1$. Using these values, we see that

$$h[0] = 0 = \frac{1}{2} + c_1 + c_2$$
$$h[1] = 1 = 0 + jc_1 - jc_2.$$

We use MATLAB to find c_1 and c_2.

```
>>  c = inv([1 1;1j -1j])*[-0.5;1]
    c = -0.2500 - 0.5000i
        -0.2500 + 0.5000i
```

Thus,

$$h[n] = \tfrac{1}{2}\delta[n] + \left[(-\tfrac{1}{4} - \tfrac{j}{2})(j)^n + (-\tfrac{1}{4} + \tfrac{j}{2})(-j)^n\right] u[n].$$

This can be simplified to

$$h[n] = \tfrac{1}{2}\delta[n] + \left[-\tfrac{1}{2}\cos(\pi n/2) + \sin(\pi n/2)\right] u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = -1$. We can verify this value by continuing our previous recursion, $h[2] = -h[0] + \delta[1] + \frac{1}{2}\delta[0] = \frac{1}{2}$ and $h[3] = -h[1] + \delta[2] + \frac{1}{2}\delta[1] = -1$. The recursion and the closed-form expressions match at $n = 3$, as expected.

(b) The characteristic equation is $\gamma^2 - \gamma + \frac{1}{4} = (\gamma - \frac{1}{2})^2 = 0$, and the characteristic roots are both $\gamma = \frac{1}{2}$. The form of the impulse response is

$$h[n] = \left[c_1 (\tfrac{1}{2})^n + c_2 n (\tfrac{1}{2})^n \right] u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = h[n-1] - \frac{1}{4}h[n-2] + \delta[n].$$

Thus, $h[0] = h[-1] - \frac{1}{4}h[-2] + \delta[0] = 1$ and $h[1] = h[0] - \frac{1}{4}h[-1] + \delta[1] = 1$. Using these values, we see that

$$\left. \begin{array}{l} h[0] = 1 = c_1 + 0c_2 \\ h[1] = 1 = \frac{1}{2}c_1 + \frac{1}{2}c_2 \end{array} \right\} \implies \begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array}.$$

Thus,

$$h[n] = (1+n)(\tfrac{1}{2})^n u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = \frac{1}{2}$. We can verify this value by continuing our previous recursion, $h[2] = h[1] - \frac{1}{4}h[0] + \delta[2] = \frac{3}{4}$ and $h[3] = h[2] - \frac{1}{4}h[1] + \delta[3] = \frac{1}{2}$. The recursion and the closed-form expressions match at $n = 3$ (and elsewhere), as expected.

(c) The characteristic equation is $\gamma^2 - \frac{1}{6}\gamma - \frac{1}{6} = (\gamma - \frac{1}{2})(\gamma + \frac{1}{3}) = 0$, and the characteristic roots are $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = -\frac{1}{3}$. The form of the impulse response is

$$h[n] = \frac{\frac{1}{3}}{-\frac{1}{6}}\delta[n] + \left[c_1 (\tfrac{1}{2})^n + c_2 (-\tfrac{1}{3})^n \right] u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = \frac{1}{6}h[n-1] + \frac{1}{6}h[n-2] + \frac{1}{3}\delta[n-2].$$

Thus, $h[0] = \frac{1}{6}h[-1] + \frac{1}{6}h[-2] + \frac{1}{3}\delta[-2] = 0$ and $h[1] = \frac{1}{6}h[0] + \frac{1}{6}h[-1] + \frac{1}{3}\delta[-1] = 0$. Using these values, we see that

$$h[0] = 0 = -2 + c_1 + c_2$$
$$h[1] = 0 = 0 + \frac{1}{2}c_1 - \frac{1}{3}c_2.$$

We use MATLAB to find c_1 and c_2.

```
>>   format rat; c = inv([1 1;1/2 -1/3])*[2;0]
     c =   4/5
           6/5
```

Thus,

$$h[n] = -2\delta[n] + \left[\tfrac{4}{5}(\tfrac{1}{2})^n + \tfrac{6}{5}(-\tfrac{1}{3})^n \right] u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = \frac{1}{18}$. We can verify this value by continuing our previous recursion, $h[2] = \frac{1}{6}h[1] + \frac{1}{6}h[0] + \frac{1}{3}\delta[0] = \frac{1}{3}$ and $h[3] = \frac{1}{6}h[2] + \frac{1}{6}h[1] + \frac{1}{3}\delta[1] = \frac{1}{18}$. The recursion and the closed-form expressions match at $n = 3$ (and elsewhere).

(d) The characteristic equation is $\gamma^2 + \frac{1}{6}\gamma - \frac{1}{6} = (\gamma + \frac{1}{2})(\gamma - \frac{1}{3}) = 0$, and the characteristic roots are $\gamma_1 = -\frac{1}{2}$ and $\gamma_2 = \frac{1}{3}$. The form of the impulse response is

$$h[n] = \left[c_1 (-\tfrac{1}{2})^n + c_2 (\tfrac{1}{3})^n \right] u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = -\frac{1}{6}h[n-1] + \frac{1}{6}h[n-2] + \frac{1}{3}\delta[n].$$

Thus, $h[0] = -\frac{1}{6}h[-1] + \frac{1}{6}h[-2] + \frac{1}{3}\delta[0] = \frac{1}{3}$ and $h[1] = -\frac{1}{6}h[0] + \frac{1}{6}h[-1] + \frac{1}{3}\delta[1] = -\frac{1}{18}$. Using these values, we see that

$$h[0] = \frac{1}{3} = c_1 + c_2$$

$$h[1] = -\frac{1}{18} = -\frac{1}{2}c_1 + \frac{1}{3}c_2.$$

We use MATLAB to find c_1 and c_2.

```
>>  format rat; c = inv([1 1;-1/2 1/3])*[1/3;-1/18]
    c =  1/5
         2/15
```

Thus,

$$h[n] = \left[\tfrac{1}{5}(-\tfrac{1}{2})^n + \tfrac{2}{15}(\tfrac{1}{3})^n\right] u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = -\frac{13}{648}$. We can verify this value by continuing our previous recursion, $h[2] = -\frac{1}{6}h[1] + \frac{1}{6}h[0] + \frac{1}{3}\delta[2] = \frac{7}{108}$ and $h[3] = -\frac{1}{6}h[0] + \frac{1}{6}h[-1] + \frac{1}{3}\delta[1] = -\frac{13}{648}$. The recursion and the closed-form expressions match at $n = 3$ (and elsewhere).

(e) The characteristic equation is $\gamma^2 + \frac{1}{4} = 0$, and the characteristic roots are $\gamma_1 = \frac{j}{2}$ and $\gamma_2 = -\frac{j}{2}$. The form of the impulse response is

$$h[n] = \left[c_1(\tfrac{j}{2})^n + c_2(-\tfrac{j}{2})^n\right] u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = -\frac{1}{4}h[n-2] + \delta[n].$$

Thus, $h[0] = -\frac{1}{4}h[-2] + \delta[0] = 1$ and $h[1] = -\frac{1}{4}h[-1] + \delta[1] = 0$. Using these values, we see that

$$h[0] = 1 = c_1 + c_2$$

$$h[1] = 0 = \frac{j}{2}c_1 - \frac{j}{2}c_2.$$

We use MATLAB to find c_1 and c_2.

```
>>  format rat; c = inv([1 1;1j/2 -1j/2])*[1;0]
    c = 1/2
        1/2
```

Thus,

$$h[n] = \left[\tfrac{1}{2}(\tfrac{j}{2})^n + \tfrac{1}{2}(-\tfrac{j}{2})^n\right] u[n].$$

This can be simplified to

$$h[n] = (\tfrac{1}{2})^n \cos(\tfrac{\pi n}{2})u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = 0$. We can verify this value by continuing our previous recursion, $h[2] = -\frac{1}{4}h[0] + \delta[2] = -\frac{1}{4}$ and $h[3] = -\frac{1}{4}h[1] + \delta[3] = 0$. The recursion and the closed-form expressions match at $n = 3$, as expected.

(f) The characteristic equation is $\gamma^2 - \frac{4}{9} = (\gamma - \frac{2}{3})(\gamma + \frac{2}{3}) = 0$, and the characteristic roots are $\gamma_1 = \frac{2}{3}$ and $\gamma_2 = -\frac{2}{3}$. The form of the impulse response is

$$h[n] = \frac{1}{-\frac{4}{9}}\delta[n] + \left[c_1(\tfrac{2}{3})^n + c_2(-\tfrac{2}{3})^n\right]u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = \frac{4}{9}h[n-2] + \delta[n] + \delta[n-2].$$

Thus, $h[0] = \frac{4}{9}h[-2] + \delta[0] + \delta[-2] = 1$ and $h[1] = \frac{4}{9}h[-1] + \delta[1] + \delta[-1] = 0$. Using these values, we see that

$$h[0] = 1 = -\frac{9}{4} + c_1 + c_2$$

$$h[1] = 0 = \frac{2}{3}c_1 - \frac{2}{3}c_2.$$

We use MATLAB to find c_1 and c_2.

```
>>  format rat; c = inv([1 1;2/3 -2/3])*[13/4;0]
    c =   13/8
          13/8
```

Thus,

$$h[n] = -\frac{9}{4}\delta[n] + \left[\frac{13}{8}(\tfrac{2}{3})^n + \frac{13}{8}(-\tfrac{2}{3})^n\right]u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = 0$. We can verify this value by continuing our previous recursion, $h[2] = \frac{4}{9}h[0] + \delta[2] + \delta[0] = \frac{13}{9}$ and $h[3] = \frac{4}{9}h[1] + \delta[3] + \delta[1] = 0$. The recursion and the closed-form expressions match at $n = 3$ (and elsewhere).

(g) In this problem, we have

$$(E^2 - \frac{1}{4})(E + \frac{1}{2})\{y[n]\} = E^3\{x[n]\}$$

or

$$(E^3 + \frac{1}{2}E^2 - \frac{1}{4}E - \frac{1}{8})\{y[n]\} = E^3\{x[n]\}.$$

The characteristic equation is $(\gamma^2 - \frac{1}{4})(\gamma + \frac{1}{2}) = (\gamma - \frac{1}{2})(\gamma + \frac{1}{2})^2 = 0$, and the characteristic roots are $\gamma_1 = \frac{1}{2}$, $\gamma_2 = -\frac{1}{2}$, and $\gamma_3 = -\frac{1}{2}$. The form of the impulse response is

$$h[n] = \left[c_1(\tfrac{1}{2})^n + c_2(-\tfrac{1}{2})^n + c_3 n(-\tfrac{1}{2})^n\right]u[n].$$

To determine c_1, c_2, and c_3, we calculate $h[0]$, $h[1]$, and $h[2]$ by recursion using

$$h[n] = -\frac{1}{2}h[n-1] + \frac{1}{4}h[n-2] + \frac{1}{8}h[n-3] + \delta[n].$$

Thus, $h[0] = -\frac{1}{2}h[-1] + \frac{1}{4}h[-2] + \frac{1}{8}h[-3] + \delta[0] = 1$, $h[1] = -\frac{1}{2}h[0] + \frac{1}{4}h[-1] + \frac{1}{8}h[-2] + \delta[1] = -\frac{1}{2}$, and $h[2] = -\frac{1}{2}h[1] + \frac{1}{4}h[0] + \frac{1}{8}h[-1] + \delta[2] = \frac{1}{2}$. Using these values, we see that

$$h[0] = 1 = c_1 + c_2 + 0c_3$$

$$h[1] = -\frac{1}{2} = \frac{1}{2}c_1 - \frac{1}{2}c_2 - \frac{1}{2}c_3$$

$$h[2] = \frac{1}{2} = \frac{1}{4}c_1 + \frac{1}{4}c_2 + (\frac{1}{4})2c_3.$$

We use MATLAB to find c_1, c_2, and c_3.

```
>> format rat; c = inv([1 1 0;1/2 -1/2 -1/2;1/4 1/4 1/2])*[1;-1/2;1/2]
   c =   1/4
         3/4
         1/2
```

Thus,

$$h[n] = \left[\tfrac{1}{4}(\tfrac{1}{2})^n + \tfrac{3}{4}(-\tfrac{1}{2})^n + \tfrac{1}{2}n(-\tfrac{1}{2})^n\right] u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = -\tfrac{1}{4}$. We can verify this value by continuing our previous recursion, $h[3] = -\tfrac{1}{2}h[2] + \tfrac{1}{4}h[1] + \tfrac{1}{8}h[0] + \delta[3] = -\tfrac{1}{4}$. The recursion and the closed-form expressions match at $n = 3$ (and elsewhere).

(h) The characteristic equation is $\gamma^2 - \gamma + \tfrac{1}{4} = (\gamma - \tfrac{1}{2})^2 = 0$, and the characteristic roots are both $\gamma = \tfrac{1}{2}$. The form of the impulse response is

$$h[n] = \tfrac{1}{4}\delta[n] + \left[c_1(\tfrac{1}{2})^n + c_2 n(\tfrac{1}{2})^n\right] u[n].$$

To determine c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = h[n-1] - \frac{1}{4}h[n-2] + \delta[n-2].$$

Thus, $h[0] = h[-1] - \tfrac{1}{4}h[-2] + \delta[-2] = 0$ and $h[1] = h[0] - \tfrac{1}{4}h[-1] + \delta[-1] = 0$. Using these values, we see that

$$\left. \begin{array}{l} h[0] = 0 = 4 + c_1 + 0c_2 \\ h[1] = 0 = \tfrac{1}{2}c_1 + \tfrac{1}{2}c_2 \end{array} \right\} \implies \begin{array}{l} c_1 = -4 \\ c_2 = 4 \end{array}.$$

Thus,

$$h[n] = 4\delta[n] + \left[-4(\tfrac{1}{2})^n + 4n(\tfrac{1}{2})^n\right] u[n].$$

Evaluated at $n = 3$, this expression yields $h[3] = 1$. We can verify this value by continuing our previous recursion, $h[2] = h[1] - \tfrac{1}{4}h[0] + \delta[0] = 1$ and $h[3] = h[2] - \tfrac{1}{4}h[1] + \delta[1] = 1$. The recursion and the closed-form expressions match at $n = 3$ (and elsewhere), as expected.

Solution 3.7-3

In standard form, the system's difference equation is

$$y[n] + \frac{1}{4}y[n-2] = 2x[n] + 2x[n-1].$$

(a) By inspection, we see that the system is second order.

(b) The characteristic equation is $\gamma^2 + \tfrac{1}{4} = (\gamma - \tfrac{j}{2})(\gamma + \tfrac{j}{2}) = 0$. The characteristic roots are $\gamma_1 = \tfrac{j}{2}$ and $\gamma_2 = -\tfrac{j}{2}$. Thus,

$$\text{the characteristic modes are } (\tfrac{j}{2})^n \text{ and } (-\tfrac{j}{2})^n.$$

(c) The form of the impulse response is

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$
$$= \tfrac{0}{\frac{1}{4}}\delta[n] + \left[c_1(\tfrac{j}{2})^n + c_2(-\tfrac{j}{2})^n\right] u[n].$$

To find c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = -\tfrac{1}{4}h[n-2] + 2\delta[n] + 2\delta[n-1].$$

Thus, $h[0] = -\frac{1}{4}h[-2] + 2\delta[0] + 2\delta[-1] = 2$ and $h[1] = -\frac{1}{4}h[-1] + 2\delta[1] + 2\delta[0] = 2$. Using these values, we see that

$$\left.\begin{array}{l} h[0] = 2 = c_1 + c_2 \\ h[1] = 2 = \frac{j}{2}c_1 - \frac{j}{2}c_2 \end{array}\right\} \implies \begin{array}{l} c_1 = 1 - 2j \\ c_2 = 1 + 2j \end{array}.$$

Thus,

$$h[n] = \left[(1 - 2j)(\tfrac{j}{2})^n + (1 + 2j)(-\tfrac{j}{2})^n\right] u[n].$$

This can be simplified to

$$h[n] = 2(\tfrac{1}{2})^n \left[\cos(\pi n/2) + 2\sin(\pi n/2)\right] u[n].$$

Solution 3.7-4

In standard form, the system's difference equation is

$$y[n] - \frac{3}{10}y[n-1] - \frac{1}{10}y[n-2] = 2x[n-2].$$

(a) By inspection, we see that the system is second order.

(b) The characteristic equation is $\gamma^2 - \frac{3}{10}\gamma - \frac{1}{10} = (\gamma - \frac{1}{2})(\gamma + \frac{1}{5}) = 0$. The characteristic roots are $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = -\frac{1}{5}$. Thus,

$$\text{the characteristic modes are } (\tfrac{1}{2})^n \text{ and } (-\tfrac{1}{5})^n.$$

(c) The form of the impulse response is

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$
$$= \frac{2}{-\frac{1}{10}}\delta[n] + \left[c_1(\tfrac{1}{2})^n + c_2(-\tfrac{1}{5})^n\right] u[n].$$

To find c_1 and c_2, we calculate $h[0]$ and $h[1]$ by recursion using

$$h[n] = \frac{3}{10}h[n-1] + \frac{1}{10}h[n-2] + 2\delta[n-2].$$

Thus, $h[0] = \frac{3}{10}h[-2] + \frac{1}{10}h[-1] + 2\delta[-2] = 0$ and $h[1] = \frac{3}{10}h[-1] + \frac{1}{10}h[0] + 2\delta[-1] = 0$. Using these values, we see that

$$\left.\begin{array}{l} h[0] = 0 = -20 + c_1 + c_2 \\ h[1] = 0 = \frac{1}{2}c_1 - \frac{1}{5}c_2 \end{array}\right\} \implies \begin{array}{l} c_1 = \frac{40}{7} \\ c_2 = \frac{100}{7} \end{array}.$$

Thus,

$$h[n] = -20\delta[n] + \left[\tfrac{40}{7}(\tfrac{1}{2})^n + \tfrac{100}{7}(-\tfrac{1}{5})^n\right] u[n].$$

Solution 3.7-5

Characteristic equation is $\gamma^2 - 6\gamma + 9 = (\gamma - 3)^2 = 0$. Also $a_2 = 9$ and $b_2 = 0$. Therefore,

$$h[n] = (c_1 + c_2 n)3^n u[n].$$

We need two values of $h[n]$ to determine c_1 and c_2. This is found from iterative solution of

$$(E^2 - 6E + 9)h[n] = E\delta[n]$$

or

$$h[n+2] - 6h[n+1] + 9h[n] = \delta[n+1].$$

Also $h[-1] = h[-2] = \delta[-1] = 0$ and $\delta[0] = 1$. Setting $n = -2$ yields

$$h[0] - 6(0) + 9(0) = 0 \quad \Longrightarrow \quad h[0] = 0.$$

Setting $n = -1$ yields

$$h[1] - 6(0) + 9(0) = 1 \quad \Longrightarrow \quad h[1] = 1.$$

Using these values, we see that

$$\left. \begin{array}{l} 0 = c_1 \\ 1 = 3(c_1 + c_2) \end{array} \right\} \Longrightarrow \quad \begin{array}{l} c_1 = 0 \\ c_2 = \frac{1}{3} \end{array}.$$

Thus,

$$h[n] = \frac{1}{3}n(3)^n u[n].$$

Solution 3.7-6

Here,

$$(E^2 - 6E + 25)y[n] = (2E^2 - 4E)x[n].$$

The characteristic roots are $5e^{\pm j0.9273}$. Since $b_2 = 0$, there is no $\delta[n]$ term in $h[n]$, and

$$h[n] = c(5)^n \cos(0.9273n + \theta)u[n].$$

We need two values of $h[n]$ to determine c and θ. This is done by solving iteratively

$$h[n] - 6h[n-1] + 25h[n-2] = 2\delta[n] - 4\delta[n-1].$$

Setting $n = 0$ yields

$$h[0] - 6(0) + 25(0) = 2(1) - 4(0) \quad \Longrightarrow \quad h[0] = 2.$$

Setting $n = 1$ in (2) yields

$$h[1] - 6(2) + 25(0) = 2(0) - 4 \quad \Longrightarrow \quad h[1] = 8.$$

Using these results, we see that

$$2 = c\cos\theta$$
$$8 = 5c\cos(0.9273 + \theta) = 3c\cos\theta - 4c\sin\theta.$$

Solution of these two equations yields

$$\left. \begin{array}{l} c\cos\theta = 2 \\ c\sin\theta = -0.5 \end{array} \right\} \Longrightarrow \quad \begin{array}{l} c = 2.0616 \\ \theta = -0.245\,\text{rad} \end{array}.$$

Thus,

$$h[n] = 2.0616(5)^n \cos(0.9273n - 0.245)u[n].$$

Solution 3.7-7

(a) Here,

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N].$$

Letting $x[n] = \delta[n]$ and $y[n] = h[n]$ yields

$$h[n] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_N\delta[n-N].$$

(b) From the result in part (a), we can immediately write $h[n]$ for this case as

$$h[n] = 3\delta[n] - 5\delta[n-1] - 2\delta[n-3].$$

Solution 3.8-1

Calling $x[n] = \frac{5}{2^n}u[n+5]$ and $h[n] = 3^n u[-n-2]$, we see that $y[n] = x[n] * h[n]$ is computed as

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m].$$

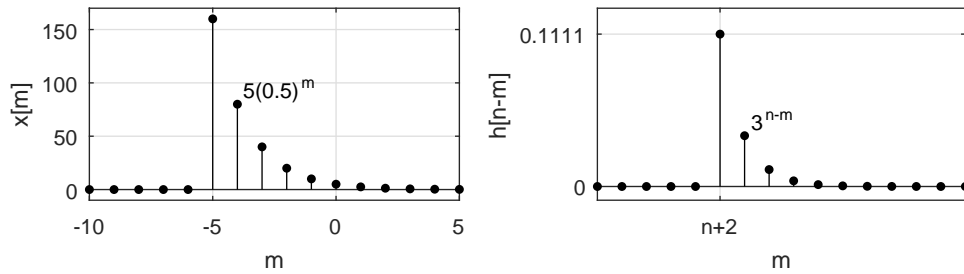To help visualize this problem, plots of $x[m]$ and $h[n-m]$ are shown in Fig. S3.8-1.

Figure S3.8-1

There are two regions to this convolution. The first region, when $n + 2 < -5$ or $n < -7$, has

$$y[n] = \sum_{m=-5}^{\infty} 5(\tfrac{1}{2})^m 3^n 3^{-m} = 5(3)^n \sum_{m=-5}^{\infty} (\tfrac{1}{6})^m$$

$$= 5(3)^n \frac{(\tfrac{1}{6})^{-5} - 0}{1 - \tfrac{1}{6}} = 5(3)^n \frac{6^5}{\tfrac{5}{6}} = 6^6 (3)^n.$$

The second region, when $n \geq -7$, has

$$y[n] = \sum_{m=n+2}^{\infty} 5(\tfrac{1}{2})^m 3^n 3^{-m} = 5(3)^n \sum_{m=n+2}^{\infty} (\tfrac{1}{6})^m$$

$$= 5(3)^n \frac{(\tfrac{1}{6})^{n+2} - 0}{1 - \tfrac{1}{6}} = 5(3)^n \frac{(\tfrac{1}{6})^2 (\tfrac{1}{2})^n (\tfrac{1}{3})^n}{\tfrac{5}{6}} = \frac{1}{6}(\tfrac{1}{2})^n.$$

Comparing these results to

$$y[n] = \begin{cases} C_1(\gamma_1)^n & n < N \\ C_2(\gamma_2)^n & n \geq N \end{cases}$$

we see that

$$C_1 = 6^6, \ C_2 = \tfrac{1}{6}, \ \gamma_1 = 3, \ \gamma_2 = \tfrac{1}{2}, \text{ and } N = -7.$$

The value $N = -6$ also works.

Solution 3.8-2

(a) Calling $x[n] = u[n-5] - u[n-9] + (0.5)^{(n-8)}u[n-9]$ and $h[n] = u[n]$, we see that $y_a[n] = x[n] * h[n]$ is computed as

$$y_a[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m].$$

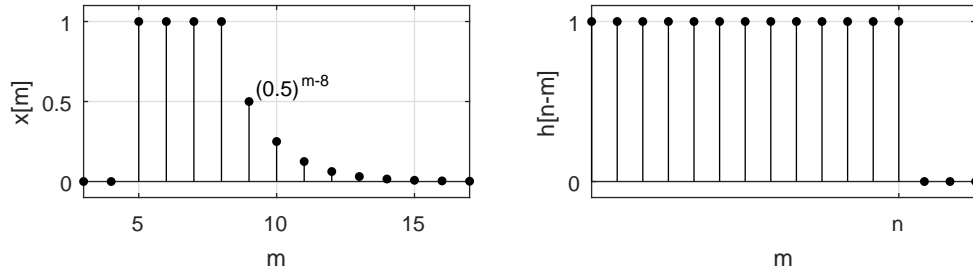To help visualize this problem, plots of $x[m]$ and $h[n-m]$ are shown in Fig. S3.8-2a.

Figure S3.8-2a

There are three regions to this convolution. The first region, when $n < 5$, has

$$y_a[n] = \sum 0 = 0.$$

The second region, when $5 \le n < 9$, has

$$y_a[n] = \sum_{m=5}^{n} 1 = n - 5 + 1 = n - 4.$$

The third region, when $n \ge 9$, has

$$y_a[n] = \sum_{m=5}^{8} 1 + \sum_{m=9}^{n} \left(\tfrac{1}{2}\right)^{m-8} = 4 + \left(\tfrac{1}{2}\right)^{-8} \frac{\left(\tfrac{1}{2}\right)^9 - \left(\tfrac{1}{2}\right)^{n+1}}{1 - \tfrac{1}{2}}$$

$$= 4 + \frac{\tfrac{1}{2} - \left(\tfrac{1}{2}\right)^{n-7}}{\tfrac{1}{2}} = 4 + 1 - \left(\tfrac{1}{2}\right)^{n-8} = 5 - \left(\tfrac{1}{2}\right)^{n-8}.$$

Thus,

$$y_a[n] = \begin{cases} 0 & n < 5 \\ n - 4 & 5 \le n < 9 \\ 5 - \left(\tfrac{1}{2}\right)^{n-8} & n \ge 9 \end{cases}.$$

(b) Calling $x[n] = \left(\tfrac{1}{2}\right)^{|n|}$ and $h[n] = u[-n + 5]$, we see that $y_b[n] = x[n] * h[n]$ is computed as

$$y_b[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m].$$

To help visualize this problem, plots of $x[m]$ and $h[n - m]$ are shown in Fig. S3.8-2b.
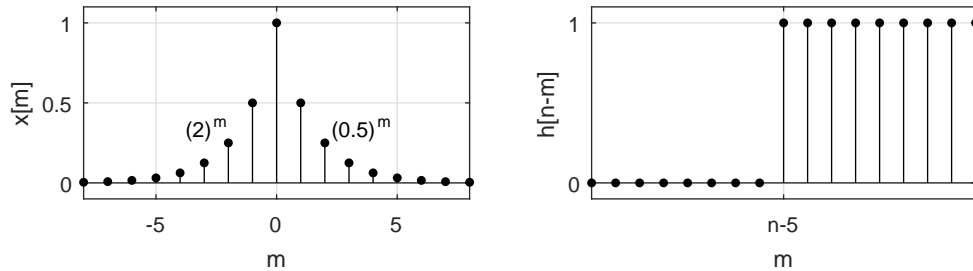
Figure S3.8-2b

There are two regions to this convolution. The first region, when $n - 5 < 0$ or $n < 5$, has

$$y_b[n] = \sum_{m=n-5}^{-1} 2^m + \sum_{m=0}^{\infty} \left(\tfrac{1}{2}\right)^m = \frac{2^{n-5} - 2^0}{1 - 2} + \frac{\left(\tfrac{1}{2}\right)^0 - 0}{1 - \tfrac{1}{2}}$$

$$= 1 - 2^{n-5} + 2 = 3 - 2^{n-5}.$$

The second region, when $n \geq 5$, has

$$y_b[n] = \sum_{m=n-5}^{\infty} (\tfrac{1}{2})^m = \frac{(\frac{1}{2})^{n-5} - 0}{1 - \frac{1}{2}} = 2(\tfrac{1}{2})^{n-5}.$$

Thus,

$$y_b[n] = \begin{cases} 3 - 2^{n-5} & n < 5 \\ 2(\tfrac{1}{2})^{n-5} & n \geq 5 \end{cases}.$$

Solution 3.8-3

(a) The characteristic equation is $\gamma^6 - 1 = 0$. Thus, $\gamma^6 = e^{j2\pi k}$ and

$$\gamma = e^{j\pi k/3} \quad \text{for integer } k.$$

This results in six unique characteristic roots:

$$\gamma_1 = e^{j\pi/3}, \; \gamma_2 = e^{j2\pi/3}, \; \gamma_3 = -1, \; \gamma_4 = e^{j4\pi/3}, \; \gamma_5 = e^{j5\pi/3}, \text{ and } \gamma_6 = 1.$$

(b) For LTID systems, $y[n] = x[n] * h[n]$. Let us "flip and shift" $x[n]$ so that $y[n]$ is computed as

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m].$$

In vector form,

$$x[n-m] = [\; \overset{\overset{m=n-3}{\downarrow}}{\tfrac{1}{8}} \;, \tfrac{1}{4}, \tfrac{1}{2}, 1, 2, 4, 8, \overset{\overset{m=n+4}{\downarrow}}{16}\;].$$

At $n = 10$, $x[10 - m]$ ranges over $7 \leq m \leq 14$ and overlaps $h[m]$ as

$$h[m] = [\; \overset{\overset{m=0}{\downarrow}}{2} \;, 2, 2, 2, 0, 0, 2, \underbrace{2, 2, 2, 0, 0, 2, 2, 2}_{\text{overlaps } x[10-m]}, 2, 0, 0, 2, \cdots].$$

Thus,

$$y[10] = 2(\tfrac{1}{8}) + 2(\tfrac{1}{4}) + 2(\tfrac{1}{2}) + 0(1) + 0(2) + 2(4) + 2(8) + 2(16) = 57.75.$$

Solution 3.8-4

(a) In vector form, the impulse response is

$$h[n] = [-8, -4, -2, -1, -\tfrac{1}{2}, -\tfrac{1}{4}, \overset{\overset{n=0}{\downarrow}}{0}\;].$$

Since $h[n] \neq 0$ for all $n < 0$, we see that

the system is not causal.

(b) For LTID systems, $y[n] = x[n] * h[n]$. Let us "flip and shift" $h[n]$ so that $y[n]$ is computed as

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m].$$

In vector form,

$$h[n-m] = [\; \overset{\overset{m=n+1}{\downarrow}}{-\tfrac{1}{4}} \;, -\tfrac{1}{2}, -1, -2, -4, \overset{\overset{m=n+6}{\downarrow}}{-8}\;].$$

At $n = 12$, $h[12 - m]$ ranges over $13 \le m \le 18$ and overlaps $x[m]$ as

$$x[m] = [\cdots, \overset{\overset{m=0}{\downarrow}}{1}, 2, 3, 1, 0, 0, 1, 2, 3, 1, 0, 0, 1, \underbrace{2, 3, 1, 0, 0, 1,}_{\substack{\text{overlaps} \\ h[12-m]}} 2, 3, 1, 0, \cdots]$$

Thus,

$$y[12] = 2(-\tfrac{1}{4}) + 3(-\tfrac{1}{2}) + 1(-1) + 0(-2) + 0(-4) + 1(-8) = -11.$$

Solution 3.8-5

In this case,

$$y[n] = (-2)^n u[n-1] * e^{-n} u[n+1]$$

$$= \sum_{m=-\infty}^{\infty} (-2)^m u[m-1] e^{-(n-m)} u[n - m - 1].$$

However, $u[m - 1] = 0$ for $m < 1$ and $u[n - m + 1] = 0$ for $m > n + 1$. Hence the summation limits may be restricted for $1 \le m \le n + 1$, and

$$y[n] = e^{-n} \sum_{m=1}^{n+1} (-2e)^m = e^{-n} \left[\frac{(-2e)^{n+2} + 2e}{-2e - 1} \right]$$

$$= \frac{2e^2}{2e + 1} \left[(-2)^{n+1} - e^{-(n+1)} \right] u[n].$$

We can also obtain this answer by using Table 3.1 and the shift property of convolution. If we advance impulse response $h[n]$ by one unit and delay the input by one unit, the convolution remains unchanged according to the shift property. Hence, we can obtain the desired convolution by using

$$h[n] = (-2)^{n+1} u[n] \qquad \text{and} \qquad x[n] = e^{-(n-1)} u[n].$$

The desired convolution is therefore given by

$$y[n] = (-2)^{n+1} u[n] * e^{-(n-1)} u[n]$$
$$= -2e \left\{ (-2)^n u[n] * e^{-n} u[n] \right\}.$$

From Table 3.1, we obtain

$$y[n] = -2e \left[\frac{(-2)^{n+1} - e^{-(n+1)}}{-1 - e^{-1}} \right] u[n]$$

$$= \frac{2e^2}{2e + 1} \left[(-2)^{n+1} - e^{-(n+1)} \right] u[n],$$

which confirms our earlier result.

Solution 3.8-6

Here,

$$y[n] = \frac{1}{2} \left[\delta[n-2] - (-2)^{n+1} \right] u[n-3] * 3^{n-1} u[n+2].$$

Because $\delta[n - 2] u[n - 3] = 0$, we have

$$y[n] = -\frac{1}{2} (-2)^{n+1} u[n-3] * 3^{n-1} u[n+2].$$

If we advance the first term by 3 units and delay the second term by 2 units, the resulting convolution yields $y[n + 1]$. Hence,

$$y[n + 1] = -\frac{1}{2}(-2)^{n+4}u[n] * 3^{n-3}u[n]$$

$$= -8(-2)^n u[n] * \frac{1}{27}3^n u[n]$$

$$= -\frac{8}{27}(-2)^n * 3^n u[n]$$

From Table 3.1, we obtain

$$y[n + 1] = -\frac{8}{27}\left[\frac{(-2)^{n+1} - (3)^{n+1}}{-2 - 3}\right] u[n]$$

$$= \frac{8}{135}\left[(-2)^{n+1} - (3)^{n+1}\right] u[n]$$

$$\text{and} \qquad y[n] = \frac{8}{135}\left[(-2)^n - (3)^n\right] u[n - 1].$$

Solution 3.8-7

Here let us delay $x[n]$ by one unit and advance $h[n]$ by one unit to obtain $y[n]$.

$$y[n] = 3^{n+1}u[n] * \left[2^{n-1} + 3(-5)^{n+1}\right] u[n]$$

$$= 3^{n+1}u[n] * 2^{n-1}u[n] + 3(3)^{n+1}u[n] * (-5)^{n+1}u[n]$$

$$= \frac{3}{2}\left\{3^n u[n] * 2^n u[n]\right\} - 45\left\{3^n u[n] * (-5)^n u[n]\right\}$$

$$= \frac{3}{2}\left[\frac{3^{n+1} - 2^{n+1}}{1}\right] u[n] - 45\left[\frac{3^{n+1} - (-5)^{n+1}}{8}\right] u[n]$$

$$= \left[\frac{99}{8}(3)^n - 3(2)^n - 225(-5)^n\right] u[n]$$

Solution 3.8-8

Here let us delay the input $x[n]$ by 3 units and advance $h[n]$ by 4 units. The resulting convolution yields $y[n - 3 + 4] = y[n + 1]$,

$$y[n + 1] = 3^{-(n+1)}u[n] * 3(n + 2)(2)^{n+1}u[n]$$

$$= 2\left\{3^{-n}u[n] * (n2^n u[n] + 2(2)^n u[n])\right\}$$

$$= 4\left\{3^{-n}u[n] * 2^n u[n]\right\} + 2\left\{3^{-n}u[n] * n2^n u[n]\right\}.$$

From Table 3.1, we obtain

$$y[n + 1] = -\frac{20}{3}\left[3^{-(n+1)} - 2^{(n+1)}\right] u[n] + \frac{6}{25}\left[3^{-n} + (5n - 1)2^n\right] u[n].$$

Hence,

$$y[n] = -\frac{20}{3}\left[3^{-n} - 2^n\right] u[n - 1] + \frac{6}{25}\left[3^{-(n-1)} + (5n - 6)2^{(n-1)}\right] u[n - 1]$$

$$= -\frac{446}{75}\left\{3^{-n}u[n - 1] - 2^n u[n - 1]\right\} + \frac{3}{5}n2^n u[n - 1].$$

Solution 3.8-9

Here, we advance $x[n]$ by one unit and leave $h[n]$ unchanged. The resulting convolution is $y[n-1]$. Hence,

$$y[n-1] = 2^{n+1}u[n] * 3^n \cos\left(\frac{\pi n}{3} - 0.5\right)u[n]$$

$$= 2\left\{2^n u[n] * 3^n \cos\left(\frac{\pi n}{3} - 0.5\right)u[n]\right\}$$

To determine this convolution, we use pair 10 of Table 3.1 with

$$R = \left[(3)^2 + (2)^2 - 2(3)(2)(0.5)\right]^{1/2} = \sqrt{7}$$

$$\text{and} \quad \phi = \tan^{-1}\left[\frac{3\sqrt{3}/2}{1.5 - 2}\right] = 1.761 \quad \text{radians.}$$

Hence,

$$y[n-1] = 2\left\{\frac{1}{\sqrt{7}}\left[(3)^{n+1} \cos\left[\frac{\pi}{3}(n+1) - 2.26\right] - 2^{n+1}\cos(2.261)\right]\right\}u[n]$$

and

$$y[n] = \left\{\frac{2}{\sqrt{7}}(3)^n \cos\left[\frac{\pi n}{3} - 2.26\right] + 1.273(2)^n\right\}u[n-1].$$

Solution 3.8-10

(a) The characteristic equation is $(\gamma - \frac{1}{2}) = 0$ and the characteristic root is $\gamma = \frac{1}{2}$. The form of the impulse response is

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n] = \frac{1}{-\frac{1}{2}}\delta[n] + c(\frac{1}{2})^n u[n].$$

From the difference equation, we know that $h[n] = \frac{1}{2}h[n-1] + \delta[n-1]$. Thus, $h[0] = \frac{1}{2}h[-1] + \delta[-1] = 0$. Using this value, we see that

$$h[0] = 0 = -2\delta[0] + c(\frac{1}{2})^0 u[0] = -2 + c \quad \Rightarrow \quad c = 2.$$

Thus,

$$h[n] = -2\delta[n] + 2(\frac{1}{2})^n u[n] = (\frac{1}{2})^{n-1}u[n-1].$$

(b) For LTID systems, $s[n] = u[n] * h[n]$. Let us "flip and shift" $u[n]$ so that $s[n]$ is computed as

$$s[n] = \sum_{m=-\infty}^{\infty} h[m]u[n-m].$$

In vector form,

$$h[m] = [\ \overset{\overset{m=0}{\downarrow}}{0}\ , 1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, (\tfrac{1}{2})^{m-1}, \ldots]$$

and

$$u[n-m] = [\cdots, 1, 1, \ \overset{\overset{m=n}{\downarrow}}{1}\ ].$$

There are two regions to consider. In the first region, when $n \leq 0$, we see that $s[n] = \sum 0 = 0$. In the second region, when $n \geq 1$, we see that

$$s[n] = \sum_{m=1}^{n}(\tfrac{1}{2})^{m-1} = 2\frac{(\frac{1}{2})^1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = 2[1 - (\tfrac{1}{2})^n].$$

Thus,

$$s[n] = 2[1 - (\frac{1}{2})^n]u[n-1].$$

(c) Here,
$$h_{\text{cascade}}[n] = h_1[n] * h_2[n] = h_1[n] * (-3u[n-13]).$$

Using linearity, the shift property, and the result from part (b), we see that

$$h_{\text{cascade}}[n] = -3s[n-13] = -6[1 - (\tfrac{1}{2})^{n-13}]u[n-14].$$

Solution 3.8-11

#1

$$\delta[n-k] * x[n] = x[n-k]$$

$$\delta[n-k] * x[n] = \sum_{m=0}^{\infty} x[m]\delta[n-m-k]$$

Since $\delta[n-m-k] = 1$ for $m = n-k$ and is zero for all other values of m, the right-side sum is given by $x[n-k]$.

#2

$$\gamma^n u[n] * u[n] = \sum_{m=0}^{n} \gamma^m u[n-m]$$

Because $u[n-m] = 1$ for all $0 \le m \le n$, we have

$$\gamma^n u[n] * u[n] = \sum_{m=0}^{n} \gamma^m = \frac{\gamma^{n+1} - 1}{\gamma - 1} u[n] \qquad \gamma \ne 1.$$

We multiply the result with $u[n]$ because the convolution is zero for $n < 0$.

#3

$$u[n] * u[n] = \sum_{m=0}^{n} u[m]u[n-m]$$

Over the range $0 \le m \le n$, $u[m] = u[n-m] = 1$. Hence,

$$u[n] * u[n] = \sum_{m=0}^{n} 1 = (n+1)u[n].$$

Solution 3.8-12

#4

$$\gamma_1^n u[n] * \gamma_2^n u[n] = \sum_{m=0}^{n} \gamma_1^m \gamma_2^{n-m}$$

$$= \gamma_2^n \sum_{m=0}^{n} (\gamma_1/\gamma_2)^m$$

$$= \gamma_2^n \left[\frac{(\gamma_1/\gamma_2)^{n+1} - 1}{1 - (\gamma_1/\gamma_2)} \right] \qquad \gamma_1 \ne \gamma_2$$

$$= \left[\frac{\gamma_1^{n+1} - \gamma_2^{n+1}}{\gamma_1 - \gamma_2} \right] u[n] \qquad \gamma_1 \ne \gamma_2$$

We multiply the result by $u[n]$ because convolution of two causal sequences is zero for $n < 0$.

#5

$$nu[n] * u[n] = \sum_{m=0}^{n} m = \frac{n(n+1)}{2} u[n]$$

#6

$$\gamma^n u[n] * nu[n] = \sum_{m=0}^{n} \gamma^m (n - m)$$

$$= n \sum_{m=0}^{n} \gamma^m - \sum_{m=0}^{n} m\gamma^m$$

$$= n \frac{\gamma^{n+1} - 1}{\gamma - 1} - \frac{\gamma + [n(\gamma - 1) - 1]\gamma^{n+1}}{(\gamma - 1)^2}$$

$$= \left[\frac{\gamma(\gamma^n - 1) + n(1 - \gamma)}{(1 - \gamma)^2} \right] u[n]$$

Solution 3.8-13

#7

$$nu[n] * nu[n] = \sum_{m=0}^{n} m(n - m)$$

$$= n \sum_{m=0}^{n} m - \sum_{m=0}^{n} m^2$$

$$= \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n^2 - 1)}{6} u[n]$$

#8

$$\gamma^n u[n] * \gamma^n u[n] = \sum_{m=0}^{n} \gamma^m \gamma^{n-m}$$

$$= \gamma^n \sum_{m=0}^{n} 1$$

$$= (n + 1)\gamma^n u[n]$$

Solution 3.8-14

#9

$$n\gamma_1^n u[n] * \gamma_2^n u[n] = \sum_{m=0}^{n} m\gamma_1^m \gamma_2^{n-m}$$

$$= \gamma_2^2 \sum_{m=0}^{n} n(\gamma_1/\gamma_2)^m$$

$$= \gamma_2^n \frac{(\gamma_1/\gamma_2) + [(n\gamma_1/\gamma_2) - n - 1](\gamma_1/\gamma_2)^{n+1}}{[(\gamma_1/\gamma_2) - 1]^2}$$

$$= \frac{\gamma_1\gamma_2}{(\gamma_1 - \gamma_2)^2} \left[\gamma_2^n - \gamma_1^n + \frac{\gamma_1 - \gamma_2}{\gamma_2} n\gamma_1^n \right] u[n] \qquad \gamma_1 \neq \gamma_2$$

#11 Let

$$c[n] = \gamma_1^n u[n] * \gamma_2^n u[-(n+1)]$$

$$= \sum_{m=-\infty}^{\infty} \gamma_1^m \gamma_2^{n-m} u[m] u[-(n-m+1)].$$

Consider $c[n]$ for $n \geq 0$. In this case $-(n-m+1) \geq 0$ for $m \geq n+1$. Therefore $u[-(n-m+1)] = 0$ for $m < n+1$ and equal to 1 for $m \geq n+1$. Also $u[m] = 1$ for $m \geq n+1$ (for positive n). Hence,

$$c[n] = \gamma_2^n \sum_{m=n+1}^{\infty} \left(\frac{\gamma_1}{\gamma_2}\right)^m = \frac{\gamma_1^{n+1}}{\gamma_2 - \gamma_1} \qquad \text{for} \quad n \geq 0 \quad \text{and} \quad |\gamma_2| > |\gamma_1|.$$

When $n \leq -1$, $-(n-m+1) \geq 0$ for $m \geq 0$. Hence, $u[m] u[-(n-m+1)] = 1$ for $m \geq 0$ and is zero otherwise. Thus,

$$c[n] = \gamma_2^n \sum_{m=0}^{\infty} \left(\frac{\gamma_1}{\gamma_2}\right)^m = \frac{\gamma_2^{n+1}}{\gamma_2 - \gamma_1} \qquad \text{for} \quad n \leq -1 \quad \text{and} \quad |\gamma_2| > |\gamma_1|$$

Therefore

$$c[n] = \frac{\gamma_1^{n+1}}{\gamma_2 - \gamma_1} u[n] + \frac{\gamma_2^{n+1}}{\gamma_2 - \gamma_1} u[-(n+1)] \qquad |\gamma_2| > |\gamma_1|.$$

Solution 3.8-15

The characteristic root is -2. Therefore,

$$y_0[n] = c(-2)^n.$$

Setting $n = -1$ and substituting $y[-1] = 10$, yields

$$10 = -\frac{c}{2} \implies c = -20.$$

Therefore,

$$y_0[n] = -20(-2)^n \qquad n \geq 0.$$

For this system $h[n]$, the unit impulse response is found in Prob. 3.7-1b to be

$$h[n] = (-2)^n u[n].$$

The zero-state response is

$$y[n] = e^{-n} u[n] * (-2)^n u[n].$$

Using Table 3.1, this convolution is found to be

$$y[n] = \frac{e}{2e+1}[e^{-(n+1)} - (-2)^{n+1}] u[n]$$

$$= \frac{e}{2e+1}[\frac{1}{e}(e)^{-n} + 2(-2)^n] u[n]$$

$$= \left[\frac{1}{2e+1}(e)^{-n} + \frac{2e}{2e+1}(-2)^n\right] u[n].$$

Thus,

$$y_{\text{total}}[n] = y_0[n] + y[n]$$

$$= [-20(-2)^n + \frac{1}{2e+1}(e)^{-n} + \frac{2e}{2e+1}(-2)^n] u[n]$$

$$= \frac{1}{2e+1}[-(38e+20)(-2)^n + (e)^{-n}] u[n].$$

Solution 3.8-16

(a)

$$y[n] = 2^n u[n] * (0.5)^n u[n]$$
$$= \frac{2^{n+1} - (0.5)^{n+1}}{2 - 0.5} u[n] = \frac{2}{3}[2^{n+1} - (0.5)^{n+1}]u[n]$$

(b)

$$x[n] = 2^{(n-3)}u[n] = 2^{-3}2^n u[n] = \frac{1}{8}2^n u[n]$$

From the result in part (a), it follows that

$$y[n] = \frac{1}{8}\frac{2}{3}[2^{n+1} - (0.5)^{n+1}]u[n] = \frac{1}{12}[2^{n+1} - (0.5)^{n+1}]u[n].$$

(c)
$$x[n] = 2^n u[n-2] = 4\{2^{(n-2)}u[n-2]\}$$

Note that $2^{(n-2)}u[n-2]$ is the same as the input $2^n u[n]$ in part (a) delayed by 2 units. From the shift property of convolution, its response will therefore be the same as in part (a) delayed by 2 units. The input here is $4\{2^{(n-2)}u[n-2]\}$. Therefore,

$$y[n] = 4\frac{2}{3}[2^{n+1-2} - (0.5)^{n+1-2}]u[n-2] = \frac{8}{3}[2^{n-1} - (0.5)^{n-1}]u[n-2].$$

Solution 3.8-17

For $x[n] = u[n]$
$$y[n] = u[n] - 2u[n-1].$$

The highest order difference is one. Hence, this is a first-order system. This is also a nonrecursive system, whose output at any instant depends only on the input. Thus, initial conditions are not needed to find the system response.

Solution 3.8-18

(a) Figure S3.8-18a represents the system in Fig P3.8-18 in a more convenient fashion. A parallel connection requires the individual impulse responses to be added, and a series connection requires the individual impulse responses to be convolved. Thus, the overall impulse response is given by
$$h[n] = h_1[n] * h_2[n] + (h_1[n] * h_5[n] + h_4[n]) * h_3[n].$$
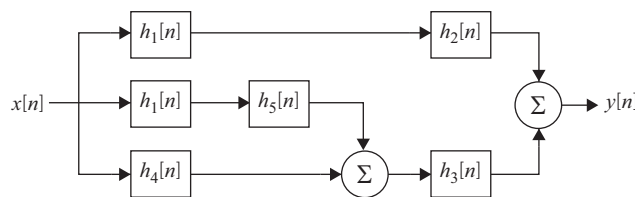
Figure S3.8-18a

(b) First, we shall simplify expressions for $h_1[n]$ and $h_2[n]$ by using the facts that $u[n-1] = u[n] - \delta[n]$, $(0.9)^{n-1} = \frac{1}{0.9}(0.9)^n$, and $(0.5)^{n-1} = \frac{1}{0.5}(0.5)^n$. Now,

$$h_1[n] = 0.9^n u[n] - \frac{0.5}{0.9}(0.9)^n(u[n] - \delta[n]) = \frac{4}{9}0.9^n u[n] + \frac{5}{9}\delta[n].$$

Similarly,

$$h_2[n] = 0.5^n u[n] - \frac{0.9}{0.5}(0.5)^n(u[n] - \delta[n]) = -0.8(0.5)^n u[n] + 1.8\delta[n].$$

Recall that $x[n] * \delta[n] = x[n]$ and $\delta[n] * \delta[n] = \delta[n]$. Hence,

$$h_1[n] * h_2[n] = \frac{4}{9}(0.9)^n u[n] * (-0.8)(0.5)^n u[n] - 0.8(\frac{5}{9})(0.5)^n u[n] + 1.8(\frac{4}{9})(0.9)^n u[n] + \delta[n]$$

$$= -\frac{16}{45}\left[\frac{0.9^{n+1} - 0.5^{n+1}}{0.4}\right]u[n] - \frac{4}{9}(0.5)^n u[n] + \frac{4}{5}(0.9)^n u[n] + \delta[n]$$

$$= \delta[n].$$

Since the impulse response is $\delta[n]$, the cascade of the two systems is an identity system.

Solution 3.8-19

(a) From Eq. (3.37), for a causal system

$$g[n] = \sum_{k=0}^{n} h[k].$$

Let $k = n - m$. The limits of summation change from $m = n$ to $m = 0$, resulting in

$$g[n] = \sum_{m=n}^{0} h[n - m].$$

In summation, we can sum from either direction. Hence,

$$g[n] = \sum_{m=0}^{n} h[n - m].$$

(b) When the system is noncausal,

$$g[n] = \sum_{k=-\infty}^{n} h[k].$$

Let $k = n - m$. The original sum limits $k = -\infty$ and $k = n$ thus become $m = \infty$ and $m = 0$. Hence,

$$g[n] = \sum_{m=\infty}^{0} h[n - m] = \sum_{m=0}^{\infty} h[n - m].$$

Solution 3.8-20

(a) First, we express $h[n]$ in terms of delta functions $\delta[n]$ as

$$h[n] = 2(u[n+2] - u[n-3]) = 2\delta[n+2] + 2\delta[n+1] + 2\delta[n] + 2\delta[n-1] + 2\delta[n-2].$$

Substituting $x[n]$ for $\delta[n]$ and $y[n]$ for $h[n]$, a suitable constant coefficient linear difference equation is

$$y[n] = 2x[n+2] + 2x[n+1] + 2x[n] + 2x[n-1] + 2x[n-2].$$

(b) Let us compute $y_{\text{zsr}}[n] = x[n] * h[n]$ as

$$y_{\text{zsr}}[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m].$$

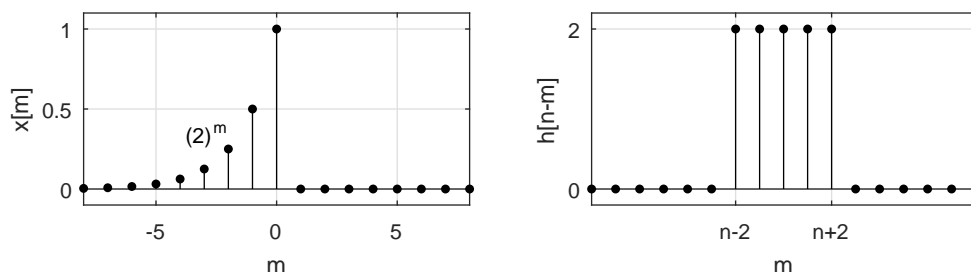To help visualize this problem, plots of $x[m]$ and $h[n-m]$ are shown in Fig. S3.8-20.

Figure S3.8-20

There are three regions to this convolution. In the first region, when $n + 2 \leq 0$ or $n \leq -2$, we have

$$y_{\text{zsr}}[n] = \sum_{m=n-2}^{n+2} 2(2)^m = 2\frac{2^{n-2} - 2^{n+3}}{1-2} = 2^{n+4} - 2^{n-1} = 15.5(2)^n.$$

In the second region, when $-2 < n \leq 2$, we have

$$y_{\text{zsr}}[n] = \sum_{m=n-2}^{0} 2(2)^m = 2\frac{2^{n-2} - 2}{1-2} = 4 - 2^{n-1}.$$

The last region, when $n > 2$, has

$$y_{\text{zsr}}[n] = \sum 0 = 0.$$

Thus,

$$y_{\text{zsr}}[n] = \begin{cases} 15.5(2)^n & n \leq -2 \\ 4 - 2^{n-1} & -2 < n \leq 2 \\ 0 & n > 2 \end{cases}.$$

Solution 3.8-21

(a) As an identity system, the impulse response of system 3 is $h_3[n] = \delta[n]$. A parallel connection of system 2 and 3 has an impulse response

$$h_{2||3}[n] = h_2[n] + h_3[n] = [\overset{\downarrow}{1}, 0, -6, -9, 3].$$

Putting $h_{2||3}[n]$ in series with $h_1[n]$, the overall impulse response is $h[n] = h_1[n] * h_{2||3}[n]$. Let us use MATLAB to compute this convolution.

```
>>  h = conv([2 -3 4],[1 0 -6 -9 3])
    h =  2    -3    -8    0    9    -45    12
```

Thus,

$$h[n] = [\overset{\downarrow}{2}, -3, -8, 0, 9, -45, 12].$$

(b) The zero-state response of system 2 to input $x[n] = u[-n]$ is given by $y_{zsr}[n] = x[n] * h_2[n]$. Let us "flip and shift" $x[n]$ so that $y_{zsr}[n]$ is computed as

$$y_{zsr}[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m].$$

In vector form,

$$x[n-m] = [\overset{\underset{\downarrow}{m=n}}{1}, 1, 1, 1, 1, \cdots].$$

and

$$h[m] = [\overset{\underset{\downarrow}{m=2}}{-6}, -9, 3].$$

For $n \le 2$,

$$y_{zsr}[n] = -6 - 9 + 3 = -12.$$

For $n = 3$,

$$y_{zsr}[n] = -9 + 3 = -6.$$

For $n = 4$,

$$y_{zsr}[n] = 3.$$

For $n > 4$,

$$y_{zsr}[n] = 0.$$

Thus,

$$y_{zsr}[n] = \begin{cases} -12 & n \le 2 \\ -6 & n = 3 \\ 3 & n = 4 \\ 0 & n > 4 \end{cases}.$$

Solution 3.8-22

The equation describing this situation is [see Eq. (3.4)]

$$(E - a)y[n] = Ex[n] \qquad a = 1 + r = 1.01.$$

The initial condition $y[-1] = 0$. Hence, there is only the zero-state component. The input is $500u[n] - 1500\delta[n-4]$ because at $n = 4$, instead of depositing the usual $500, she withdraws $1000. To find $h[n]$, we solve iteratively

$$(E - a)h[n] = E\delta[n]$$

or

$$h[n+1] - ah[n] = \delta[n+1].$$

Setting $n = -1$ and substituting $h[-1] = 0$ and $\delta[0] = 1$ yield

$$h[0] = 1.$$

Also, the characteristic root is a and $b_0 = 0$. Therefore,

$$h[n] = ca^n u[n].$$

Setting $n = 0$ and substituting $h[0] = 1$ yield

$$1 = c.$$

Therefore,

$$h[n] = (a)^n u[n] = (1.01)^n u[n].$$

The (zero-state) response is

$$
\begin{aligned}
y[n] &= (1.01)^n u[n] * x[n] \\
&= (1.01)^n u[n] * \{500u[n] - 1500\delta[n-4]\} \\
&= 500(1.01)^n u[n] * u[n] - 1500(1.01)^{n-4} u[n-4] \\
&= \frac{500}{0.01}[(1.01)^{n+1} - 1]u[n] - 1500(1.01)^{n-4} u[n-4] \\
&= 50000[(1.01)^{n+1} - 1]u[n] - 1500(1.01)^{n-4} u[n-4].
\end{aligned}
$$

Solution 3.8-23

This problem is identical to the savings account problem with negative initial deposit (loan). If M is the initial loan, then $y[0] = -M$. If $y[n]$ is the loan balance, then [see Eq. (3.4)]

$$
y[n+1] - ay[n] = x[n+1] \qquad a = 1 + r
$$

or

$$
(E - a)y[n] = Ex[n].
$$

The characteristic root is a, and the impulse response for this system is easily found to be

$$
h[n] = a^n u[n].
$$

This problem can be solved in two ways.

First method: We may consider the loan of M dollars as an a negative input $-M\delta[n]$. The monthly payment of P starting at $n = 1$ also is an input. Thus the total input is $x[n] = -M\delta[n] + Pu[n-1]$ with zero initial conditions. Because $u[n] = \delta[n] + u[n-1]$, we can express the input in a more convenient form as $x[n] = -(M + P)\delta[n] + Pu[n]$. The loan balance (response) $y[n]$ is

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= a^n u[n] * \{-(M+P)\delta[n] + Pu[n]\} \\
&= -(M+P)a^n u[n] + Pa^n u[n] * u[n] \\
&= -(M+P)a^n u[n] + P\left[\frac{a^{n+1} - 1}{a - 1}\right] u[n] \\
&= -Ma^n u[n] - P\left[a^n - \frac{a^{n+1} - 1}{a - 1}\right] u[n] \\
&= \left\{-Ma^n + P\left[\frac{a^n - 1}{a - 1}\right]\right\} u[n].
\end{aligned}
$$

Also $a = 1 + r$ and $a - 1 = r$ where r is the interest rate per dollar per month. At $n = N$, the loan balance is zero. Therefore

$$
y[N] = -Ma^N + P\left[\frac{a^N - 1}{r}\right] = 0
$$

or

$$
P = \frac{ra^N}{a^N - 1}M = \frac{rM}{1 + a^{-N}} = \frac{rM}{1 + (1+r)^{-N}}.
$$

Second method: In this approach, the initial condition is $y[0] = -M$, and the input is $x[n] = Pu[n-1]$ because the monthly payment of P starts at $n = 1$. The characteristic root is a, and The zero-input response is

$$
y_0[n] = ca^n u[n].
$$

Setting $n = 0$ and substituting $y_0[0] = -M$ yield $c = -M$ and

$$
y_0[n] = -Ma^n u[n].
$$

The zero-state response $y[n]$ is

$$y[n] = h[n] * x[n] = h[n] * Pu[n-1] = Pa^n u[n] * u[n-1].$$

Here we use shift property of convolution. If we let

$$x[n] = a^n u[n] * u[n] = [\frac{a^{n+1} - 1}{a - 1}]u[n],$$

then the shift property yields

$$Pa^n u[n] * u[n-1] = x[n-1] = P\left[\frac{a^n - 1}{a - 1}\right]u[n-1].$$

The total balance is

$$y_0[n] + y[n] = -Ma^n u[n] + P\left[\frac{a^n - 1}{a - 1}\right]u[n-1].$$

For $n > 1$, $u[n] = u[n-1] = 1$. Therefore,

$$\text{Loan balance} = -Ma^n + P\left[\frac{a^n - 1}{a - 1}\right] \qquad n > 1,$$

which confirms the result obtained by the first method. From here on the procedure is identical to that of the first method.

Solution 3.8-24

We use the result in Prob. 3.8-23 with $r = 0.015$, $a = 1.015$, $P = 500$, and $M = 10000$. Therefore,

$$500 = 10000\frac{(1.015)^N(0.015)}{(1.015)^N - 1}$$

or

$$(1.015)^N = 1.42857$$
$$N\ln(1.015) = \ln(1.42857)$$
$$N = \frac{\ln(1.42857)}{\ln(1.015)} = 23.956.$$

Hence $N = 23$ payments are needed. The residual balance (remainder) at the 23rd payment is

$$y[23] = -10000(1.015)^{23} + 500[\frac{(1.015)^{23} - 1}{0.015}] = -471.2.$$

Solution 3.8-25

In the following convolutions, we call the first function $x[n]$ and the second function $h[n]$ so that

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m].$$

(a) For $y_a[n] = x[n] * h[n] = [\overset{\downarrow}{2}, 3, -2, -3] * [-10, \overset{\downarrow}{0}, -5]$, we see that

$$h[n-m] = [-5, 0, \overset{\overset{m=n+1}{\downarrow}}{-10}] \quad \text{and} \quad x[m] = [\overset{\overset{m=0}{\downarrow}}{2}, 3, -2, -3].$$

Sliding $h[n - m]$ across $x[m]$, the convolution is computed as

n	$y_a[n]$
< -1	0
-1	$-10(2) = -20$
0	$-10(3) + 0(2) = -30$
1	$-10(-2) + 0(3) - 5(2) = 10$
2	$-10(-3) + 0(-2) - 5(3) = 15$
3	$0(3) - 5(-2) = 10$
4	$-5(-3) = 15$
> 4	0

Thus,

$$y_a[n] = [-20, \overset{\underset{n=0}{\downarrow}}{-30}, 10, 15, 10, 15].$$

(b) For $y_b[n] = x[n] * h[n] = [2, \overset{\downarrow}{-1}, 3, -2] * [-1, -4, 1, \overset{\downarrow}{-2}]$, we see that

$$h[n - m] = [-2, 1, -4, \overset{\underset{m=n+3}{\downarrow}}{-1}\ ] \quad \text{and} \quad x[m] = [\ \overset{\underset{m=-1}{\downarrow}}{2}\ , -1, 3, -2].$$

Sliding $h[n - m]$ across $x[m]$, the convolution is computed as

n	$y_b[n]$
< -4	0
-4	$-1(2) = -2$
-3	$-1(-1) - 4(2) = -7$
-2	$-1(3) - 4(-1) + 1(2) = 3$
-1	$-1(-2) - 4(3) + 1(-1) - 2(2) = -15$
0	$-4(-2) + 1(3) - 2(-1) = 13$
1	$1(-2) - 2(3) = -8$
2	$-2(-2) = 4$
> 2	0

Thus,

$$y_b[n] = [-2, -7, 3, -15, \overset{\underset{n=0}{\downarrow}}{13}, -8, 4].$$

(c) For $y_c[n] = x[n] * h[n] = [\overset{\downarrow}{0}, 0, 3, 2, 1, 2, 3] * [2, 3, -2, \overset{\downarrow}{1}]$, we see that

$$h[n - m] = [1, -2, 3, \overset{\underset{m=n+3}{\downarrow}}{2}\ ] \quad \text{and} \quad x[m] = [\ \overset{\underset{m=2}{\downarrow}}{3}\ , 2, 1, 2, 3].$$

Sliding $h[n - m]$ across $x[m]$, the convolution is computed as

n	$y_c[n]$
< -1	0
-1	$2(3) = 6$
0	$2(2) + 3(3) = 13$
1	$2(1) + 3(2) - 2(3) = 2$
2	$2(2) + 3(1) - 2(2) + 1(3) = 6$
3	$2(3) + 3(2) - 2(1) + 1(2) = 12$
4	$3(3) - 2(2) + 1(1) = 6$
5	$-2(3) + 1(2) = -4$
6	$1(3) = 3$
> 6	0

Thus,

$$y_c[n] = [6, \overset{n=0}{\underset{\downarrow}{13}}, 2, 6, 12, 6, -4, 3].$$

(d) For $y_d[n] = x[n] * h[n] = [5, 0, 0, \overset{\downarrow}{-2}, 8] * [-1, 1, \overset{\downarrow}{3}, 3, -2, 3]$, we see that

$$h[n - m] = [3, -2, 3, 3, 1, \overset{m=n+2}{\underset{\downarrow}{-1}}] \quad \text{and} \quad x[m] = [\overset{m=-3}{\underset{\downarrow}{5}}, 0, 0, -2, 8].$$

Sliding $h[n - m]$ across $x[m]$, the convolution is computed as

n	$y_d[n]$
< -5	0
-5	$-1(5) = -5$
-4	$-1(0) + 1(5) = 5$
-3	$-1(0) + 1(0) + 3(5) = 15$
-2	$-1(-2) + 1(0) + 3(0) + 3(5) = 17$
-1	$-1(8) + 1(-2) + 3(0) + 3(0) - 2(5) = -20$
0	$1(8) + 3(-2) + 3(0) - 2(0) + 3(5) = 17$
1	$3(8) + 3(-2) - 2(0) + 3(0) = 18$
2	$3(8) - 2(-2) + 3(0) = 28$
3	$-2(8) + 3(-2) = -22$
4	$3(8) = 24$
> 4	0

Thus,

$$y_d[n] = [-5, 5, 15, 17, -20, \overset{n=0}{\underset{\downarrow}{17}}, 18, 28, -22, 24].$$

(e) For $y_e[n] = ([1, \overset{\downarrow}{-1}] * [\overset{\downarrow}{1}, -1]) * ([\overset{\downarrow}{1}, -1] * [1, \overset{\downarrow}{-1}])$, we first compute $[1, \overset{\downarrow}{-1}] * [\overset{\downarrow}{1}, -1]$ and $[\overset{\downarrow}{1}, -1] * [1, \overset{\downarrow}{-1}]$. These trivial convolutions are

$$[1, \overset{\downarrow}{-1}] * [\overset{\downarrow}{1}, -1] = [\overset{\downarrow}{1}, -1] * [1, \overset{\downarrow}{-1}] = [1, \overset{\downarrow}{-2}, 1].$$

Thus, $y_e[n] = x[n] * h[n] = [1, \overset{\downarrow}{-2}, 1] * [1, \overset{\downarrow}{-2}, 1]$, and we see that

$$h[n - m] = [1, -2, \overset{m=n+1}{\underset{\downarrow}{1}}] \quad \text{and} \quad x[m] = [\overset{m=-1}{\underset{\downarrow}{1}}, -2, 1].$$

Sliding $h[n - m]$ across $x[m]$, the convolution is computed as

n	$y_e[n]$
< -2	0
-2	$1(1) = 1$
-1	$1(-2) - 2(1) = -4$
0	$1(1) - 2(-2) + 1(1) = 6$
1	$-2(1) + 1(-2) = -4$
2	$1(1) = 1$
> 2	0

Thus,

$$y_e[n] = [1, -4, \overset{n=0}{\underset{\downarrow}{6}}, -4, 1].$$

(f) For $y_f[n] = ([2, \overset{\downarrow}{-1}] * [\overset{\downarrow}{1}, -2]) * ([\overset{\downarrow}{1}, -2] * [2, \overset{\downarrow}{-1}])$, we first compute $[2, \overset{\downarrow}{-1}] * [\overset{\downarrow}{1}, -2]$ and $[\overset{\downarrow}{1}, -2] * [2, \overset{\downarrow}{-1}]$. These trivial convolutions are

$$[2, \overset{\downarrow}{-1}] * [\overset{\downarrow}{1}, -2] = [\overset{\downarrow}{1}, -2] * [2, \overset{\downarrow}{-1}] = [2, \overset{\downarrow}{-5}, 2].$$

Thus, $y_f[n] = x[n] * h[n] = [2, \overset{\downarrow}{-5}, 2] * [2, \overset{\downarrow}{-5}, 2]$, and we see that

$$h[n - m] = [2, -5, \overset{\overset{m=n+1}{\downarrow}}{2} \;] \quad \text{and} \quad x[m] = [\; \overset{\overset{m=-1}{\downarrow}}{2} \;, -5, 2].$$

Sliding $h[n - m]$ across $x[m]$, the convolution is computed as

n	$y_e[n]$
< -2	0
-2	$2(2) = 4$
-1	$2(-5) - 5(2) = -20$
0	$2(2) - 5(-5) + 2(2) = 33$
1	$-5(2) + 2(-5) = -20$
2	$2(2) = 4$
> 2	0

Thus,

$$y_f[n] = [4, -20, \overset{\overset{n=0}{\downarrow}}{33}, -20, 4].$$

Solution 3.8-26

By definition, we know that

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m].$$

Since $x[m]$ is nonzero only for $-3 \le m \le 1$, this convolution is the sum of five (shifted-and-scaled) versions of $h[n]$. That is,

							$\overset{n=0}{\downarrow}$	
$x[-3]h[n - (-3)]$	-2	-1	1	2	0	0	0	0
$x[-2]h[n - (-2)]$		-4	-2	2	4	0	0	0
$x[-1]h[n - (-1)]$			-6	-3	3	6	0	0
$x[0]h[n - 0]$				-8	-4	4	8	0
$x[1]h[n - 1]$					-10	-5	5	10
$x[n] * h[n]$	-2	-5	-7	-7	-7	5	13	10

Thus,

$$x[n] * h[n] = [-2, -5, -7, -7, -7, 5, \overset{\overset{n=0}{\downarrow}}{13}, 10].$$

Using the linearity and shift properties, we know that $y[n] = (2x[n - 30]) * \left(-\frac{3}{2}h[n - 10]\right)$ is just -3 times $x[n] * h[n]$ righted-shifted by 40. Thus,

$$y[n] = (2x[n - 30]) * \left(-\tfrac{3}{2}h[n - 10]\right)$$

$$= -3[-2, -5, -7, -7, -7, 5, \overset{\overset{n=40}{\downarrow}}{13}, 10]$$

$$= [6, 15, 21, 21, 21, -15, \overset{\overset{n=40}{\downarrow}}{-39}, -30].$$

Solution 3.8-27

In this problem, notice that m designates a fixed shift parameter and does not reflect the convolution sum variable typically used in the chapter.

(a) The strips corresponding to $u[n]$ and the flipped and shifted $u[n]$ are shown in the Fig. S3.8-27a for $n = 0$ (no shift)) and $n = 1$ (shift by one). We see that if $c[n] = u[n] * u[n]$, then

$$c[0] = 1, \quad c[1] = 2, \quad c[2] = 3, \quad c[3] = 4, \quad c[4] = 5, \quad c[5] = 6, \cdots, c[n] = n + 1.$$

Hence,
$$u[n] * u[n] = (n + 1)u[n].$$

(b) The appropriate strips for the two functions $u[n] - u[n-m]$ and $u[n]$ are shown in Figure S??b. The upper strip corresponding to $u[n] - u[n-m]$ has the first m slots with value 1 and all the remaining slots have value 0. The lower strip corresponding to a flipped and shifted $u[n]$ has values of 1 for all slots left of position n. From this figure it follows that

$$c[0] = 1, \quad c[1] = 2, \quad c[2] = 3, \cdots, c[m-1] = m = c[m] = c[m+1] = \cdots.$$

Hence,
$$c[n] = (n + 1)u[n] - (n - m + 1)u[n - m].$$
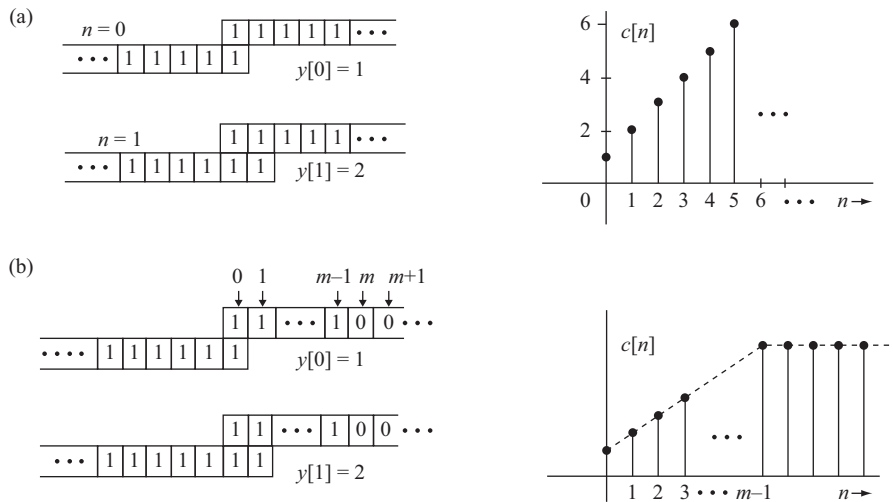
Figure S3.8-27

Solution 3.8-28

From Fig. S3.8-28 we observe that:

n	$y[n]$	
0	$0 + 1 + 2 + 3 + 4 + 5 = 15$	$y[n] = 0 \quad k \geq 6$
1	$1 + 2 + 3 + 4 + 5 = 15$	
2	$2 + 3 + 4 + 5 = 14$	$y[n] = 15 \quad n < 0$
3	$3 + 4 + 5 = 12$	
4	$4 + 5 = 9$	
5	5	
6	0	
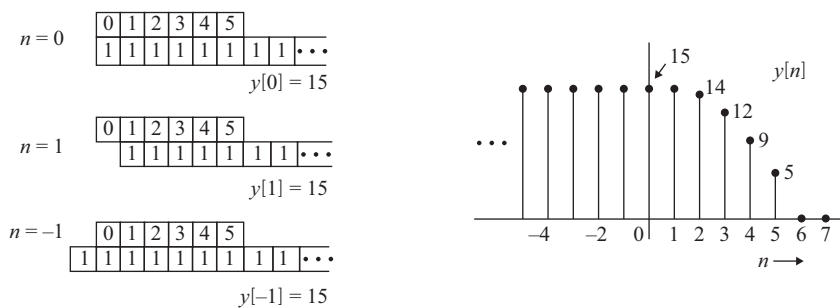
Figure S3.8-28

Solution 3.8-29

From Fig. S3.8-29, we observe the following values for $y[n]$:

n	$y[n]$	n	$y[n]$
0	$5 \times 5 + 5 \times 5 = 50$	± 11	$0 \times 0 + 5 \times 4 = 20$
± 1	$5 \times 4 + 0 = 20$	± 12	$0 \times 0 + 5 \times 3 = 15$
± 2	$5 \times 3 + 0 = 15$	± 13	$0 \times 0 + 5 \times 2 = 10$
± 3	$5 \times 2 + 0 = 10$	± 14	$0 \times 0 + 5 \times 1 = 5$
± 4	$5 \times 1 + 0 = 5$	± 15	$0 \times 0 + 0 \times 0 = 0$
± 5	$5 \times 0 + 0 = 0$	± 16	0
$\cdots$	$\cdots$	± 17	0
± 9	$0 \times 0 + 0 \times 0 = 0$	± 18	0
± 10	$0 \times 0 + 5 \times 5 = 25$		

Further, observe that

$$y[n] = 0 \qquad 5 \le |n| \le 9 \quad \text{and} \quad |n| \ge 15.$$
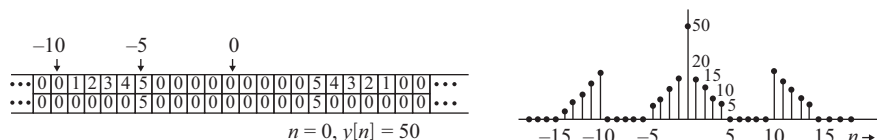
Figure S3.8-29

Solution 3.8-30

(a) From Fig. S3.8-30, we observe the following values of $y[n]$:

n	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	$|n| > 7$
$y[n]$	7	6	5	4	3	2	1	0	0

(b) The answer is identical to that of (a). This is because when we lay the tapes $x[m]$ and $g[-m]$ together, the situation is identical to that in (a).
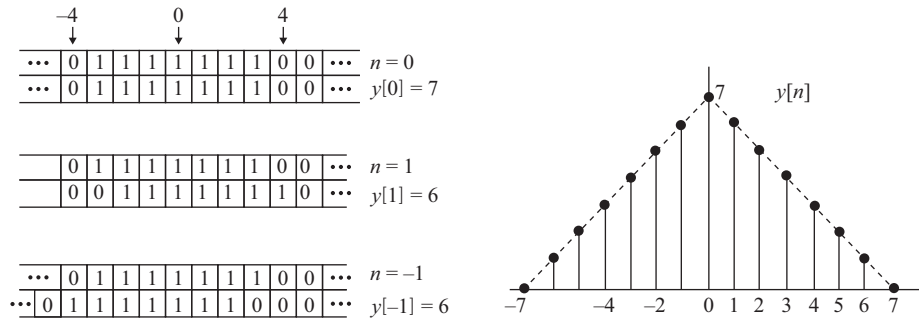
Figure S3.8-30

Solution 3.8-31

This problem considers the convolution

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m],$$

where $x[n]$ and $y[n]$ are known and $h[n]$ is unknown. The nonzero values of signal $x[n]$ are $[1, 2, \overset{\downarrow}{2}]$ and the nonzero values of signal $y[n]$ are $[3, 4, 6, 6, 11, \overset{\downarrow}{2}, -2]$. By the convolution width property, $h[n]$ must have length $7 - 3 + 1 = 5$. Following the sliding tape method, we see that

$$x[n-m] = [2, 2, \overset{\overset{m=n+2}{\downarrow}}{1}] \quad \text{and} \quad h[m] = [\overset{\overset{m=M}{\downarrow}}{a}, b, c, d, e].$$

To produce the given output $y[n]$ with leftmost nonzero value at $n = -5$ requires that

$$n + 2\big|_{n=-5} = M \quad \Rightarrow \quad M = -3.$$

Sliding $x[n-m]$ across $h[m]$ five times, we use the known values of $x[n]$ and $y[n]$ to compute the five unknown values of $h[n]$. That is,

n	$y[n]$
< -3	0
-3	$a = 3$
-2	$b + 2(a) = b + 6 = 4 \quad \Rightarrow b = -2$
-1	$c + 2(b) + 2(a) = c - 4 + 6 = 6 \quad \Rightarrow c = 4$
0	$d + 2(c) + 2(b) = d + 8 - 4 = 6 \quad \Rightarrow d = 2$
1	$e + 2(d) + 2(c) = e + 4 + 8 = 11 \quad \Rightarrow e = -1$

Continuing to slide $x[n-m]$ across $h[m]$, we see our computed values of $h[n]$ confirm the final values of $y[n]$,

n	$y[n]$
2	$1(0) + 2(e) + 2(d) = -2 + 4 = 2$
3	$1(0) + 2(0) + 2(e) = -2$

Thus,

$$h[n] = [3, -2, 4, \overset{\overset{n=0}{\downarrow}}{2}, -1].$$

Solution 3.8-32

(a)

$$8 = h[0]$$
$$12 = h[1] + h[0] \implies h[1] = 12 - 8 = 4$$
$$14 = h[2] + h[1] + h[0] \implies h[2] = 2$$
$$15 = h[3] + h[2] + h[1] + h[0] \implies h[3] = 1$$
$$15.5 = h[4] + h[3] + h[2] + h[1] + h[0] \implies h[4] = 0.5$$
$$15.75 = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] \implies h[5] = 0.25$$

Thus,

$$h[n] = [\ \overset{\overset{n=0}{\downarrow}}{8}\ , 4, 2, 1, 0.5, 0.25, \cdots].$$

(b)

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \implies \mathbf{H}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{H}^{-1}\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7/3 \\ 43/9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ 1/9 \end{bmatrix}$$

Hence the input sequence is: $(1,\ 1/3,\ 1/9,\ \cdots)$

Solution 3.8-33

Using the sifting property, the zero-input response is rewritten as $y_0[n] = 2u[n] + (1/3)^n u[n]$.

(a) Since the system is second-order, it has two characteristic roots. Both roots appear in the zero-input response, and are easily identified as $\gamma_1 = 1$ and $\gamma_2 = 1/3$. Let $a_0 = 1$ (standard form), so $\gamma^2 + a_1\gamma + a_2 = (\gamma - 1)(\gamma - 1/3) = \gamma_2 - 4/3\gamma + 1/3$. Thus,

$$a_0 = 1, a_1 = -4/3, \text{ and } a_2 = 1/3.$$

(b) To produce a strong response, the input should be close to a natural mode. Since the system has two bounded modes, any linear combination of these modes will produce a strong response. That is, a strong response is generated in response to

$$x[n] = (c_1\gamma_1^n + c_2\gamma_2^n)\,u[n] = (c_1 + c_2(1/3)^n)\,u[n],$$

where c_1 and c_2 are arbitrary (nonzero) constants.

(c) A causal, bounded, and infinite duration input is conveniently chosen in the form $x[n] = \gamma^n u[n]$, where $|\gamma| \leq 1$. To produce a weak response, the input should be far from the system's natural modes. In the complex plane, the farthest point γ from the modes $\gamma_1 = 1$ and $\gamma_2 = 1/3$ with $|\gamma| \leq 1$ is $\gamma = -1$. Thus, a relatively weak response is generated in response to

$$x[n] = c_1(-1)^n u[n],$$

where c_1 is an arbitrary nonzero constant.

Solution 3.8-34

(a) MATLAB is used to plot $h_1[n]$ and $h_2[n]$ (see Fig. S3.8-34).

```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0);
>>  u = @(n) 1.0*(n>=0).*(mod(n,1)==0); n = -10:10;
>>  h1 = @(n) delta(n+2)-delta(n-2); h2 = @(n) n.*(u(n+4)-u(n-4));
>>  subplot(221); stem(n,h1(n),'k.'); axis on; grid on;
>>  xlabel('n'); ylabel('h_1[n]'); axis([-10.5 10.5 -5 5]);
>>  set(gca,'xtick',-10:2:10,'ytick',-5:2:5);
>>  subplot(222); stem(n,h2(n),'k.'); axis on; grid on;
>>  xlabel('n'); ylabel('h_2[n]'); axis([-10.5 10.5 -5 5]);
>>  set(gca,'xtick',-10:2:10,'ytick',-5:2:5);
```

(b) The impulse response of two systems in parallel is just the sum of the individual impulse responses,
$$h_p[n] = h_1[n] + h_2[n].$$
MATLAB easily computes and plots $h_p[n]$ (see Fig. S3.8-34).

```
>>  hp = @(n) h1(n)+h2(n); subplot(223); stem(n,hp(n),'k.'); axis on; grid on;
>>  xlabel('n'); ylabel('h_p[n]'); axis([-10.5 10.5 -5 5]);
>>  set(gca,'xtick',-10:2:10,'ytick',-5:2:5);
```

(c) The impulse response of two systems in series is just the convolution of the individual impulse responses,
$$h_s[n] = h_1[n] * h_2[n].$$
MATLAB easily computes and plots $h_s[n]$ (see Fig. S3.8-34).

```
>>  hs = conv(h1(n),h2(n)); n = -20:20; subplot(224); stem(n,hs,'k.'); axis on;
>>  grid on; xlabel('n'); ylabel('h_s[n]'); axis([-10.5 10.5 -5 5]);
>>  set(gca,'xtick',-10:2:10,'ytick',-5:2:5);
```
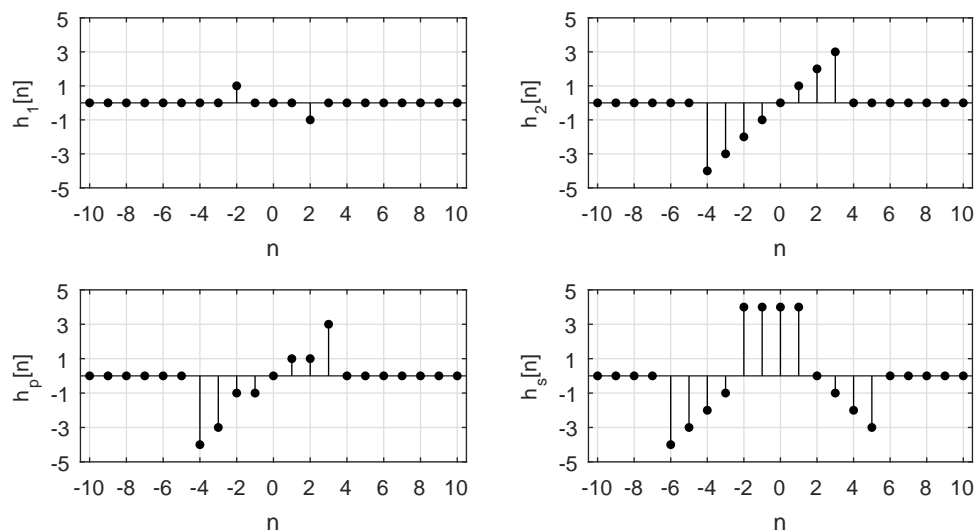

Figure S3.8-34

Solution 3.8-35

(a) $(z^3 + z^2 + z + 1)^2 = z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1$. MATLAB is used to compute the convolution $[1111] * [1111]$.

```
>>  conv([1 1 1 1],[1 1 1 1])
    ans = 1 2 3 4 3 2 1
```

Notice, the coefficients of the polynomial expansion are the same as the terms from the convolution.

(b) It is possible to expand the product of two polynomial expressions by convolving the coefficients from each ordered polynomial expression. Repeated convolution can be used to expand the product of more than two polynomial expressions. Care must be used to include all coefficient terms, including zeros, from the highest power to the lowest, and coefficients need to be ordered in a consistent manner.

(c) $(z^{-4} - 2z^{-3} + 3z^{-2})^4 = ((z^{-4} - 2z^{-3} + 3z^{-2})^2)^2$. First, use MATLAB and convolution to compute $(z^{-4} - 2z^{-3} + 3z^{-2})^2$.

```
>>  temp = conv([1 -2 3],[1 -2 3]);
```

Convolution is used to square this intermediate result and obtain $(z^{-4} - 2z^{-3} + 3z^{-2})^4$.

```
>>  conv(temp,temp)
    ans = 1 -8 36 -104 214 -312 324 -216 81
```

Recognizing that the highest power of z^{-1} in the result must be 16, the expansion is

$$z^{-16} - 8z^{-15} + 36z^{-14} - 104z^{-13} + 214z^{-12} - 312z^{-11} + 324z^{-10} - 216z^{-9} + 81z^{-8}.$$

(d) First, use convolution to expand $(z^5 + 2z^4 + 3z^2 + 5)^2$.

```
>>  temp = conv([1 2 0 3 0 5],[1 2 0 3 0 5]);
```

Convolution is again used to multiply the resulting polynomial by $(13 - 5z^{-2} + z^{-4})$. It is important to order the coefficients in descending powers of z to be compatible with the first result.

```
>>  conv(temp,[13 0 -5 0 1])
    ans = 13 52 47 58 137 104 321 -44 257 10 204 0 -95 0 25
```

Recognizing that the highest power of z in the result must be 10, the expansion is

$$13z^{10} + 52z^9 + 47z^8 + 58z^7 + 137z^6 + 104z^5 + 321z^4 - 44z^3 + 257z^2 + 10z + 204 - 95z^{-2} + 25z^{-4}.$$

Solution 3.8-36

(a) Actually, only a first-order difference equation is necessary to describe this system. On refill n, the amount of sugar $y[n]$ is equal to the amount added $x[n]$ plus whatever was still in the mug. Since Joe drinks $2/3$ of his cup of coffee before each refill, one-third of the sugar from the previous cup remains. Thus, $y[n] = y[n-1]/3 + x[n]$. Thus,

$$a_1 = -1/3, a_2 = 0, b_0 = 1, b_1 = 0, \text{ and } b_2 = 0.$$

In standard form, the difference equation is

$$y[n] - y[n-1]/3 = x[n].$$

(b) Since Joe adds two teaspoons of sugar each time he fills his cup,

$$x[n] = 2u[n].$$

(c) The total solution to a difference equation is the sum of the zero-input response and the zero-state response. Since Joe starts with a clean mug, $y[-1] = 0$ and the zero-input response is necessarily zero, $y_0[n] = 0$. Thus, the total solution is just the zero-state solution.

To obtain the zero-state solution, the impulse response $h[n]$ is needed. In this case, $h[n] = \frac{0}{1/3} + \tilde{y}_0[n]u[n]$, where $\tilde{y}_0[n] = c\gamma^n$. To determine c, input $x[n] = \delta[n]$ into the original difference equation to yield $h[n] - h[n-1]/3 = \delta[n]$. Since $h[n]$ is causal, $h[0] - h[-1]/3 = h[0] = \delta[0] = 1 = \tilde{y}_0[0]u[0] = c$. Thus, $h[n] = 3^{-n}u[n]$. The zero-state solution is

$$x[n] * h[n] = \left(\sum_{k=0}^n 2(3)^{-n}\right)u[n] = 2\frac{1-3^{-(n+1)}}{1-1/3}u[n] = (3 - 3^{-n})u[n],$$

and

$$y[n] = \left(3 - 3^{-n}\right)u[n].$$

(d)

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty}\left(3 - 3^{-n}\right) = 3.$$

That is, after many cups of coffee, Joe's mug reaches a steady-state of three teaspoons of sugar.

One way to make Joe's coffee remain a constant for all non-negative n is begin at the steady-state value of three and then add two teaspoons of sugar at each refill. That is,

$$x[n] = 2u[n] + \delta[n].$$

The added $\delta[n]$ "jump starts" the sugar content of the first cup to the steady-state value.

If Joe desires a steady value of two teaspoons, which is two-thirds the original steady-state value, the input is simply scaled by 2/3. That is, the steady level $y[n] = 2u[n]$ is achieved using the input

$$x[n] = \frac{4}{3}u[n] + \frac{2}{3}\delta[n].$$

Solution 3.8-37

(a) The characteristic equation is $j\gamma + 0.5 = 0$. Thus, the characteristic root of the system is $\gamma = j0.5$. The impulse response has form $h[n] = \frac{b_N}{a_N}\delta[n] + c\gamma^n u[n] = c(j0.5)^n u[n]$. To determine c, evaluate the original difference equation according to $jh[0] + 0.5h[-1] = jh[0] = -5\delta[0] = -5$. Thus, $h[0] = j5 = c$. Taken together,

$$h[n] = j5(j0.5)^n u[n].$$

(b) First, compute the zero-input response $y_0(t) = c(jy0.5)^n$. To find c, use the initial condition $y_0[-1] = j = c(jy0.5)^{-1}$. Thus, $c = jy^2 0.5 = -0.5$ and $y_0[n] = -0.5(j0.5)^n$.

The zero-state response is

$$h[n] * x[n] = \left(\sum_{k=0}^{n-5} j5(j0.5)^k\right)u[n-5] = \left(j5\frac{1-(j0.5)^{n-4}}{1-j0.5}\right)u[n-5].$$

Adding the zero-input and zero-state responses yields the total response for $n \geq 0$,

$$y[n] = -0.5(j0.5)^n + \left(j5\frac{1-(j0.5)^{n-4}}{1-j0.5}\right)u[n-5].$$

Solution 3.8-38

(a) MATLAB is used to plot the function $h[n] = n(u[n-2] - u[n+2])$ (see Fig. S3.8-38).

```
>>  u = @(n) 1.0*(n>=0).*(mod(n,1)==0);
>>  n = -5:5; h = @(n) n.*(u(n-2)-u(n+2));
>>  stem(n,h(n),'k.'); axis on; grid on;
>>  xlabel('n'); ylabel('h[n]'); axis([-5.5 5.5 -2.5 2.5]);
```
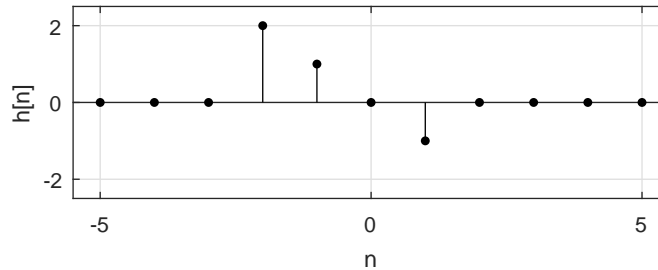

Figure S3.8-38

(b) Using Fig. S3.8-38, write $h[n] = 2\delta[n+2] + \delta[n+1] - \delta[n-1]$. A difference equation representation immediately follows from this form,

$$y[n] = 2x[n+2] + x[n+1] - x[n-1].$$

Solution 3.8-39

For this solution, consider the signals $x[n] = \delta[n] - \delta[n-1] = [1, -1]$, $y[n] = z[n] = \delta[n] + \delta[n-1] = [1, 1]$. In this case, $x[n]\,(y[n] * z[n]) = [1, -2]$.

(a) Not equivalent. By counter-example, $(x[n] * y[n])z[n] = [1] \neq [1, -2] = x[n]\,(y[n] * z[n])$.

(b) Not equivalent. By counter-example, $(x[n]y[n]) * (x[n]z[n]) = [1, -2, 1] \neq [1, -2] = x[n]\,(y[n] * z[n])$.

(c) Not equivalent. By counter-example, $(x[n]y[n]) * z[n] = [1, 0, -1] \neq [1, -2] = x[n]\,(y[n] * z[n])$.

(d) True. None of the above expressions is equivalent to $x[n]\,(y[n] * z[n])$.

Solution 3.8-40

In this problem, a causal system with input $x[n]$ and output $y[n]$ is described by difference equation $y[n] - ny[n-1] = x[n]$. Since the system is causal, its impulse response $h[n]$ equals 0 for $n < 0$.

(a) Substituting $\delta[n]$ for $x[n]$ and $h[n]$ for $y[n]$, we can determine $h[n]$ through recursion according to

$$h[n] = nh[n-1] + \delta[n].$$

The first six nonzero values of $h[n]$ are thus

n	0	1	2	3	4	5
$h[n]$	$1 = 0!$	$1(1) = 1 = 1!$	$2(1) = 2 = 2!$	$3(2) = 6 = 3!$	$4(6) = 24 = 4!$	$5(24) = 120 = 5!$

That is,

$$h[n] = [\overset{\downarrow}{0}, 1, 1, 2, 6, 24, 120, \ldots]$$

or, more generally,

$$h[n] = \left(\sum_{i=0}^{\infty} i!\delta[n-i]\right) u[n].$$

The system is not BIBO stable since $h[n]$ grows without bound and is not absolutely summable.

(b) Assuming all initial conditions are zero, we can determine the system output $y_R[n]$ in response to $x[n] = u[n]$ through recursion according to

$$y_R[n] = ny_{|rmR}[n-1] + u[n].$$

Thus,

n	0	1	2	3	4
$y_R[n]$	$0(0) + 1 = 1$	$1(1) + 1 = 2$	$2(2) + 1 = 5$	$3(5) + 1 = 16$	$4(16) + 1 = 65$

At $n = 4$, we see that
$$y_R[4] = 65.$$

(c) Designating $y_C[n] = h[n] * u[n]$, we compute $y_C[4]$ as

$$y_C[4] = \sum_{m=-\infty}^{\infty} h[m]u[4-m].$$

Since $h[n]$ and $x[n]$ are both causal, we see that

$$y_C[4] = \sum_{m=0}^{4} h[m] = 1 + 1 + 2 + 6 + 24 = 34.$$

(d) From parts (b) and (c), we see that $y_R[4] = 65 \neq 34 = y_C[4]$. That is, the recursive solution and the convolution solution do not agree. The recursive solution $y_R[n]$ correctly reflects the zero state response of the system. The convolution solution $y_C[n] = h[n] * x[n]$ can compute the zero state response if the system is both linear and time-invariant. Although this system is linear, it is not time-invariant. Thus, $y_C[n] = h[n] * x[n]$ cannot correctly compute the zero state response. Notice that although the system is BIBO unstable, this fact does not explain why $y_R[n] \neq y_C[n]$.

Solution 3.9-1

Assume that a system exists that violates Eq. (3.43) and yet produces bounded output for every bounded input. The system response at $n = n_1$ is

$$y[n_1] = \sum_{m=0}^{\infty} h[m]x[n_1 - m]$$

Consider a bounded input $x[n]$ such that

$$x[n_1 - m] = \begin{cases} 1 & \text{if} \quad h[m] > 0 \\ -1 & \text{if} \quad h[m] < 0 \end{cases}$$

In this case
$$h[m]x[n_1 - m] = |h[m]|$$

and
$$y[n_1] = \sum_{m=0}^{\infty} |h[m]| = \infty.$$

This violates the assumption.

Solution 3.9-2

(a) Here,
$$\gamma^2 + 0.6\gamma - 1.6 = (\gamma - 0.2)(\gamma + 0.8).$$

The characteristic roots are 0.2 and -0.8. Since both are inside the unit circle,

the system is BIBO stable and asymptotically stable.

(b) In this case,
$$\gamma^2 + 3\gamma + 2 = (\gamma + 2)(\gamma + 1).$$

The characteristic roots are -1 and -2. Since one root outside the unit circle and the other is on the unit circle,

the system is BIBO unstable and asymptotically unstable.

(c) Now,
$$(\gamma - 1)^2(\gamma + \frac{1}{2}).$$

The characteristic roots are 1 (multiplicity two) and -0.5. Since there is a repeated root on unit circle,

the system is BIBO unstable and asymptotically unstable.

(d) In this case,
$$\gamma^2 + 2\gamma + 0.96 = (\gamma + 0.8)(\gamma + 1.2).$$

The characteristic roots are -0.8 and -1.2. Since the root (-1.2) is outside the unit circle,

the system is BIBO unstable and asymptotically unstable.

(e) Here,
$$\gamma^2 + \gamma - 2 = (\gamma + 0.5 + j1.5)(\gamma + 05 - j1.5).$$

The characteristic roots are $-0.5 \pm j1.5$. Since there are roots outside the unit circle,

the system is BIBO unstable and asymptotically unstable.

(f) In the final case,
$$(\gamma^2 - 1)(\gamma^2 + 1) = (\gamma + 1)(\gamma - 1)(\gamma + j1)(\gamma - j1).$$

The characteristic roots are ± 1 and $\pm j1$. Since we have simple (unrepeated) roots on the unit circle and no roots outside the unit circle,

the system is BIBO unstable and marginally stable.

Solution 3.9-3

The system $\mathcal{S}_1$ is asymptotically (and BIBO) unstable. The system $\mathcal{S}_2$ is BIBO and asymptotically stable. If we cascade the two systems, the impulse response of the composite system is

$$h[n] = 2^n u[n] * (\delta[n] - 2\delta[n-1]) = 2^n u[n] - 2(2)^{n-1} u[n-1] = \delta[n].$$

The composite system is BIBO stable. However, the system $\mathcal{S}_1$ will burn out (or saturate) because its output contains a signal of the form 2^n.

Solution 3.9-4

(a) To be unstable, a causal mode must have magnitude greater than one. That is, at least one characteristic root must be outside the unit circle. By this criteria,

systems D, E, and I are unstable.

(b) To be real, the characteristic modes need to be either real or in complex-conjugate pairs. By this criteria,

systems A, B, C, D, E, G, I, and J are real.

(c) Oscillatory modes include sinusoids, decaying sinusoids, or exponentially growing sinusoids. Unless the characteristic roots are all real and positive, the corresponding natural mode(s) will exhibit oscillatory behavior. By this criteria,

systems A, B, C, D, E, F, G, H, and I have oscillatory natural modes.

(d) To have a mode that decays at a rate of 2^{-n}, at least one characteristic root needs to lie on the circle of radius one-half centered at the origin. By this criteria,

systems B, D, F, and J have at least one mode that decays by 2^{-n}

(e) For a second-order system with two finite roots to only have one mode, one characteristic root needs to be located at the origin. By this criteria,

systems F, H, and J have only one mode.

Solution 3.9-5

Notice, the system response can be written more simply as $h[n] = \delta[n] + \left(\frac{1}{3}\right)^n u[n-1] = \left(\frac{1}{3}\right)^n u[n]$.

(a) The system is stable since the impulse response function is absolutely summable. That is, $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1-0}{1-1/3} = 3/2 < \infty$. The system is causal since $h[n] = 0$ for $n < 0$.

(b) MATLAB is used to plot $x[n]$ (see Fig. S3.9-5).

```
>>  u = @(n) 1.0*(n>=0).*(mod(n,1)==0);
>>  n = -10:10; x = @(n) (u(n-3)-u(n+3));
>>  subplot(121); stem(n,x(n),'k.'); axis on; grid on;
>>  xlabel('n'); ylabel('x[n]'); axis([-5.5 5.5 -1.1 0.1]);
```

(c) The zero-state response is computed as $y[n] = x[n] * h[n]$. This convolution involves three regions.

For $n < -3$, $y[n] = 0$.

For $-3 \leq n < 2$, $y[n] = \sum_{k=-3}^{n} -(1/3)^{n-k} = -(1/3)^n \sum_{k=-3}^{n} 3^k = -(1/3)^n \frac{3^{-3}-3^{n+1}}{1-3} = \frac{3^{-(n+3)}-3}{2}$.

For $n \geq 2$, $y[n] = \sum_{k=-3}^{2} -(1/3)^{n-k} = -(1/3)^n \sum_{k=-3}^{2} 3^k = -(1/3)^n \frac{3^{-3}-3^3}{1-3} = -\frac{728}{54}(3)^{-n}$.

Combining yields

$$y[n] = \begin{cases} 0 & n < -3 \\ \frac{3^{-(n+3)}-3}{2} & -3 \leq n < 2 \\ -\frac{728}{54}(3)^{-n} & n \geq 2 \end{cases}.$$

MATLAB is used to plot the result (see Fig. S3.9-5).

```
>>  y = @(n) (3.^(-(n+3))-3)/2.*(u(n+3)-u(n-2))-728/54*(3).^(-n).*u(n-2);
>>  subplot(122); stem(n,y(n),'k.'); axis on; grid on;
>>  xlabel('n'); ylabel('y[n]'); axis([-10.5 10.5 -1.6 0.1]);
```
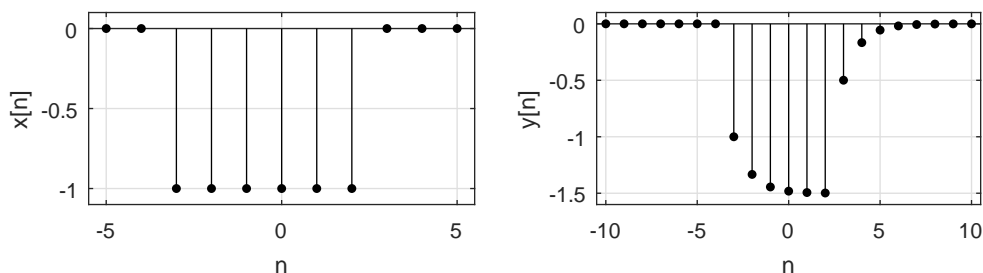
Figure S3.9-5

Solution 3.9-6

(a) No, the system is not causal since $h[n] \neq 0$ for $(n < 0)$.

(b)

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\left(\frac{1}{2}\right)^{|n|}| = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|}$$

$$= \sum_{n=-\infty}^{-1} 0.5^{-n} + \sum_{n=0}^{\infty} 0.5^n = \sum_{n=-\infty}^{-1} 2^n + \sum_{n=0}^{\infty} 0.5^n$$

$$= \frac{0 - 2^0}{1 - 2} + \frac{0.5^0 - 0}{1 - 0.5} = 1 + 2 = 3.$$

Since $h[n]$ is absolutely summable, the system is BIBO stable.

(c)

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} |x[n]|^2 = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} 9u[n-5]$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=5}^{n=N} 9 \lim_{N \to \infty} \frac{1}{2N+1} 9(N-5+1) = \lim_{N \to \infty} \frac{9N-36}{2N+1} = \frac{9}{2}.$$

Since the power P_x is finite, energy must be infinite.

$$E_x = \infty \text{ and } P_x = \frac{9}{2}.$$

(d) We know that $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$. Thus,

$$y[10] = \sum_{k=-\infty}^{\infty} h[k]x[10-k] = \sum_{k=-\infty}^{\infty} 0.5^{|k|}3u[10-5-k]$$

$$= \sum_{k=-\infty}^{-1} 3(2^k) + \sum_{k=0}^{5} 3(0.5^k) = 3\frac{0 - 2^0}{1 - 2} + 3\frac{0.5^0 - 0.5^6}{1 - 0.5}$$

$$= 3 + 3\frac{63}{32} = \frac{285}{32}.$$

Thus,

$$y[10] = \frac{285}{32} \approx 8.91.$$

Solution 3.10-1

Here, we consider just one of the many possible solutions to this problem. For the input $x[n] = 2(\frac{1}{3})^n u[-n-4]$ to cause resonance, the $(\frac{1}{3})^n$ term needs to exactly match a mode of the system. Thus, the system needs a characteristic root $\gamma = \frac{1}{3}$. A simple first-order system will do the trick:

$$y[n] - \frac{1}{3}y[n-1] = x[n].$$

Solution 3.10-2

Here, we consider just one of the many possible solutions to this problem. The causal LTID system described by $(E^2 + 1)\{y[n]\} = (E + 0.5)\{x[n]\}$ has characteristic roots $\gamma = \pm j$. To cause resonance, an input needs to match at least one of the corresponding characteristic modes, either $(j)^n = e^{j\pi n/2}$ or $(-j)^n = e^{-j\pi n/2}$. To keep the input real, we use both of modes. Thus,

real input $x[n] = 2\cos(\frac{\pi}{2}n) = e^{j\pi n/2} + e^{-j\pi n/2}$ causes resonance (both modes) in the system.

Solution 3.10-3

The time constant T_1 of the system with impulse response $h_1[n] = -(0.5)^n u[n]$ is

$$T_1 = \frac{-\sum_{n=0}^{\infty}(\frac{1}{2})^n}{-1} = \frac{-2}{-1} = 2.$$

The time constant T_2 of the system with impulse response $h_2[n] = 2(u[n] - u[n-4])$ is

$$T_2 = \frac{2\sum_{n=0}^{3}1}{2} = \frac{2(4)}{2} = 4.$$

Since $T_1 = 2 < 4 = T_2$,

system 1 would more efficiently transmit a binal communication signal.

That is, the faster system more efficiently transmits binary communication signals.

Solution 3.11-1

Here, we consider just one of the many possible coding solutions to this problem.

```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0); u = @(n) 1.0*(n>=0).*(mod(n,1)==0);
>>  x = @(n) delta(n)+u(n-50); n = -2:100; y = zeros(size(n)); y(n==-2)=2; y(n==-1)=2;
>>  for nstep = 0:100,
>>      y(n==nstep) = y(n==nstep-1)/3-y(n==nstep-2)/2+x(nstep);
>>  end
>>  stem(n,y,'k.'); grid on; axis([-.5 100.5 -1 1.5]); xlabel('n'); ylabel('y[n]');
```
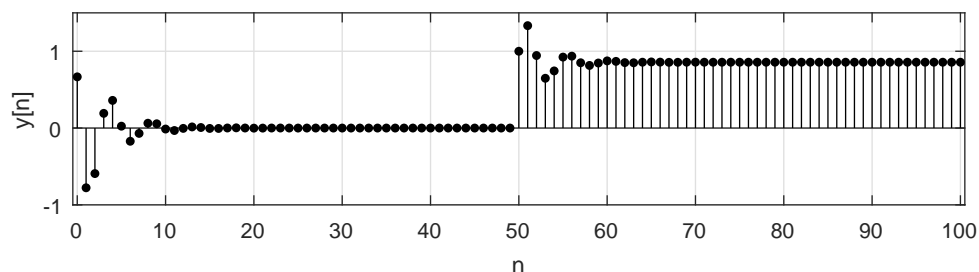

Figure S3.11-1

Solution 3.11-2

To accommodate upsampling, the function needs to be modified so that it assigns a zero for non-integer inputs.

```
>> f = @(n) exp(-n/5).*cos(pi*n/5).*(n>=0).*(n==fix(n));
```

The added term $(n==fix(n))$ is one if n is an integer and zero if n is not an integer. The modified function is easy to test.

```
>> n = -10:10; stem(n,f(n/2),'k.'); ylabel('f[n/2]'); xlabel('n'); axis([-10.5 10.5 -0.5 1.1]);
```
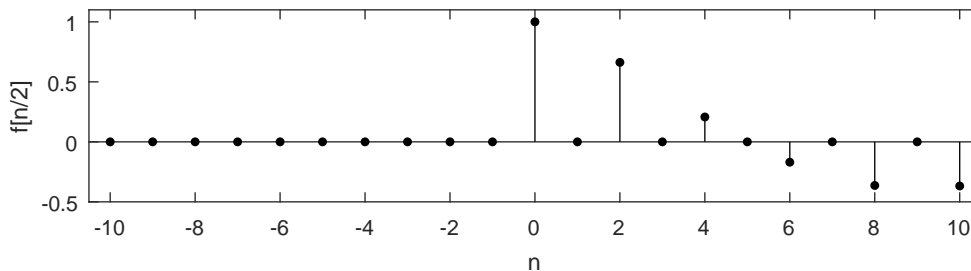

Figure S3.11-2

As shown in Fig. S3.11-2, `f[n/2]` inserts a zero between every sample of `f[n]`, which corresponds to the desired upsample-by-two operation.

Solution 3.11-3

(a)
```
>> x = [2,3,-2,-3]; h = [-10,0,-5]; nx = 0:3; nh = -1:1;
>> y = conv(x,h); n = (nx(1)+nh(1)):(nx(end)+nh(end));
>> stem(n,y,'k.'); grid on; xlabel('n'); ylabel('y_a[n]');
>> set(gca,'ytick',unique(y)); del = (max(y)-min(y))/20;
>> axis([n(1)-.5 n(end)+.5 min(y)-del max(y)+del]);
```
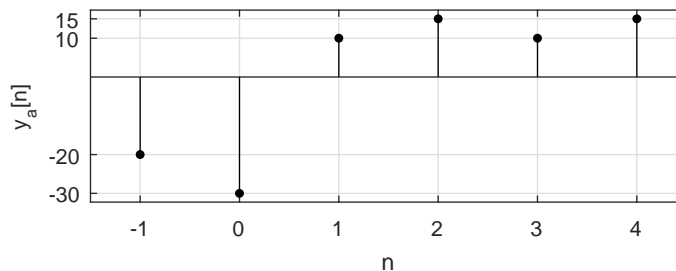

Figure S3.11-3a

From Fig. S3.11-3a,

$$y_\mathrm{a}[n] = [-20, \overset{\overset{n=0}{\downarrow}}{-30}, 10, 15, 10, 15].$$

This confirms the result of Prob. 3.8-25a.

(b)
```
>> x = [2,-1,3,-2]; h = [-1,-4,1,-2]; nx = -1:2; nh = -3:0;
>> y = conv(x,h); n = (nx(1)+nh(1)):(nx(end)+nh(end));
>> stem(n,y,'k.'); grid on; xlabel('n'); ylabel('y_b[n]');
>> set(gca,'ytick',unique(y)); del = (max(y)-min(y))/20;
>> axis([n(1)-.5 n(end)+.5 min(y)-del max(y)+del]);
```
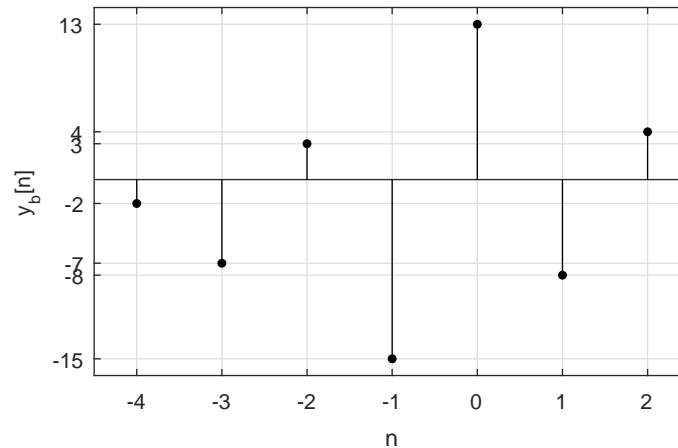
Figure S3.11-3b

From Fig. S3.11-3b,

$$y_{\mathrm{b}}[n] = [-2, -7, 3, -15, \overset{\overset{n=0}{\downarrow}}{13}, -8, 4].$$

This confirms the result of Prob. 3.8-25b.

(c)
```
>>  x = [3,2,1,2,3]; h = [2,3,-2,1]; nx = 2:6; nh = -3:0;
>>  y = conv(x,h); n = (nx(1)+nh(1)):(nx(end)+nh(end));
>>  stem(n,y,'k.'); grid on; xlabel('n'); ylabel('y_c[n]');
>>  set(gca,'ytick',unique(y)); del = (max(y)-min(y))/20;
>>  axis([n(1)-.5 n(end)+.5 min(y)-del max(y)+del]);
```
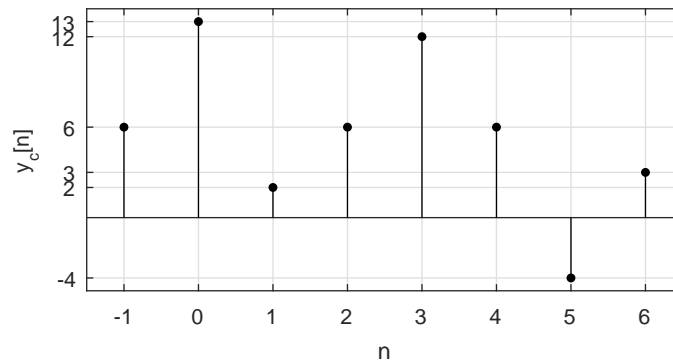

Figure S3.11-3c

From Fig. S3.11-3c,

$$y_{\mathrm{c}}[n] = [6, \overset{\overset{n=0}{\downarrow}}{13}, 2, 6, 12, 6, -4, 3].$$

This confirms the result of Prob. 3.8-25c.

(d)
```
>>  x = [5,0,0,-2,8]; h = [-1,1,3,3,-2,3]; nx = -3:1; nh = -2:3;
>>  y = conv(x,h); n = (nx(1)+nh(1)):(nx(end)+nh(end));
>>  stem(n,y,'k.'); grid on; xlabel('n'); ylabel('y_d[n]');
>>  set(gca,'ytick',unique(y)); del = (max(y)-min(y))/20;
>>  axis([n(1)-.5 n(end)+.5 min(y)-del max(y)+del]);
```
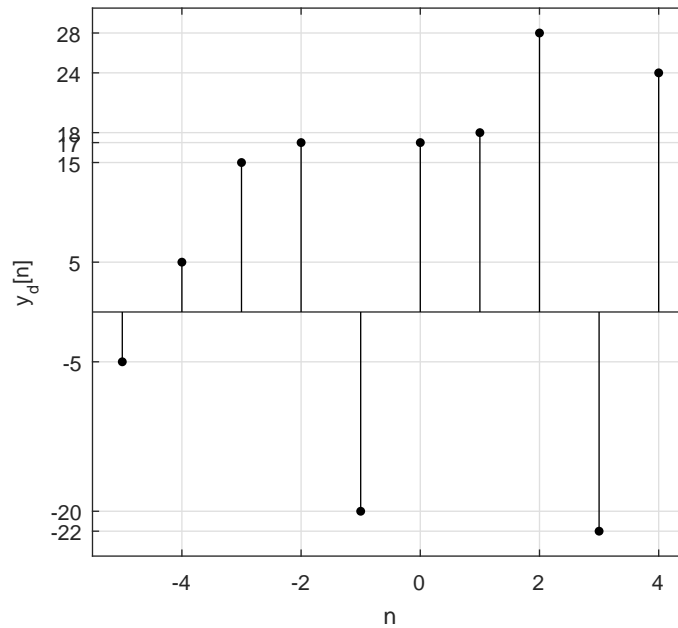
Figure S3.11-3d

From Fig. S3.11-3d,

$$y_{\mathrm{d}}[n] = [-5, 5, 15, 17, -20, \overset{\overset{n=0}{\downarrow}}{17}, 18, 28, -22, 24].$$

This confirms the result of Prob. 3.8-25d.

(e)
```
>>  x = conv([1,-1],[1,-1]); h = conv([1,-1],[1,-1]); nx = -1:1; nh = -1:1;
>>  y = conv(x,h); n = (nx(1)+nh(1)):(nx(end)+nh(end));
>>  stem(n,y,'k.'); grid on; xlabel('n'); ylabel('y_e[n]');
>>  set(gca,'ytick',unique(y)); del = (max(y)-min(y))/20;
>>  axis([n(1)-.5 n(end)+.5 min(y)-del max(y)+del]);
```
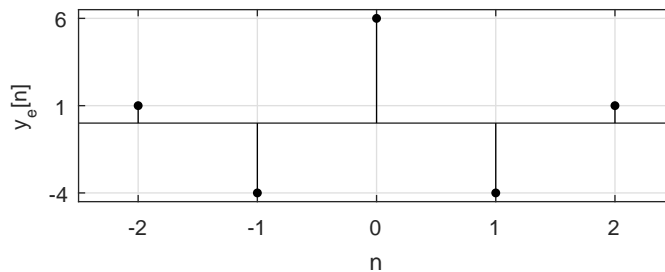

Figure S3.11-3e

From Fig. S3.11-3e,

$$y_{\mathrm{e}}[n] = [1, -4, \overset{\overset{n=0}{\downarrow}}{6}, -4, 1].$$

This confirms the result of Prob. 3.8-25e.

(f)
```
>>  x = conv([2,-1],[1,-2]); h = conv([1,-2],[2,-1]); nx = -1:1; nh = -1:1;
>>  y = conv(x,h); n = (nx(1)+nh(1)):(nx(end)+nh(end));
>>  stem(n,y,'k.'); grid on; xlabel('n'); ylabel('y_e[n]');
```

```
>>  set(gca,'ytick',unique(y)); del = (max(y)-min(y))/20;
>>  axis([n(1)-.5 n(end)+.5 min(y)-del max(y)+del]);
```
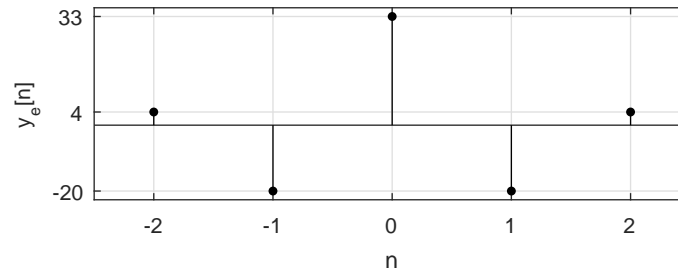

Figure S3.11-3f

From Fig. S3.11-3f,

$$y_f[n] = [4, -20, \overset{\overset{n=0}{\downarrow}}{33}, -20, 4].$$

This confirms the result of Prob. 3.8-25f.

Solution 3.11-4

There are many ways to solve this problem. For this solution, let $d[n]$ designate the distance from the student's destination, which alternates between home and the exam location, just before the student changes his mind. For even-valued n the destination is home, and for odd-valued n the destination is the exam location.

(a) Just before changing direction, the student is a distance of $d[n]$ miles from his destination. Turning around, his next destination is therefore a distance of $2-d[n]$ miles. The student travels one-half of this distance, which leaves $\frac{2-d[n]}{2}$ miles remaining. Thus, a difference equation description of this problem is $d[n+1] = \frac{2-d[n]}{2} = 1 - 0.5d[n] = u[n] - 0.5d[n]$. Rearranging and shifting by one yields

$$d[n] + 0.5d[n-1] = u[n-1].$$

For this description, $d[0] = 0$. This auxiliary condition simply states that before the student first decides to go to the exam, he is at home.

(b) MATLAB is used to iteratively simulate the difference equation.

```
>>  n = 0:20; d = zeros(size(n));
>>  for index = find(n>0), d(index) = 1-0.5*d(index-1); end; d
    d =         0 1.0000 0.5000 0.7500 0.6250 0.6875 0.6563 0.6719
           0.6641 0.6680 0.6660 0.6670 0.6665 0.6667 0.6666 0.6667
           0.6667 0.6667 0.6667 0.6667 0.6667
```

As time increases, the remaining distance to the destination (alternating home and exam) reaches a steady-state value of two-thirds of a mile. That is,

$$\lim_{n \to \infty} d[n] = 2/3.$$

Changing the problem so that the student travels two-thirds the remaining distance each time changes the result. In this case, the difference equation is $d[n] + \frac{1}{3}d[n-1] = \frac{2}{3}u[n]$.

```
>>  n = 0:20; d = zeros(size(n));
>>  for index = find(n>0), d(index) = (2-d(index-1))/3; end; d
    d =         0 0.6667 0.4444 0.5185 0.4938 0.5021 0.4993 0.5002
           0.4999 0.5000 0.5000 0.5000 0.5000 0.5000 0.5000 0.5000
           0.5000 0.5000 0.5000 0.5000 0.5000
```

In this case, the steady-state remaining distance is $\lim_{n \to \infty} d[n] = 1/2$.

(c) The closed form, or total, solution is the sum of the zero-input and zero-state responses. Since the auxiliary condition is $d[0] = 0$, the zero-input response is just zero. To compute the zero-state response, write the difference equation as $d[n] + 0.5d[n] = u[n-1] = x[n]$, where the "input" $x[n]$ is just a shifted unit step. In this way, $h[n] = (-1/2)^n u[n]$. The final solution is $d[n] = h[n] * x[n] = \left(\sum_{k=0}^{n-1} (-0.5)^k \right) u[n-1] = \frac{1-(-0.5)^n}{1-(-0.5)} u[n-1]$. Thus,

$$d[n] = \frac{2}{3} \left(1 - (-0.5)^n \right) u[n-1].$$

MATLAB is used to evaluate $d[n]$.

```
>>  n = 0:20; d = 2/3*(1-(-0.5).^n).*(n>0)
    d =
            0 1.0000 0.5000 0.7500 0.6250 0.6875 0.6563 0.6719
          0.6641 0.6680 0.6660 0.6670 0.6665 0.6667 0.6666 0.6667
          0.6667 0.6667 0.6667 0.6667 0.6667
```

These results are identical to the iterative solution, which provides good evidence that the solution is correct.

Solution 3.11-5

(a) Given $r_{xy}[k] = \sum_{n=-\infty}^{\infty} x[n]y[n-k]$, substituting $m = -n + k$ yields $r_{xy}[k] = \sum_{m=-\infty}^{\infty} y[-m]x[k-m] = y[-n] * x[n]$. Thus,

$$r_{xy}[k] = x[n] * y[-n].$$

Similarly, $r_{yx}[k] = y[n] * x[-n]$. In general, $x[n] * y[-n] \neq y[n] * x[-n]$. Thus,

$$r_{xy}[k] \neq r_{yx}[k].$$

It is true, however, that $r_{xy}[k] = r_{yx}[-k]$.

(b) Yes, cross-correlation indicates similarity between signals as a function of the *shift* between the two functions. That is, when the shift k aligns two similar signals, the two signals constructively interact and $r_{xy}[k]$ becomes large. A large negative correlation means that the first signal is very similar to the negative of the first signal.

(c) Here, we consider just one of the many possible coding solutions to this problem.

```
function [rxy,k] = crosscorr(x,y,nx,ny)
% function [rxy,k] = crosscorr(x,y,nx,ny)

% Ensure inputs are column vectors:
x=x(:); y=y(:);
% Reverse y and compute rxy using the conv command:
rxy = conv(x,flipud(y));
% Compute shifts:
k = [nx(1)-ny(end):nx(end)-ny(1)];
```

(d)
```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0); u = @(n) 1.0*(n>=0).*(mod(n,1)==0);
>>  nx = 0:20; x = u(nx-5)-u(nx-10);
>>  ny = -20:10; y = u(-ny-15)-u(-ny-10)+delta(ny-2);
>>  [rxy,k] = crosscorr(x,y,nx,ny);
>>  subplot(221); stem(nx,x,'k.'); grid on;
```

```
>>  xlabel('n'); ylabel('x[n]'); axis([0 20 -1.1 1.1]);
>>  subplot(222); stem(ny,y,'k.'); grid on;
>>  xlabel('n'); ylabel('y[n]'); axis([-20 10 -1.1 1.1]);
>>  subplot(212); stem(k,rxy,'k.'); grid on;
>>  xlabel('k'); ylabel('r_{xy}[k]');
```

As shown in Fig. S3.11-5, the largest magnitude of $r_{xy}[k]$ is five and occurs at $k = 19$. Signal $x[n]$ has a unit pulse of width five starting at $n = 5$. Signal $y[n]$ has a similar feature: a negative unit pulse of width five starting at $n = -14$. These two similar features are separated by a shift $k = 5 - (-14) = 19$. Since these are the most similar features between the two signals, it seems very sensible that the autocorrelation function has a large, negative value at shift $k = 19$.
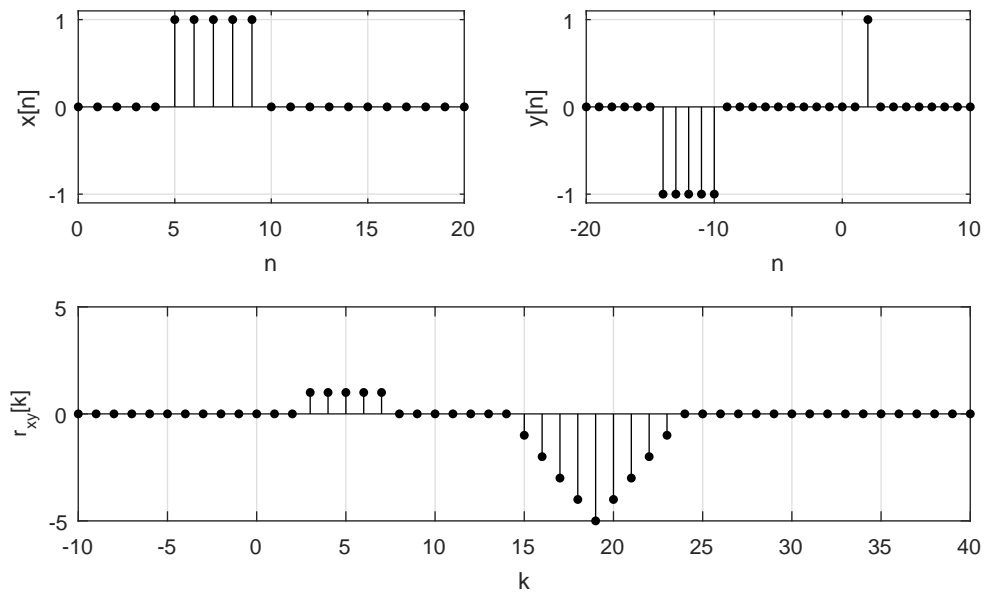
Figure S3.11-5

Solution 3.11-6

The functions to compute E_x and P_x are nearly trivial.

(a)
```
function [Ex] = Energy(x)
Ex = sum(x.*conj(x));
```

(b)
```
function [Px] = Power(x)
Px = sum(x.*conj(x))/length(x);
```

Solution 3.11-7

(a)
```
function [y] = filtermax(x,N)
% function [y] = filtermax(x,N)

M = length(x); x = x(:); x = [zeros(N-1,1);x]; y = zeros(M,1);
for m = 1:M,
    y(m) = max(x([m:m+(N-1)]));
end
```

(b)
```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0);
>>  n = 0:44; x = cos(pi*n/5)+delta(n-30)-delta(n-35);
```

```
>>  y4 = filtermax(x,4); y8 = filtermax(x,8); y12 = filtermax(x,12);
>>  subplot(221); stem(n,x,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('x[n]'); grid on;
>>  subplot(222); stem(n,y4,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=4'); grid on;
>>  subplot(223); stem(n,y8,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=8'); grid on;
>>  subplot(224); stem(n,y12,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=12'); grid on;
```
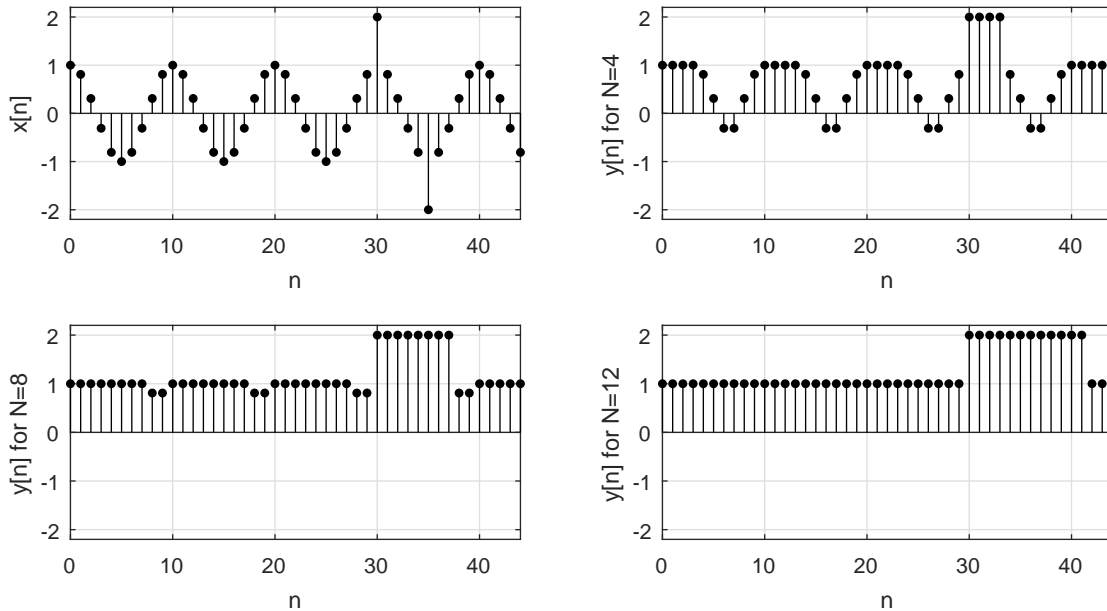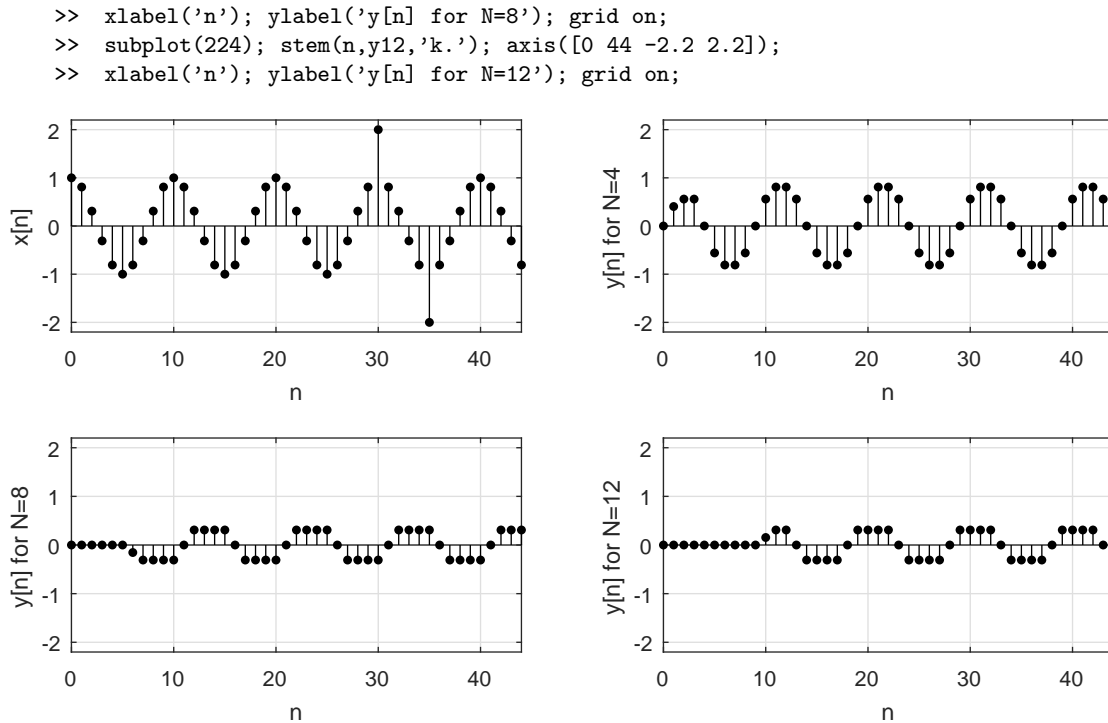

Figure S3.11-7

The plots are consistent with the expected behavior of a max filter. The output, which is always greater than or equal to the input, emphasizes large input values. Larger values of N cause particular maximum values to persist longer. The max filter is very sensitive to large, positive outliers, such as that caused by the added $\delta[n-30]$.

Also notice that the max filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. Furthermore, since a sinusoidal input does not result in a sinusoidal output, the max filter cannot be a LTI system (the max filter is TI but not linear).

Solution 3.11-8

(a)
```
function [y] = filtermin(x,N)
% function [y] = filtermin(x,N)

M = length(x); x = x(:); x = [zeros(N-1,1);x]; y = zeros(M,1);
for m = 1:M,
    y(m) = min(x([m:m+(N-1)]));
end
```

(b)
```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0);
>>  n = 0:44; x = cos(pi*n/5)+delta(n-30)-delta(n-35);
>>  y4 = filtermin(x,4); y8 = filtermin(x,8); y12 = filtermin(x,12);
>>  subplot(221); stem(n,x,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('x[n]'); grid on;
```

```
>>  subplot(222); stem(n,y4,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=4'); grid on;
>>  subplot(223); stem(n,y8,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=8'); grid on;
>>  subplot(224); stem(n,y12,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=12'); grid on;
```
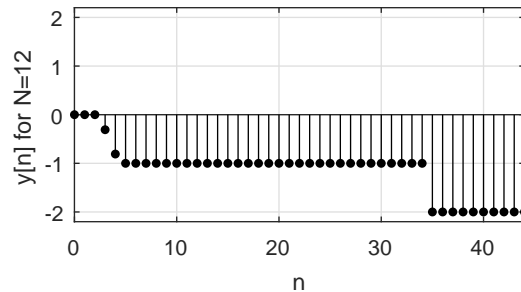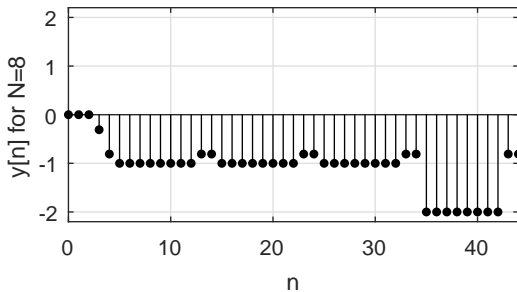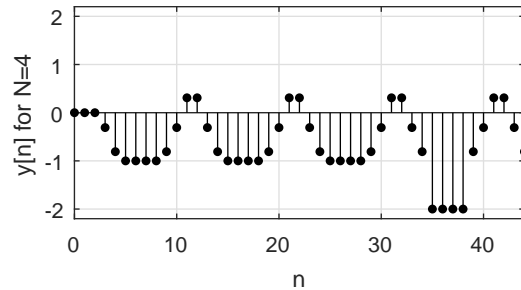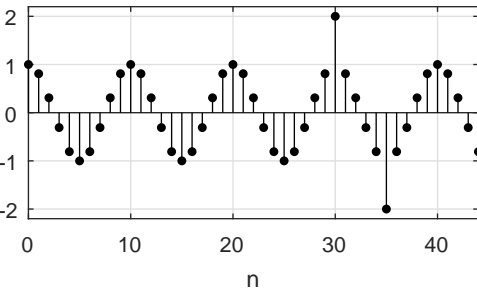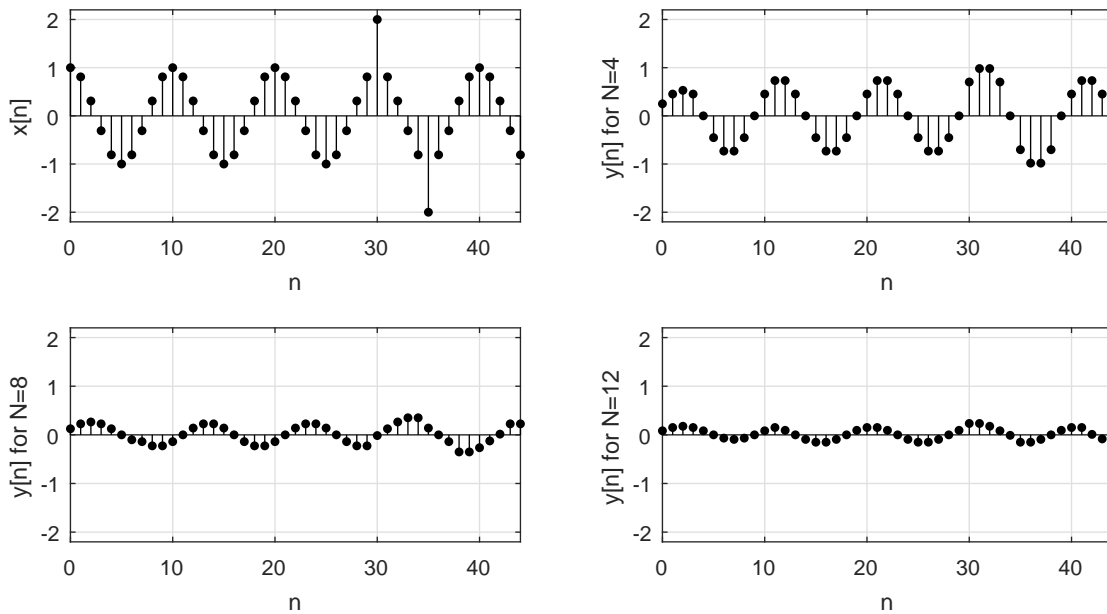

Figure S3.11-8

The plots are consistent with the expected behavior of a min filter. The output, which is always less than or equal to the input, emphasizes highly negative input values. Larger values of N cause particular minimum values to persist longer. The min filter is very sensitive to large, negative outliers, such as that caused by the added $-\delta[n-35]$.

Also notice that the min filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. Furthermore, since a sinusoidal input does not result in a sinusoidal output, the min filter cannot be a LTI system (the min filter is TI but not linear).

Solution 3.11-9

(a)
```
function [y] = filtermedian(x,N)
% function [y] = filtermedian(x,N)

M = length(x); x = x(:); x = [zeros(N-1,1);x]; y = zeros(M,1); for
m = 1:M,
    y(m) = median(x([m:m+(N-1)]));
end
```

(b)
```
>>  delta = @(n) 1.0*(n==0).*(mod(n,1)==0);
>>  n = 0:44; x = cos(pi*n/5)+delta(n-30)-delta(n-35);
>>  y4 = filtermedian(x,4); y8 = filtermedian(x,8); y12 = filtermedian(x,12);
>>  subplot(221); stem(n,x,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('x[n]'); grid on;
>>  subplot(222); stem(n,y4,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=4'); grid on;
>>  subplot(223); stem(n,y8,'k.'); axis([0 44 -2.2 2.2]);
```

```
>>  xlabel('n'); ylabel('y[n] for N=8'); grid on;
>>  subplot(224); stem(n,y12,'k.'); axis([0 44 -2.2 2.2]);
>>  xlabel('n'); ylabel('y[n] for N=12'); grid on;
```


Figure S3.11-9

The plots are consistent with the expected behavior of a median filter. The output magnitude tends to be smaller than the input magnitude. Unlike the max or min filters, the median filter is not sensitive to outliers in the input data.

Also notice that the median filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N - 1$ samples. Furthermore, since a sinusoidal input does not result in a sinusoidal output, the median filter cannot be a LTI system (the median filter is TI but not linear).

Solution 3.11-10

(a) Replacing $h[n]$ with $y[n]$ and $\delta[n]$ with $x[n]$ yields the desired difference equation,

$$ y[n] = \sum_{k=-(N-1)}^{N-1} \left(1 - \left| \frac{k}{N} \right| \right) x[n-k]. $$

(b) To make the system causal, $h[n]$ must be right-shifted by at least $(N-1)$. Since the system is time-invariant, shifting $h[n]$ causes the output to be delayed (shifted) by the same $(N-1)$ amount.

(c)
```
function [a,b] = interpfilter(N)
%function [a,b] = interpfilter(N)

a = 1; b = conv(ones(N,1),ones(N,1))/N;
```

(d)
```
>>  n = [0:9]; x = @(n) cos(n).*(fix(n)==n);
>>    = 10; nup = [0:N*length(n)-1]; xup = x(nup/N);
>>  [a,b] = interpfilter(N); y = filter(b,a,xup);
>>  subplot(311),stem(n,x(n),'k.'); axis([0 10 -1.1 1.1]);
>>  xlabel('n'); ylabel('x[n]'); grid on;
```

```
>> subplot(312),stem(nup,xup,'k.'); axis([0 100 -1.1 1.1]);
>> xlabel('n'); ylabel('x_{up}[n]'); grid on;
>> subplot(313),stem(nup,y,'k.'); axis([0 100 -1.1 1.1]);
>> xlabel('n'); ylabel('y[n]'); grid on;
```
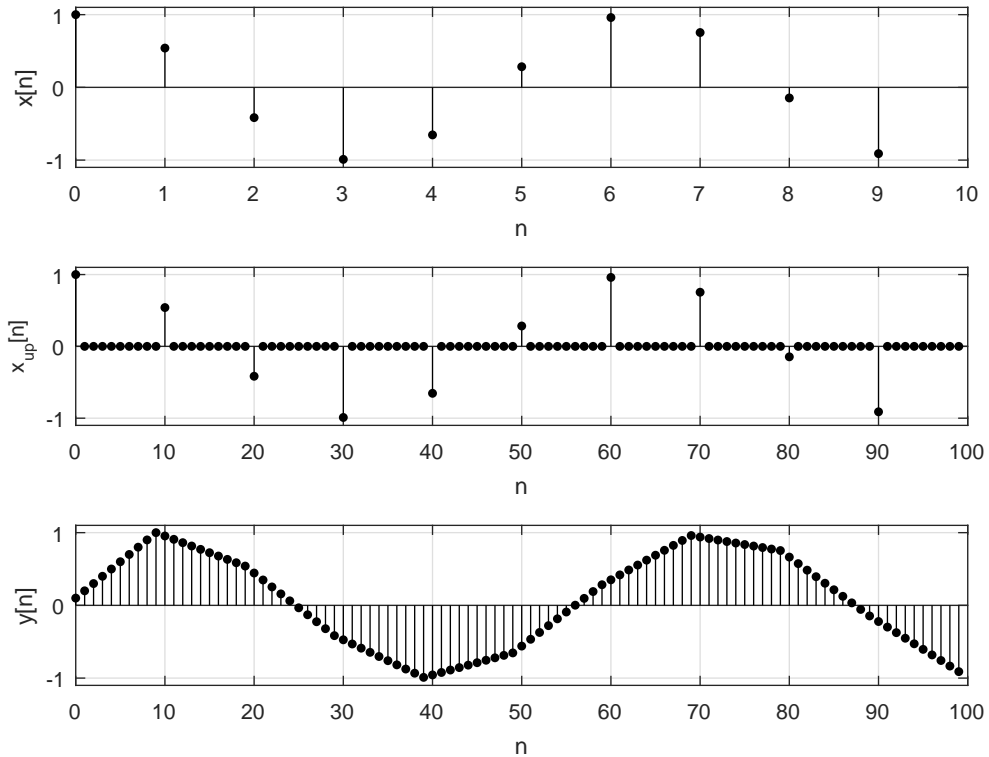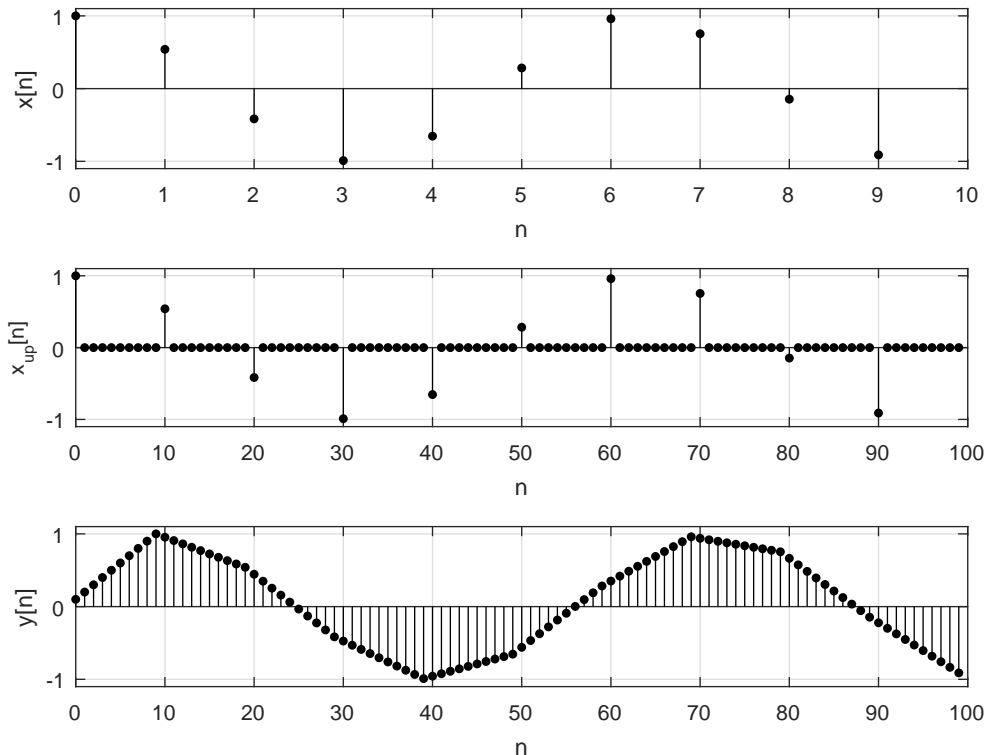

Figure S3.11-10

Indeed, the filter produces the desired linear interpolation of the up-sampled input. As expected from a causal implementation of the interpolation filter, an $(N-1)$ delay is visible in the output.

Solution 3.11-11

(a) First, rewrite the impulse response as

$$h[n] = \tfrac{1}{N}\left(u[n] - u[n-N]\right) = \tfrac{1}{N}\sum_{k=0}^{N-1}\delta[n-k].$$

Replacing $h[n]$ with $y[n]$ and $\delta[n]$ with $x[n]$ yields the desired difference equation,

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k].$$

(b)
```
function [a,b] = filterma(N)
% function [a,b] = filterma(N)

a = 1; b = ones(N,1)/N;
```

(c)
```
>> n = [0:44]; x = cos(pi*n/5)+(n==30)-(n==35);
>> N = 4; [a,b] = filterma(N);
>> y1 = filter(b,a,x);
```

```
>> N = 8; [a,b] = filterma(N);
>> y2 = filter(b,a,x);
>> N = 12; [a,b] = filterma(N);
>> y3 = filter(b,a,x);
>> subplot(221); stem(n,x,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('x[n]');
>> subplot(222); stem(n,y1,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=4');
>> subplot(223); stem(n,y2,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=8');
>> subplot(224); stem(n,y3,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=12');
```


Figure S3.11-11c

The plots are consistent with the expected behavior of a moving average filter. The output is a low-pass version of the input. Larger values of N average over a wider window; thus, larger N results in greater attenuation of the input sinusoid. Large outliers have high frequency content and are significantly attenuated.

Also notice that the moving average filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. This particular filter is a LTI filter, so a sinusoidal input will result in a steady-state sinusoidal output.

(d) The total impulse response of a cascade of two N-point moving average filters is

$$
h_{\text{cascade}}[n] = \left(\frac{u[n] - u[n-N]}{N}\right) * \left(\frac{u[n] - u[n-N]}{N}\right)
$$

$$
= \left(\sum_{k=0}^{n} N^{-2}\right)(u[n] - u[n-N]) + \left(\sum_{k=n-(N-1)}^{N-1} \frac{1}{N^2}\right)(u[n-N] - u[n-(2N-1)])
$$

$$
= \frac{n+1}{N^2}(u[n] - u[n-N]) + \frac{2N-n-1}{N^2}(u[n-N] - u[n-(2N-1)]).
$$

That is, $h_{\text{cascade}}[n]$ is a triangle-shaped function with width $2N-1$ and maximum height $1/N$.

The impulse response of a causal linear interpolation filter is

$$h_{\text{lininterp}}[n] = \sum_{k=-(N-1)}^{N-1} \left(1 - \left|\frac{k}{N}\right|\right) \delta(n - (N-1) - k).$$

This function is identical to $h_{\text{cascade}}[n]$ except that it has a maximum height of 1. Thus, the cascade of moving average filters is a factor $1/N$ different than the linear interpolation filter,

$$N h_{\text{cascade}}[n] = h_{\text{lininterp}}[n].$$

MATLAB is used to implement a linear interpolation filter using a cascade of two moving average filters.

```
>>  n = [0:9]; x = @(n) cos(n).*(fix(n)==n);
>>  N = 10; nup = [0:N*length(n)-1]; xup = x(nup/N);
>>  [a,b] = filterma(N); y = N*filter(b,a,filter(b,a,xup));
>>  subplot(311),stem(n,x(n),'k.'); axis([0 10 -1.1 1.1]);
>>  xlabel('n'); ylabel('x[n]'); grid on;
>>  subplot(312),stem(nup,xup,'k.'); axis([0 100 -1.1 1.1]);
>>  xlabel('n'); ylabel('x_{up}[n]'); grid on;
>>  subplot(313),stem(nup,y,'k.'); axis([0 100 -1.1 1.1]);
>>  xlabel('n'); ylabel('y[n]'); grid on;
```


Figure S3.11-11d

Figure S3.11-11d demonstrates that linear interpolation is possible using a cascade of two moving average filters. As expected, a delay of $N - 1$ is visible in the output.

Chapter 4 Solutions

(a) Here, $x(t) = u(t) - u(t-1)$, and

$$X(s) = \int_0^1 e^{-st}\, dt = -\frac{e^{-st}}{s}\bigg|_0^1 = -\frac{1}{s}[e^{-s} - 1]$$
$$= \frac{1}{s}[1 - e^{-s}].$$

This result is valid for all values of s. Hence the region of convergence is the entire s-plane (the abscissa of convergence is $\sigma_0 = -\infty$).

(b) In this case, $x(t) = te^{-t}u(t)$, and

$$X(s) = \int_0^\infty te^{-t}e^{-st}\, dt = \int_0^\infty te^{-(s+1)}\, dt = -\frac{e^{-(s+1)t}}{(s+1)^2}\big[-(s+1)t - 1\big]_0^\infty$$
$$= \frac{1}{(s+1)^2},$$

provided that $e^{-(s+1)\infty} = 0$ or $\mathrm{Re}(s+1) > 0$. Hence the region of convergence is $\mathrm{Re}(s) > -1$ (the abscissa of convergence is $\sigma_0 > -1$).

(c) Here, $x(t) = t\cos\omega_0 t\, u(t)$, and

$$X(s) = \int_0^\infty t\cos\omega_0 t e^{-st}\, dt$$
$$= \frac{1}{2}\left\{\int_0^\infty [te^{(j\omega_0 - s)t} + te^{-(j\omega_0 + s)t}]\, dt\right\}$$
$$= \frac{1}{2}\left[\frac{1}{(s - j\omega_0)^2} + \frac{1}{(s + j\omega_0)^2}\right], \qquad \mathrm{Re}(s) > 0.$$

Simplifying, we see that

$$X(s) = \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}, \qquad \mathrm{Re}(s) > 0.$$

(d) For $x(t) = (e^{2t} - 2e^{-t})u(t)$, we see that

$$X(s) = \int_0^\infty (e^{2t} - 2e^{-t})e^{-st} \, dt$$

$$= \int_0^\infty e^{2t}e^{-st} \, dt - 2\int_0^\infty e^{-t}e^{-st} \, dt$$

$$= \int_0^\infty e^{-(s-2)t} \, dt - 2\int_0^\infty e^{-(s+1)t} \, dt$$

$$= \frac{1}{s-2} - \frac{2}{s+1}.$$

We get the first term only if $\text{Re}(s) > 2$, and we get the second term only if $\text{Re}(s) > -1$. Both conditions will be satisfied if $\text{Re}(s) > 2$. Hence,

$$X(s) = \frac{1}{s-2} - \frac{2}{s+1}, \qquad \text{Re}(s) > 2.$$

(e) In this case, we see that

$$x(t) = \cos\omega_1 t \cos\omega_2 t \, u(t) = \left[\frac{1}{2}\cos(\omega_1 + \omega_2)t + \frac{1}{2}\cos(\omega_1 - \omega_2)t\right]u(t).$$

Taking the bilateral Laplace transform, we obtain

$$X(s) = \frac{1}{2}\int_0^\infty \cos(\omega_1 + \omega_2)te^{-st} \, dt + \frac{1}{2}\int_0^\infty \cos(\omega_1 - \omega_2)te^{-st} \, dt$$

$$= \frac{1}{2}\left[\frac{s}{s^2 + (\omega_1 + \omega_2)^2} + \frac{s}{s^2 + (\omega_1 - \omega_2)^2}\right], \qquad \text{Re}(s) > 0.$$

(f) For $x(t) = \cosh(at)u(t)$,

$$X(s) = \frac{1}{2}\left[\int_0^\infty e^{at}e^{-st} \, dt + \int_0^\infty e^{-at}e^{-st} \, dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} \, dt + \int_0^\infty e^{-(s+a)t} \, dt\right]$$

$$= \frac{s}{s^2 - a^2}, \qquad \text{Re}(s) > |a|.$$

(g) For $x(t) = \sinh(at)u(t)$,

$$X(s) = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} \, dt - \int_0^\infty e^{-(s+a)t} \, dt\right]$$

$$= \frac{a}{s^2 - a^2}, \qquad \text{Re}(s) > |a|.$$

(h) In this case, we have

$$x(t) = e^{-2t}\cos(5t + \theta)u(t)$$

$$= \frac{1}{2}\left[e^{-2t+j(5t+\theta)} + e^{-2t-j(5t+\theta)}\right]$$

$$= \frac{1}{2}e^{j\theta}e^{-(2-j5)t} + \frac{1}{2}e^{-j\theta}e^{-(2+j5)t}.$$

Hence,

$$X(s) = \frac{1}{2}e^{j\theta}\left(\frac{1}{s+2-j5}\right) + \frac{1}{2}e^{-j\theta}\left(\frac{1}{s+2+j5}\right),$$

provided that $\text{Re}(s) > -2$ (necessary for both terms). Simplifying, we obtain

$$X(s) = \frac{(s+2)\cos\theta - 5\sin\theta}{s^2+4s+29}, \qquad \text{Re}(s) > -2.$$

Solution 4.1-2

(a) For $x(t) = e^{-2t}u(t-5) + \delta(t-1)$,

$$X(s) = \int_{-\infty}^{\infty}\left[e^{-2t}u(t-5) + \delta(t-1)\right]e^{-st}\,dt$$

$$= \int_{5}^{\infty}e^{-t(s+2)}\,dt + \int_{-\infty}^{\infty}\delta(t-1)e^{-st}\,dt$$

$$= \left.-\frac{1}{s+2}e^{-t(s+2)}\right|_{t=5}^{\infty} + e^{-s}.$$

Convergence requires $\text{Re}(s) > -2$. Thus,

$$X(s) = \frac{1}{s+2}e^{-5(s+2)} + e^{-s}, \qquad \text{Re}(s) > -2.$$

(b) For $x(t) = \pi e^{3t}u(t+5) - \delta(2t)$,

$$X(s) = \int_{-\infty}^{\infty}\left[\pi e^{3t}u(t+5) - \delta(2t)\right]e^{-st}\,dt$$

$$= \pi\int_{-5}^{\infty}e^{-t(s-3)}\,dt - \int_{-\infty}^{\infty}\delta(2t)e^{-st}\,dt.$$

Letter $t' = 2t$ and $dt' = 2dt$ in the second integral, we obtain

$$X(s) = \pi\int_{-5}^{\infty}e^{-t(s-3)}\,dt - \frac{1}{2}\int_{-\infty}^{\infty}\delta(t')e^{-st'/2}\,dt' = \left.-\frac{\pi}{s-3}e^{-t(s-3)}\right|_{t=-5}^{\infty} - \frac{1}{2}.$$

Convergence requires $\text{Re}(s) > 3$. Thus,

$$X(s) = \frac{\pi}{s-3}e^{5(s-3)} - \frac{1}{2}, \qquad \text{Re}(s) > 3.$$

(c) For $x(t) = \sum_{k=0}^{\infty}\delta(t-kT)$ (where $T > 0$),

$$X(s) = \int_{-\infty}^{\infty}\left[\sum_{k=0}^{\infty}\delta(t-kT)\right]e^{-st}\,dt = \sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\delta(t-kT)e^{-st}\,dt = \sum_{k=0}^{\infty}(e^{-sT})^k.$$

For $T > 0$, convergence requires that $\text{Re}(s) > 0$. Thus,

$$X(s) = \frac{1}{1-e^{-sT}}, \qquad T > 0 \text{ and } \text{Re}(s) > 0.$$

Solution 4.1-3

(a)

$$X(s) = \int_0^1 t e^{-st} \, dt = \frac{e^{-st}}{s}(-st - 1)\bigg|_0^1 = \frac{1}{s^2}(1 - e^{-s} - se^{-s})$$

(b)

$$X(s) = \int_0^\pi \sin t \, e^{-st} \, dt = \frac{e^{-st}}{s^2 + 1}(-s \sin t - \cos t)\bigg|_0^\pi = \frac{1 + e^{-\pi s}}{s^2 + 1}$$

(c)

$$X(s) = \int_0^1 \frac{t}{e} e^{-st} \, dt + \int_1^\infty e^{-t} e^{-st} \, dt = \frac{1}{e}\int_0^1 t e^{-st} \, dt + \int_1^\infty e^{-(s+1)t} \, dt$$

$$= \frac{e^{-st}}{es}(-st - 1)\bigg|_0^1 - \frac{1}{s+1} e^{-(s+1)}\bigg|_1^\infty$$

$$= \frac{1}{es^2}(1 - e^{-s} - se^{-s}) + \frac{1}{s+1} e^{-(s+1)}$$

Solution 4.1-4

(a)

$$X(s) = \frac{2s + 5}{s^2 + 5s + 6} = \frac{2s + 5}{(s + 2)(s + 3)} = \frac{1}{s + 2} + \frac{1}{s + 3}$$

$$x(t) = (e^{-2t} + e^{-3t})u(t)$$

(b)

$$X(s) = \frac{3s + 5}{s^2 + 4s + 13}$$

Here $A = 3$, $B = 5$, $a = 2$, $c = 13$, $b = \sqrt{13 - 4} = 3$.

$$r = \sqrt{\frac{117 + 25 - 60}{13 - 4}} = 3.018, \qquad \theta = \tan^{-1}\left(\frac{1}{9}\right) = 6.34°$$

$$x(t) = 3.018 e^{-2t} \cos(3t + 6.34°)u(t)$$

(c)

$$X(s) = \frac{(s + 1)^2}{s^2 - s - 6} = \frac{(s + 1)^2}{(s + 2)(s - 3)}$$

This is an improper fraction with $b_n = b_2 = 1$. Therefore,

$$X(s) = 1 + \frac{a}{s + 2} + \frac{b}{s - 3} = 1 - \frac{0.2}{s + 2} + \frac{3.2}{s - 3}$$

$$x(t) = \delta(t) + (3.2e^{3t} - 0.2e^{-2t})u(t)$$

(d)

$$X(s) = \frac{5}{s^2(s + 2)} = \frac{k}{s} + \frac{2.5}{s^2} + \frac{1.25}{s + 2}$$

To find k, set $s = 1$ on both sides to obtain

$$\frac{5}{3} = k + 2.5 + \frac{5}{12} \quad \Longrightarrow \quad k = -1.25$$

and

$$X(s) = -\frac{1.25}{s} + \frac{2.5}{s^2} + \frac{1.25}{s+2}$$
$$x(t) = 1.25(-1 + 2t + e^{-2t})u(t)$$

(e)

$$X(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{-1}{s+1} + \frac{As+B}{s^2+2s+2}$$

Multiply both sides by s and let $s \to \infty$. This yields

$$0 = -1 + A \quad \Longrightarrow \quad A = 1$$

Setting $s = 0$ on both sides yields

$$\frac{1}{2} = -1 + \frac{B}{2} \quad \Longrightarrow \quad B = 3$$

$$X(s) = -\frac{1}{s+1} + \frac{s+3}{s^2+2s+2}$$

In the second fraction, $A = 1$, $B = 3$, $a = 1$, $c = 2$, and $b = \sqrt{2-1} = 1$.

$$r = \sqrt{\frac{2+9-6}{2-1}} = \sqrt{5}, \qquad \theta = \tan^{-1}(\frac{-2}{1}) = -63.4°$$

$$x(t) = [-e^{-t} + \sqrt{5}e^{-t}\cos(t - 63.4°)]u(t)$$

(f)

$$X(s) = \frac{s+2}{s(s+1)^2} = \frac{2}{s} + \frac{k}{s+1} - \frac{1}{(s+1)^2}$$

To compute k, multiply both sides by s and let $s \to \infty$. This yields

$$0 = 2 + k + 0 \quad \Longrightarrow \quad k = -2$$

and

$$X(s) = \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2}$$
$$x(t) = [2 - (2+t)e^{-t}]u(t)$$

(g)

$$X(s) = \frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} + \frac{k_1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Multiply both sides by s and let $s \to \infty$. This yields

$$0 = 1 + k_1 \quad \Longrightarrow \quad k_1 = -1$$

$$\frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Setting $s = 0$ and -3 on both sides yields

$$\frac{1}{16} = 1 - \frac{1}{2} + \frac{k_2}{4} + \frac{k_3}{8} - \frac{1}{16} \implies 4k_2 + 2k_3 = -6$$

$$-\frac{1}{2} = -\frac{1}{2} + 1 + k_2 - k_3 - 1 \implies k_2 - k_3 = 0$$

Solving these two equations simultaneously yields $k_2 = k_3 = -1$. Therefore,

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} - \frac{1}{(s+2)^3} - \frac{1}{(s+2)^4}$$

$$x(t) = [e^{-t} - (1 + t + \frac{t^2}{2} + \frac{t^3}{6})e^{-2t}]u(t)$$

Comment: This problem could be tackled in many ways. We could have used Eq. (B.30), or after determining first two coefficients by Heaviside method, we could have cleared fractions. Also instead of letting $s = 0$ and -3, we could have selected any other set of values. However, in this case these values appear most suitable for numerical work.

(h)

$$X(s) = \frac{s+1}{s(s+2)^2(s^2+4s+5)} = \frac{(1/20)}{s} + \frac{k}{s+2} + \frac{(1/2)}{(s+2)^2} + \frac{As+B}{s^2+4s+5}$$

Multiplying both sides by s and letting $s \to \infty$ yield

$$0 = \frac{1}{20} + k + A \implies k + A = -\frac{1}{20}$$

Setting $s = 1$ and -1 yields

$$\frac{2}{90} = \frac{1}{20} + \frac{k}{3} + \frac{1}{18} + \frac{A+B}{10} \implies 20k + 6A + 6B = -5$$
$$0 = -\frac{1}{20} + k + \frac{1}{2} + \frac{-A+B}{2} \implies 20k - 10A + 10B = -9$$

Solving these three equations in k, A and B yields $k = -\frac{1}{4}$, $A = \frac{1}{5}$ and $B = -\frac{1}{5}$. Therefore,

$$X(s) = \frac{1/20}{s} - \frac{1/4}{s+2} + \frac{(1/2)}{(s+2)^2} + \frac{1}{5}(\frac{s-1}{s^2+4s+5})$$

For the last fraction in parenthesis on the right-hand side $A = 1$, $B = -1$, $a = 2$, $c = 5$, and $b = \sqrt{5-4} = 1$.

$$r = \sqrt{\frac{5+1+4}{5-4}} = \sqrt{10} \qquad \theta = \tan^{-1}(\frac{3}{1}) = 71.56°$$

$$x(t) = [\frac{1}{20} - \frac{1}{4}(1 - 2t)e^{-2t} + \frac{\sqrt{10}}{5}e^{-2t}\cos(t + 71.56°)]u(t)$$

(i)

$$X(s) = \frac{s^3}{(s+1)^2(s^2+2s+5)} = \frac{k}{s+1} - \frac{1/4}{(s+1)^2} + \frac{As+B}{s^2+2s+5}$$

Multiply both sides by s and let $s \to \infty$ to obtain

$$1 = k + A$$

Setting $s = 0$ and 1 yields

$$0 = k - \frac{1}{4} + \frac{B}{5} \implies 20k + 4B = 5$$
$$\frac{1}{32} = \frac{k}{2} - \frac{1}{16} + \frac{A+B}{8} \implies 16k + 4A + 4B = 3$$

Solving these three equations in k, A and B yields $k = \frac{3}{4}$, $A = \frac{1}{4}$ and $B = -\frac{5}{2}$.

$$X(s) = \frac{3/4}{s+1} - \frac{1/4}{(s+1)^2} + \frac{1}{4}\left(\frac{s-10}{s^2+2s+5}\right)$$

For the last fraction in parenthesis, $A = 1$, $B = -10$, $a = 1$, $c = 5$, and $b = \sqrt{5-1} = 2$.

$$r = \sqrt{\frac{5+100+20}{5-1}} = 5.59 \qquad \theta = \tan^{-1}\left(\frac{11}{4}\right) = 70°$$

Therefore,

$$x(t) = [(\frac{3}{4} - \frac{1}{4}t)e^{-t} + \frac{5.59}{4}e^{-t}\cos(2t + 70°)]u(t)$$
$$= [\frac{1}{4}(3-t) + 1.3975\cos(2t + 70°)]e^{-t}u(t)$$

Solution 4.2-1

(a) Using properties, we establish that

$$X(s) \Longleftrightarrow x(t) \qquad \text{(starting fact)}$$
$$X(s + \tfrac{1}{2}) \Longleftrightarrow e^{-t/2}x(t) \qquad \text{(frequency shift property)}$$
$$e^{-5s}X(s + \tfrac{1}{2}) \Longleftrightarrow e^{-(t-5)/2}x(t-5) \qquad \text{(time shift property)}$$
$$Y_a(s) = se^{-5s}X(s + \tfrac{1}{2}) \Longleftrightarrow \tfrac{d}{dt}\left\{e^{-(t-5)/2}x(t-5)\right\} \qquad \text{(time differentiation property)}$$

Since $x(t) = 2\left[u(t-2) - u(t+1)\right]$, we see that

$$y_a(t) = \frac{d}{dt}\left\{e^{-(t-5)/2}2\left[u(t-7) - u(t-4)\right]\right\}$$
$$= -e^{-(t-5)/2}\left[u(t-7) - u(t-4)\right] + 2e^{-1}\delta(t-7) - 2e^{1/2}\delta(t-4).$$

(b) Again using properties, we see that

$$X(s) \Longleftrightarrow x(t) \qquad \text{(starting fact)}$$
$$X(s - 2) \Longleftrightarrow e^{2t}x(t) \qquad \text{(frequency shift property)}$$
$$G(s) = sX(s - 2) \Longleftrightarrow \frac{d}{dt}\left\{e^{2t}x(t)\right\} \qquad \text{(time differentiation property)}$$
$$Y_b(s) = 2^{-s}G(s) = e^{-\ln(2)s}G(s) \Longleftrightarrow g(t - \ln(2)) \qquad \text{(time shift property)}$$

Since $x(t) = 2\left[u(t-2) - u(t+1)\right]$, we see that

$$g(t) = \frac{d}{dt}\left\{e^{2t}2\left[u(t-2) - u(t+1)\right]\right\} = 4e^{2t}\left[u(t-2) - u(t+1)\right] + 2e^4\delta(t-2) - 2e^{-2}\delta(t+1).$$

Since $y_b(t) = g(t - \ln(2))$, we see that

$$y_b(t) = 4e^{2(t-\ln(2))}\left[u(t-2-\ln(2)) - u(t+1-\ln(2))\right] + 2e^4\delta(t-2-\ln(2)) - 2e^{-2}\delta(t+1-\ln(2)).$$

Figure S4.2-1 shows $y_a(t)$ and $y_b(t)$. The leftmost delta of $y_b(t)$ is almost too small to visually represent.
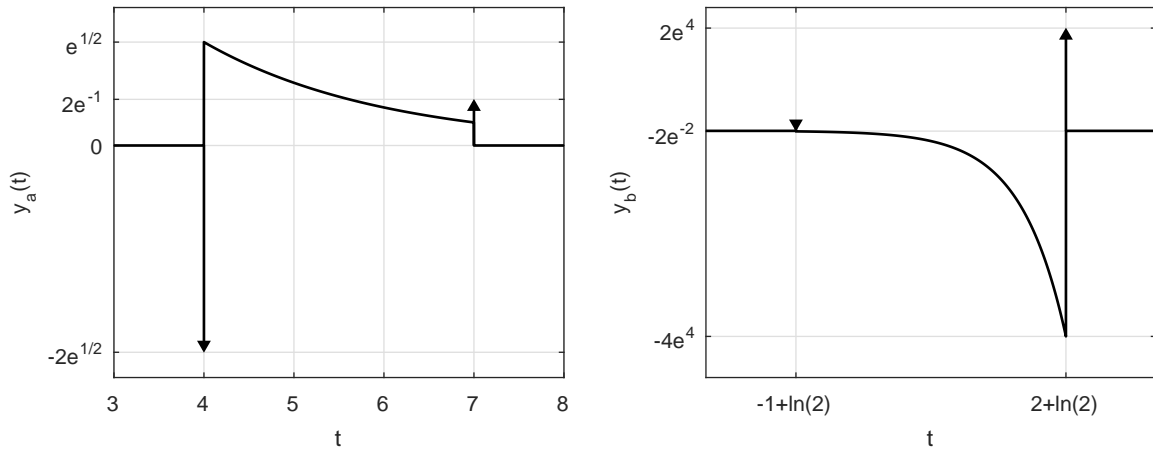
Figure S4.2-1

Solution 4.2-2

In this problem, we use of Table 4.1 and the time-shifting property.

(a)

$$x(t) = u(t) - u(t-1)$$

$$X(s) = \mathcal{L}[u(t)] - \mathcal{L}[u(t-1)] = \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1}{s}(1 - e^{-s})$$

(b)

$$x(t) = e^{-(t-\tau)}u(t-\tau)$$

$$X(s) = \frac{1}{s+1}e^{-s\tau}$$

(c)

$$x(t) = e^{-(t-\tau)}u(t) = e^{\tau}e^{-t}u(t)$$

Therefore, $\qquad X(s) = e^{\tau}\frac{1}{s+1}$

(d)

$$x(t) = e^{-t}u(t-\tau) = e^{-\tau}e^{-(t-\tau)}u(t-\tau)$$

Observe that $e^{-(t-\tau)}u(t-\tau)$ is $e^{-t}u(t)$ delayed by τ. Therefore,

$$X(s) = e^{-\tau}\left(\frac{1}{s+1}\right)e^{-s\tau} = \left(\frac{1}{s+1}\right)e^{-(s+1)\tau}$$

(e)

$$x(t) = te^{-t}u(t-\tau) = (t-\tau+\tau)e^{-(t-\tau+\tau)}u(t-\tau)$$

$$= e^{-\tau}\left[(t-\tau)e^{-(t-\tau)}u(t-\tau) + \tau e^{-(t-\tau)}u(t-\tau)\right]$$

Therefore,

$$X(s) = e^{-\tau}\left[\frac{1}{(s+1)^2}e^{-s\tau} + \frac{\tau}{(s+1)}e^{-s\tau}\right]$$

$$= \frac{e^{-(s+1)\tau}\left[1 + \tau(s+1)\right]}{(s+1)^2}$$

(f)
$$x(t) = \sin \omega_0 (t - \tau) u(t - \tau)$$

Note that this is $\sin \omega_0 t$ shifted by τ. Hence,

$$X(s) = \left(\frac{\omega_0}{s^2 + \omega_0^2} \right) e^{-s\tau}$$

(g)
$$x(t) = \sin \omega_0 (t - \tau) u(t) = [\sin \omega_0 t \cos \omega_0 \tau - \cos \omega_0 t \sin \omega_0 \tau] u(t)$$
$$X(s) = \frac{\omega_0 \cos \omega_0 \tau - s \sin \omega_0 \tau}{s^2 + \omega_0^2}$$

(h)
$$x(t) = \sin \omega_0 t \, u(t - \tau) = \sin[\omega_0 (t - \tau + \tau)] u(t - \tau)$$
$$= \cos \omega_0 \tau \sin[\omega_0 (t - \tau)] u(t - \tau) + \sin \omega_0 \tau \cos[\omega_0 (t - \tau)] u(t - \tau)$$

Therefore,

$$X(s) = \left[\cos \omega_0 \tau \left(\frac{\omega_0}{s^2 + \omega_0^2} \right) + \sin \omega_0 \tau \left(\frac{s}{s^2 + \omega_0^2} \right) \right] e^{-s\tau}$$

(i) From Euler's, we know that

$$x(t) = t \sin(t) u(t) = \frac{1}{2j} [t e^{jt} - t e^{-jt}].$$

Using entry 6 of Table 4.1, we see that

$$X(s) = \frac{1}{2j} \left[\frac{1}{(s - j)^2} - \frac{1}{(s + j)^2} \right]$$
$$= \frac{1}{2j} \left[\frac{s^2 + 2js - 1 - (s^2 - 2js - 1)}{(s - j)^2 (s + j)^2} \right].$$

Thus,

$$X(s) = \frac{2s}{(s^2 + 1)^2}.$$

(j) From Euler's, we know that

$$t \cos(t) u(t) = \frac{1}{2} [t e^{jt} + t e^{-jt}].$$

Using entry 6 of Table 4.1, we see that

$$\mathcal{L}\{-t \cos(t) u(t)\} = -\frac{1}{2} \left[\frac{1}{(s - j)^2} + \frac{1}{(s + j)^2} \right]$$
$$= -\frac{1}{2} \left[\frac{s^2 + 2js - 1 + s^2 - 2js - 1}{(s - j)^2 (s + j)^2} \right]$$
$$= -\frac{s^2 - 1}{(s^2 + 1)^2}.$$

Delaying $-t \cos(t) u(t)$ by 1 yields the desired time domain signal

$$x(t) = (1 - t) \cos(t - 1) u(t - 1)$$

From the time-shifting property, it follows that

$$X(s) = e^{-s} \left[\frac{-s^2 + 1}{(s^2 + 1)^2} \right].$$

Solution 4.2-3

In this problem, we use of Table 4.1 and the time-shifting property.

(a)

$$x(t) = t[u(t) - u(t-1)] = tu(t) - (t-1)u(t-1) - u(t-1)$$

$$X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}$$

(b)

$$x(t) = \sin t\, u(t) + \sin(t-\pi)\, u(t-\pi)$$

$$X(s) = \frac{1}{s^2+1}(1 + e^{-\pi s})$$

(c)

$$x(t) = [u(t) - u(t-1)] + e^{-t}u(t-1)$$
$$= tu(t) - (t-1)u(t-1) - u(t-1) + e^{-1}e^{-(t-1)}u(t-1)$$

Therefore,

$$X(s) = \frac{1}{s^2}(1 - e^{-s} - se^{-s}) + \frac{e^{-s}}{e(s+1)}$$

Solution 4.2-4

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left\{\int_{-\infty}^{\infty} x(t)e^{-st}\, dt\right\} = \int_{-\infty}^{\infty} x(t)\frac{d}{ds}\left\{e^{-st}\right\}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\left\{-te^{-st}\right\}\, dt = \int_{-\infty}^{\infty}\left\{-tx(t)\right\}e^{-st}\, dt = \mathcal{L}\left\{-tx(t)\right\}.$$

Thus,

$$-tx(t) \Longleftrightarrow \frac{d}{ds}X(s).$$

Solution 4.2-5

(a) Using properties, we establish that

$$e^{-2t}u(t) \Longleftrightarrow \frac{1}{s+2} \qquad \text{(Table 4.1, entry 5)}$$
$$e^{-2(t-2)}u(t-2) \Longleftrightarrow e^{-2s}\frac{1}{s+2} \qquad \text{(time-shifting property)}$$
$$-e^2 e^{-2(t-2)}u(t-2) \Longleftrightarrow -e^2 e^{-2s}\frac{1}{s+2} \qquad \text{(scaling property)}$$
$$x(t) = -t\left(-e^{-2(t-3)}u(t-2)\right) \Longleftrightarrow \frac{d}{ds}\left\{-e^2 e^{-2s}\frac{1}{s+2}\right\} \qquad \text{(s-differentiation)}$$

Thus,

$$X_{\mathrm{u}}(s) = -e^2\left\{-2e^{-2s}\frac{1}{s+2} + e^{-2s}(-1)\frac{1}{(s+2)^2}\right\} = e^{2(1-s)}\frac{2s+5}{(s+2)^2}.$$

(b) Since $x(t) = x(t)u(t)$, the unilateral and bilateral Laplace transforms will be the same. The only difference is that the ROC needs to be explicitly specified in the bilateral Laplace transform. In the present case, $x(t)$ is a right-sided signal with a pole at -2. Thus, the ROC is $\mathrm{Re}(s) > -2$. Taken together, the bilateral Laplace transform of $x(t)$ is

$$X(s) = e^{2(1-s)}\frac{2s+5}{(s+2)^2}, \qquad \mathrm{Re}(s) > -2.$$

Solution 4.2-6

(a) By definition,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt = \int_{0}^{1} e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{1}$$

$$= \frac{1 - e^{-s}}{s}, \qquad \text{ROC: all } s.$$

(b) To begin, we notice that $y(t) = (\frac{t-1}{2})x(\frac{t-1}{2})$. Using properties, we see that

$$x(t) \Longleftrightarrow X(s) \qquad \text{(starting fact)}$$
$$x(\tfrac{t}{2}) \Longleftrightarrow 2X(2s) \qquad \text{(time scale)}$$
$$\tfrac{t}{2}x(\tfrac{t}{2}) \Longleftrightarrow -\tfrac{1}{2}\tfrac{d}{ds}\{2X(2s)\} \qquad \text{(differentiation in } s)$$
$$y(t) = \tfrac{t-1}{2}x(\tfrac{t-1}{2}) \Longleftrightarrow -e^{-s}\tfrac{d}{ds}\{X(2s)\} \qquad \text{(time shift)}$$

Thus,

$$Y(s) = -e^{-s}\frac{d}{ds}X(2s), \qquad \text{ROC: all } s.$$

Solution 4.2-7

(a)

$$X(s) = \frac{(2s+5)e^{-2s}}{s^2 + 5s + 6} = \hat{X}(s)e^{-2s}$$

It is clear that $x(t) = \hat{x}(t-2)$.

$$\hat{X}(s) = \frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$
$$\hat{x}(t) = (e^{-2t} + e^{-3t})u(t)$$
$$x(t) = \hat{x}(t-2) = [e^{-2(t-2)} + e^{-3(t-2)}]u(t-2)$$

(b)

$$X(s) = \frac{s}{s^2 + 2s + 2}e^{-3s} + \frac{2}{s^2 + 2s + 2} = X_1(s)e^{-3s} + X_2(s)$$

where

$$X_1(s) = \frac{s}{s^2 + 2s + 2} \quad \left\{ \begin{array}{l} A = 1,\ B = 0,\ a = 1,\ c = 2,\ b = 1 \\ r = \sqrt{2},\ \theta = \tan^{-1}(1) = \pi/4 \end{array} \right.$$

$$x_1(t) = \sqrt{2}e^{-t}\cos(t + \frac{\pi}{4})$$

$$X_2(s) = \frac{2}{s^2 + 2s + 2} \quad \text{and} \quad x_2(t) = 2e^{-t}\sin t$$

Also

$$x(t) = x_1(t-3) + x_2(t)$$
$$= \sqrt{2}e^{-(t-3)}\cos(t - 3 + \frac{\pi}{4})u(t-3) + 2e^{-t}\sin t\, u(t)$$

(c)

$$X(s) = \frac{(e)e^{-s}}{s^2 - 2s + 5} + \frac{3}{s^2 - 2s + 5}$$

$$= e\frac{1}{s^2 - 2s + 5}e^{-s} + \frac{3}{s^2 - 2s + 5}$$

$$= eX_1(s)e^{-s} + X_2(s)$$

where

$$X_1(s) = \frac{1}{s^2 - 2s + 2} \quad \text{and} \quad x_1(t) = \frac{1}{2}e^t \sin 2t \, u(t)$$

$$X_2(s) = \frac{3}{s^2 - 2s + 2} \quad \text{and} \quad x_2(t) = \frac{3}{2}e^t \sin 2t \, u(t)$$

Therefore

$$x(t) = ex_1(t - 1) + x_2(t)$$

$$= \frac{e}{2}e^{(t-1)} \sin 2(t - 1)u(t - 1) + \frac{3}{2}e^t \sin 2t \, u(t)$$

(d)

$$X(s) = \frac{e^{-s} + e^{-2s} + 1}{s^2 + 3s + 2} = (e^{-s} + e^{-2s} + 1)\left[\frac{1}{s^2 + 3s + 2}\right]$$

$$= (e^{-s} + e^{-2s} + 1)\left[\frac{1}{s + 1} - \frac{1}{s + 2}\right]$$

$$X(s) = (e^{-s} + e^{-2s} + 1)\hat{X}(s)$$

where

$$\hat{X}(s) = \frac{1}{s + 1} - \frac{1}{s + 2} \quad \text{and} \quad \hat{x}(t) = (e^{-t} - e^{-2t})u(t)$$

Moreover

$$x(t) = \hat{x}(t - 1) + \hat{x}(t - 2) + \hat{x}(t)$$

$$= [e^{-(t-1)} - e^{-2(t-1)}]u(t - 1) + [e^{-(t-2)} - e^{-2(t-2)}]u(t - 2) + (e^{-t} - e^{-2t})u(t)$$

Using properties, we see that

$$1 \Longleftrightarrow \delta(t) \qquad \text{(Table 4.1, entry 1)}$$

$$e^{-2s} \Longleftrightarrow \delta(t - 2) \qquad \text{(time shift)}$$

$$\frac{1}{s}e^{-2s} \Longleftrightarrow \int_{-\infty}^t \delta(\tau - 2)\,d\tau = u(t - 2) \qquad \text{(time integration)}$$

$$\frac{d}{ds}\left(\frac{e^{-2s}}{s}\right) \Longleftrightarrow -tu(t - 2) \qquad \text{(s differentiation)}$$

$$X(s) = s^{-1}\frac{d}{ds}\left(\frac{e^{-2s}}{s}\right) \Longleftrightarrow -\int_{-\infty}^t \tau u(\tau - 2)\,d\tau \qquad \text{(time integration)}$$

Thus,

$$x(t) = -\int_{-\infty}^t \tau u(\tau - 2)\,d\tau = \begin{cases} -\int_2^t \tau\,d\tau & t > 2 \\ 0 & t < 2 \end{cases} = \left(-\frac{\tau^2}{2}\Big|_{\tau=2}^t\right)u(t - 2)$$

or

$$x(t) = \left(2 - \frac{t^2}{2}\right)u(t - 2).$$

Solution 4.2-9

(a)

$$g(t) = x(t) + x(t - T_0) + x(t - 2T_0) + \cdots$$

and

$$
\begin{aligned}
G(s) &= X(s) + X(s)e^{-sT_0} + X(s)e^{-2sT_0} + \cdots \\
&= X(s)[1 + e^{-sT_0} + e^{-2sT_0} + e^{-3sT_0} + \cdots \\
&= \frac{X(s)}{1 - e^{-sT_0}} \qquad |e^{-sT_0}| < 1 \text{ or } \operatorname{Re}(s) > 0.
\end{aligned}
$$

(b)

$$x(t) = u(t) - u(t - 2) \quad \text{and} \quad X(s) = \frac{1}{s}(1 - e^{-2s})$$

$$G(s) = \frac{X(s)}{1 - e^{-8s}} = \frac{1}{s}\left(\frac{1 - e^{-2s}}{1 - e^{-8s}}\right)$$

Solution 4.2-10

Pair 2

$$u(t) = \int_{0^-}^{t} \delta(\tau)\, d\tau \iff \frac{1}{s}(1) = \frac{1}{s}$$

Pair 3

$$tu(t) = \int_{0^-}^{t} u(\tau)\, d\tau \iff \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Pair 4: Use successive integration of $tu(t)$

Pair 5: Applying the frequency-shifting property of Eq. (4.14) to $u(t) \iff \frac{1}{s}$, we obtain

$$e^{\lambda t} u(t) \iff \frac{1}{s - \lambda}.$$

Pair 6: Applying the frequency-shifting property of Eq. (4.14) to $tu(t) \iff \frac{1}{s^2}$, we obtain

$$te^{\lambda t} u(t) \iff \frac{1}{(s - \lambda)^2}.$$

Pair 7: As with pairs 5 and 6, applying the frequency-shifting property of Eq. (4.14) to $t^2 u(t)$, $t^3 u(t)$, ..., generates

$$t^n e^{\lambda t} u(t) \iff \frac{n!}{(s - \lambda)^{n+1}}.$$

Pair 8a:

$$\cos bt\, u(t) = \frac{1}{2}(e^{jbt} + e^{-jbt})u(t) \iff \frac{1}{2}\left(\frac{1}{s - jb} + \frac{1}{s + jb}\right) = \frac{s}{s^2 + b^2}$$

Pair 8b:

$$\sin bt\, u(t) = \frac{1}{2j}(e^{jbt} - e^{-jbt})u(t) \iff \frac{1}{2j}\left(\frac{1}{s - jb} - \frac{1}{s + jb}\right) = \frac{b}{s^2 + b^2}$$

Pair 9a: Application of the frequency-shift property of Eq. (4.14) to pair 8a yields

$$e^{-at}\cos bt\, u(t) \iff \frac{s + a}{(s + a)^2 + b^2}.$$

Pair 9b: Application of the frequency-shift property of Eq. (4.14) to pair 8b yields

$$e^{-at} \sin bt \, u(t) \Longleftrightarrow \frac{b}{(s+a)^2 + b^2}.$$

Pairs 10a and 10b: Recognize that

$$re^{-at} \cos(bt + \theta) = re^{-at}[\cos\theta \cos bt - \sin\theta \sin bt].$$

Now use results in pairs 9a and 9b to obtain pair 10a. Pair 10b is equivalent to pair 10a.

Solution 4.2-11

(a) For the first pulse $x(t) = u(t) - u(t-2)$, we see that

$$\frac{dx}{dt} = \delta(t) - \delta(t-2)$$
$$sX(s) = 1 - e^{-2s}$$
$$X(s) = \frac{1}{s}(1 - e^{-2s})$$

For the second pulse $x(t) = u(t-2) - u(t-4)$, we see that

$$\frac{dx}{dt} = \delta(t-2) - \delta(t-4)$$
$$sX(s) = e^{-2s} - e^{-4s}$$
$$X(s) = \frac{1}{s}(e^{-2s} - e^{-4s})$$

(b)

$$\frac{dx}{dt} = u(t) - 3u(t-2) + 2u(t-3)$$
$$sX(s) = \frac{1}{s} - \frac{3}{s}e^{-2s} + \frac{2}{s}e^{-3s} \quad [x(0^-) = 0]$$
$$X(s) = \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$$

Solution 4.2-12

$X(s) = e^{-3}e^{-s}\left[\frac{s^2}{(s+1)(s+2)}\right] = e^{-3}e^{-s}\left[1 + \frac{-3s-2}{(s+1)(s+2)}\right] = e^{-3}e^{-s}\left[1 + \frac{1}{(s+1)} + \frac{-4}{(s+2)}\right]$. Thus,

$$x(t) = e^{-3}\left[\delta(t-1) + e^{-(t-1)}u(t-1) - 4e^{-2(t-1)}u(t-1)\right].$$

Solution 4.2-13

First, note that the nth derivative of $\frac{1}{s+a}$ is $\frac{(-1)^n n!}{(s+1)^{n+1}}$. Thus, rewrite the transform as

$$X(s) = \frac{1}{(s+1)^{13}} = \frac{1}{12!}\frac{12!}{(s+1)^{13}} = \frac{1}{12!}\frac{d^{12}}{ds^{12}}\left(\frac{1}{s+1}\right).$$

Since $\sigma > -1$, the time-domain signal $x(t)$ must be right sided. Repeated use of the frequency differentiation property provides the resulting inverse transform.

$$x(t) = \frac{1}{12!}(-t)^{12}e^{-t}u(t) = \frac{t^{12}}{12!}e^{-t}u(t).$$

Solution 4.2-14

(a) Using the frequency differentiation property,

$$\mathcal{L}\left[tx(t)\right] = -\frac{d}{ds}X(s).$$

(b) Here, $y(t) = tx(t) = t\frac{1}{t}u(t) = u(t)$. Thus,

$$Y(s) = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{\infty}.$$

Convergence requires $\mathrm{Re}(s) > 0$, resulting in

$$Y(s) = \frac{1}{s}.$$

(c) Combining the previous two parts yields $-\frac{d}{ds}X(s) = \frac{1}{s}$. Thus,

$$X(s) = -\int \frac{1}{s}\,ds = -\ln(s).$$

Solution 4.3-1

(a)

$$(s^2 + 3s + 2)Y(s) = s\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

(b)

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or

$$(s^2 + 4s + 4)Y(s) = 2s + 10$$

and

$$Y(s) = \frac{2s+10}{s^2+4s+4} = \frac{2s+10}{(s+2)^2} = \frac{2}{s+2} + \frac{6}{(s+2)^2}$$

$$y(t) = (2 + 6t)e^{-2t}u(t)$$

(c)

$$(s^2Y(s) - s - 1) + 6(sY(s) - 1) + 25Y(s) = (s+2)\frac{25}{s} = 25 + \frac{50}{s}$$

or

$$(s^2 + 6s + 25)Y(s) = s + 32 + \frac{50}{s} = \frac{s^2 + 32s + 50}{s}$$

and

$$Y(s) = \frac{s^2 + 32s + 50}{s(s^2 + 6s + 25)} = \frac{2}{s} + \frac{-s + 20}{s^2 + 6s + 25}$$

$$y(t) = [2 + 5.836e^{-3t}\cos(4t - 99.86°)]u(t)$$

Solution 4.3-2

(a)

$$(s^2 + 3s + 2)Y(s) = s\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

Since all initial conditions are zero, the zero-state response equals the total response. Thus,

$$y_{zsr}(t) = (e^{-t} - e^{-2t})u(t)$$

$$y_{zir}(t) = 0$$

(b) The Laplace transform of the differential equation is

$$(s^2 Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or

$$(s^2 + 4s + 4)Y(s) - (2s + 9) = 1$$

or

$$(s^2 + 4s + 4)Y(s) = \underbrace{2s + 9}_{\text{i.c. terms}} + \underbrace{1}_{\text{input}}.$$

$$Y(s) = \underbrace{\frac{2s + 9}{s^2 + 4s + 4}}_{\text{zir}} + \underbrace{\frac{1}{s^2 + 4s + 4}}_{\text{zsr}}$$

$$= \underbrace{\frac{2}{s+2} + \frac{5}{(s+2)^2}}_{\text{zir}} + \underbrace{\frac{1}{(s+2)^2}}_{\text{zsr}}$$

$$y(t) = \underbrace{(2 + 5t)e^{-2t}}_{y_{zir}(t)} + \underbrace{te^{-2t}}_{y_{zsr}(t)}$$

(c) The Laplace transform of the equation is

$$(s^2 Y(s) - s - 1) + 6(sY(s) - 1) + 25Y(s) = 25 + \frac{50}{s}$$

or

$$(s^2 + 6s + 25)Y(s) = \underbrace{s + 7}_{\text{i.c. terms}} + \underbrace{25 + \frac{50}{s}}_{\text{input}}.$$

$$Y(s) = \underbrace{\frac{s + 7}{s^2 + 6s + 25}}_{\text{zir}} + \underbrace{\frac{25s + 50}{s(s^2 + 6s + 25)}}_{\text{zsr}}$$

$$= \left(\frac{s + 7}{s^2 + 6s + 25}\right) + \left(\frac{2}{s} + \frac{-2s + 13}{s^2 + 6s + 25}\right)$$

$$y(t) = \underbrace{[\sqrt{2}e^{-3t}\cos(4t - \frac{\pi}{4})]}_{y_{zir}(t)} + \underbrace{[2 + 5.154e^{-3t}\cos(4t - 112.83°)]}_{y_{zsr}(t)}$$

Solution 4.3-3

(a) Setting $x(t) = 0$ and taking the unilateral Laplace transform of $2\dot{y}(t) + 6y(t) = \dot{x}(t) - 4x(t)$ yield

$$2\left[sY_{\text{zir}}(s) - y(0^-)\right] + 6Y_{\text{zir}}(s) = 0.$$

Thus,

$$Y_{\text{zir}}(s) = \frac{y(0^-)}{s+3} = \frac{-3}{s+3}.$$

Inverting,

$$y_{\text{zir}}(t) = -3e^{-3t}u(t).$$

(b) The input is $x(t) = e\delta(t - \pi)$. Using entry 1 of Table 4.1 and the time-shifting property,

$$X(s) = ee^{-s\pi}.$$

By inspection of $2\dot{y}(t) + 6y(t) = \dot{x}(t) - 4x(t)$, the system transfer function is

$$H(s) = \frac{\frac{1}{2}s - 2}{s+3}.$$

The transform of the zero-state response is

$$Y_{\text{zsr}}(s) = X(s)H(s) = ee^{-s\pi}\frac{\frac{1}{2}s - 2}{s+3} = ee^{-s\pi}\left[\frac{1}{2} + \frac{-\frac{7}{2}}{s+3}\right].$$

Inverting,

$$y_{\text{zsr}}(t) = e\left[-\frac{7}{2}e^{-3(t-\pi)}u(t-\pi) + \frac{1}{2}\delta(t-\pi)\right].$$

Solution 4.3-4

(a) Setting $x(t) = 0$ and taking the unilateral Laplace transform of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\dot{x}(t) - x(t)$ yield

$$s^2Y_{\text{zir}}(s) - sy(0^-) - \dot{y}(0^-) + 3sY_{\text{zir}}(s) - 3y(0^-) + 2Y_{\text{zir}}(s) = 0.$$

Since $\dot{y}(0^-) = 2$ and $y(0^-) = -3$,

$$(s^2 + 3s + 2)Y_{\text{zir}}(s) = -(3s + 7).$$

Thus,

$$Y_{\text{zir}}(s) = \frac{-3s - 7}{s^2 + 3s + 2} = \frac{-3s - 7}{(s+1)(s+2)} = \frac{-4}{s+1} + \frac{1}{s+2}.$$

Inverting,

$$y_{\text{zir}}(t) = -4e^{-t}u(t) + e^{-2t}u(t).$$

(b) The input is $x(t) = u(t)$, which has $X(s) = \frac{1}{s}$. By inspection of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2\dot{x}(t) - x(t)$, the system transfer function is

$$H(s) = \frac{2s - 1}{s^2 + 3s + 2} = \frac{2s - 1}{(s+1)(s+2)}.$$

The transform of the zero-state response is

$$Y_{\text{zsr}}(s) = X(s)H(s) = \frac{2s - 1}{s(s+1)(s+2)} = \frac{-\frac{1}{2}}{s} + \frac{3}{s+1} + \frac{-\frac{5}{2}}{s+2}.$$

Inverting,

$$y_{\text{zsr}}(t) = \left[-\frac{1}{2} + 3e^{-t} - \frac{5}{2}e^{-2t}\right]u(t).$$

Solution 4.3-5

(a) Laplace transform of the two equations yields

$$(s+3)Y_1(s) - 2Y_2(s) = \frac{1}{s}$$
$$-2Y_1(s) + (2s+4)Y_2(s) = 0$$

Using Cramer's rule, we obtain

$$Y_1(s) = \frac{s+2}{s(s^2+5s+4)} = \frac{s+2}{s(s+1)(s+4)} = \frac{1/2}{s} - \frac{1/3}{s+1} - \frac{1/6}{s+4}$$

$$Y_2(s) = \frac{1}{s(s^2+5s+4)} = \frac{1}{s(s+1)(s+4)} = \frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}.$$

Thus,

$$y_1(t) = \left(\frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{-4t}\right)u(t)$$

$$y_2(t) = \left(\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right)u(t)$$

Since $Y(s) = X(s)H(s)$ and $X(s) = \frac{1}{s}$, we see that

$$H_1(s) = \frac{s+2}{s^2+5s+4} \quad \text{and} \quad H_2(s) = \frac{1}{s^2+5s+4}.$$

(b) The Laplace transform of the equations are

$$(s+2)Y_1(s) - (s+1)Y_2(s) = 0$$
$$-(s+1)Y_1(s) + (2s+1)Y_2(s) = 0$$

Application of Cramer's rule yields

$$Y_1(s) = \frac{s+1}{s(s^2+3s+1)} = \frac{s+1}{s(s+0.382)(s+2.618)} = \frac{1}{s} - \frac{0.724}{s+0.382} - \frac{0.276}{s+2.618}$$

$$Y_2(s) = \frac{s+2}{s(s^2+3s+1)} = \frac{s+2}{s(s+0.382)(s+2.618)} = \frac{2}{s} - \frac{1.894}{s+0.382} - \frac{0.1056}{s+2.618}.$$

Consequently,

$$y_1(t) = (1 - 0.724e^{-0.382t} - 0.276e^{-2.618t})u(t)$$
$$y_2(t) = (2 - 1.894e^{-0.382t} - 0.1056e^{-2.618t})u(t)$$

Furthermore,

$$H_1(s) = \frac{s+1}{s^2+3s+1} \quad \text{and} \quad H_2(s) = \frac{s+2}{s^2+3s+1}.$$

Solution 4.3-6

(a) Setting all initial conditions to zero and taking the Laplace transform of $\dot{y}(t) + 2y(t) = \dot{x}(t)$ yield

$$sY(s) + 2Y(s) = sX(s) \Longrightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s+2}.$$

(b) First, we notice that $H(s)$ can be expressed as

$$H(s) = \frac{s}{s+2} = 1 + \frac{-2}{s+2}.$$

Inverting, we obtain the unit impulse response as

$$h(t) = \delta(t) - 2e^{-2t}u(t).$$

(c) Taking the unilateral Laplace transform of $\dot{y}(t) + 2y(t) = \dot{x}(t)$ yields

$$sY(s) - y(0^-) + 2Y(s) = sX(s) - x(0^-).$$

In this case, $x(t) = e^{-t}u(t)$ and $X(s) = \frac{1}{s+1}$. Further, $y(0^-) = \sqrt{2}$ and $x(0^-) = 0$. Thus,

$$(s+2)Y(s) = \sqrt{2} + s\frac{1}{s+1}$$

or

$$Y(s) = \frac{s}{(s+1)(s+2)} + \frac{\sqrt{2}}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2} + \frac{\sqrt{2}}{s+2}.$$

Inverting yields

$$y_{\text{total}}(t) = -e^{-t}u(t) + (2 + \sqrt{2})e^{-2t}u(t).$$

Solution 4.3-7

Expressing $3y(t) + \dot{y}(t) + \dot{x}(t) = 0$ in standard form yields

$$\dot{y}(t) + 3y(t) = -\dot{x}(t).$$

(a) Setting all initial conditions to zero and taking the Laplace transform of $\dot{y}(t) + 3y(t) = -\dot{x}(t)$ yield

$$sY(s) + 3Y(s) = -sX(s) \Longrightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{-s}{s+3}.$$

(b) First, we notice that $H(s)$ can be expressed as

$$H(s) = \frac{-s}{s+3} = -1 + \frac{3}{s+3}.$$

Inverting, we obtain the unit impulse response as

$$h(t) = -\delta(t) + 3e^{-3t}u(t).$$

(c) Taking the unilateral Laplace transform of $\dot{y}(t) + 3y(t) = -\dot{x}(t)$ yields

$$sY(s) - y(0^-) + 3Y(s) = -sX(s) + x(0^-).$$

In this case, $x(t) = e^{-t}u(t)$ and $X(s) = \frac{1}{s+1}$. Further, $y(0^-) = \sqrt{2}$ and $x(0^-) = 0$. Thus,

$$(s+3)Y(s) = \sqrt{2} - s\frac{1}{s+1}$$

or

$$Y(s) = \frac{-s}{(s+1)(s+3)} + \frac{\sqrt{2}}{s+3} = \frac{\frac{1}{2}}{s+1} + \frac{-\frac{3}{2}}{s+3} + \frac{\sqrt{2}}{s+3}.$$

Inverting yields

$$y_{\text{total}}(t) = \frac{1}{2}e^{-t}u(t) + \left(\sqrt{2} - \frac{3}{2}\right)e^{-3t}u(t).$$

Solution 4.3-8

At $t = 0$, the inductor current $y_1(0) = 4$ and the capacitor voltage is 16 volts. After $t = 0$, the loop equations are

$$2\frac{dy_1}{dt} - 2\frac{dy_2}{dt} + 5y_1(t) - 4y_2(t) = 40$$

$$-2\frac{dy_1}{dt} - 4y_1(t) + 2\frac{dy_2}{dt} + 4y_2(t) + \int_{-\infty}^{t} y_2(\tau)\, d\tau = 0.$$

Using the given initial conditions, we see that

$$y_1(t) \iff Y_1(s), \qquad \frac{dy_1}{dt} \iff sY_1(s) - 4$$
$$y_2(t) \iff Y_2(s), \qquad \frac{dy_2}{dt} \iff sY_2(s)$$

$$\int_{-\infty}^{t} y_2(\tau)\, d\tau \iff \frac{1}{s}Y_2(s) + \frac{16}{s}$$

Thus, the Laplace transform of the loop equations are

$$2(sY_1(s) - 4) - 2sY_2(s) + 5Y_1(s) - 4Y_2(s) = \frac{40}{s}$$

$$-2(sY_1(s) - 4) - 4Y_1(s) + 2sY_2(s) + 4Y_2(s) + \frac{1}{s}Y_2(s) + \frac{16}{s} = 0$$

or

$$(2s + 5)Y_1(s) - (2s + 4)Y_2(s) = 8 + \frac{40}{s}$$

$$-(2s + 4)Y_1(s) + (2s + 4 + \frac{1}{s})Y_2(s) = -8 - \frac{16}{s}.$$

Cramer's rule yields

$$Y_1(s) = \frac{4(6s^2 + 13s + 5)}{s(s^2 + 3s + 2.5)} = \frac{8}{s} + \frac{16s + 28}{s^2 + 3s + 2.5}$$

$$Y_2(s) = \frac{20(s + 2)}{(s^2 + 3s + 2.5)}.$$

Inverting yields

$$y_1(t) = [8 + 17.89e^{-1.5t}\cos(\frac{t}{2} - 26.56°)]u(t)$$

$$y_2(t) = 20\sqrt{2}e^{-1.5t}\cos(\frac{t}{2} - \frac{\pi}{4})u(t).$$

Solution 4.3-9

(a) $\frac{5s+3}{s^2+11s+24}$

(b) $\frac{3s^2+7s+5}{s^3+6s^2-11s+6}$

(c) $\frac{3s+2}{s(s^3+4)}$

(d) $\frac{1}{s+1}$

Solution 4.3-10

(a) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 8y(t) = \frac{dx}{dt} + 5x(t)$

(b) $\frac{d^3y}{dt^3} + 8\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 7y(t) = \frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 5x(t)$

(c) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y(t) = 5\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 2x(t)$

Solution 4.3-11

(a) For $x_1(t) = 10u(t)$, $X_1(s) = \frac{10}{s}$ and the zero-state response is

$$Y_1(s) = \frac{10(2s+3)}{s(s^2+2s+5)} = \frac{6}{s} + \frac{-6s+8}{s^2+2s+5}$$

$$y_1(t) = [6 + 9.22e^{-t}\cos(2t - 130.6°)]u(t)$$

For $x_2(t) = u(t-5)$, $X_2(s) = \frac{1}{s}e^{-5s}$ and the zero-state response is

$$Y_2(s) = \frac{2s+3}{s(s^2+2s+5)}e^{-5s} = \left[\frac{0.6}{s} + \frac{1}{10}\left(\frac{-6s+8}{s^2+2s+5}\right)\right]e^{-5s}$$

$$y_2(t) = \frac{1}{10}\{6 + 9.22e^{-(t-5)}\cos[2(t-5) - 130.6°]\}u(t-5)$$

(b) $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 2\dot{x}(t) + 3x(t)$

Solution 4.3-12

(a) $X(s) = \frac{1}{s(s+1)}$

$$Y(s) = \frac{1}{(s+1)(s^2+9)} = \frac{0.1}{s+1} + \frac{re^{j\theta}}{s+j3} + \frac{re^{-j\theta}}{s-j3} \qquad r = \frac{1}{3\sqrt{10}}, \; \theta = -161.56°$$

$$y(t) = 0.1e^{-t} + \frac{1}{3\sqrt{10}}\cos(3t - 161.56°)$$

(b)

$$\ddot{y}(t) + 9y(t) = \dot{x}(t)$$

Solution 4.3-13

(a) Here, $X_a(s) = \frac{1}{s+3}$ and

$$Y_a(s) = \frac{s+5}{(s+3)(s^2+5s+6)} = \frac{s+5}{(s+2)(s+3)^2} = \frac{3}{s+2} - \frac{3}{s+3} - \frac{2}{(s-3)^2}$$

$$y_a(t) = (3e^{-2t} - 3e^{-3t} - 2te^{-3t})u(t).$$

(b) In this case, $X_b(s) = \frac{1}{s+4}$. Thus,

$$Y_b(s) = \frac{s+5}{(s+2)(s+3)(s+4)} = \frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{(s+4)}$$

$$y_b(t) = \frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}u(t)$$

(c) Note that $x_c(t)$ is just $x_b(t)$ delayed by 5 seconds. Therefore, $X_c(s) = \frac{1}{s+4}e^{-5s}$ and

$$Y_c(s) = \frac{s+5}{(s+2)(s+3)(s+4)}e^{-5s} = [\frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{(s+4)}]e^{-5s}$$

$$y_c(t) = \frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}]u(t-5)$$

(d) Note that $x_d(t)$ is just $x_b(t)$ multiplied by e^{20} because $e^{-4(t-5)} = e^{20}e^{-4t}$. Therefore the output $y_d(t)$ is equal to the output $y_b(t)$ multiplied by e^{20},

$$y_d(t) = e^{20}[\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}]u(t).$$

(e) Note that $x_e(t)$ is just $x_c(t)$ multiplied by e^{-20} because $e^{-4t}u(t-5) = e^{-20}e^{-4(t-5)}u(t-5)$. Therefore,

$$y_e(t) = e^{-20}[\frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}]u(t-5).$$

(f) $(D^2 + 2D + 5)y(t) = (2D + 3)x(t)$

Solution 4.3-14

Although this problem can be solved with Laplace transforms, it is easier to solve in the time domain. Since the system step response is $s(t) = e^{-t}u(t) - e^{-2t}u(t)$, the system impulse response is $h(t) = \frac{d}{dt}s(t) = -e^{-t}u(t) + \delta(t) + 2e^{-2t}u(t) - \delta(t) = (2e^{-2t} - e^{-t})u(t)$. The input $x(t) = \delta(t - \pi) - \cos(\sqrt{3})u(t)$ is just a sum of a shifted delta function and a scaled step function. Since the system is LTI, the output is quickly computed using just $h(t)$ and $s(t)$. That is,

$$y(t) = h(t - \pi) - \cos(\sqrt{3})s(t) = (2e^{-2(t-\pi)} - e^{-(t-\pi)})u(t - \pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})u(t).$$

Solution 4.3-15

(a) Let $H(s)$ be the system transfer function,

$$Y(s) = X(s)H(s).$$

Consider the input $x_1(t) = \dot{x}(t)$. Then $X_1(s) = sX(s)$. If the output is $y_1(t)$ and its transform is $Y_1(s)$, then

$$Y_1(s) = X_1(s)H(s) = sX(s)H(s) = sY(s).$$

This shows that

$$y_1(t) = \frac{dy(t)}{dt}.$$

(b) Using similar argument we show that for the input $\int_0^t x(\tau)\,d\tau$, the output is $\int_0^t y(\tau)\,d\tau$. Because $u(t)$ is an integral of $\delta(t)$, the unit step response is the integral of the unit impulse response $h(t)$.

Solution 4.3-16

(a)
$$H(s) = \frac{s+5}{s^2 + 3s + 2} = \frac{s+5}{(s+1)(s+2)}$$

Both characteristic roots ($\lambda_1 = -1$ and $\lambda_2 = -2$) are in the LHP. Hence the system is both BIBO and asymptotically stable.

(b)

$$H(s) = \frac{s+5}{s^2(s+2)}$$

The characteristic roots are 0, 0, -2. There are repeated roots on imaginary axis. Hence the system is both BIBO and asymptotically unstable.

(c)

$$H(s) = \frac{s(s+2)}{s+5}$$

Although the characteristic root -5 is in the LHP, because $M > N$, the system is BIBO unstable.

(d)

$$H(s) = \frac{s+5}{s(s+2)}$$

The characteristic roots are 0 and -2. One of the roots is on the imaginary axis and the other is in the LHP, which makes the system BIBO unstable but marginally stable.

(e)

$$H(s) = \frac{s+5}{s^2-2s+3} = \frac{s+5}{(s-1-j\sqrt{2})(s-1+j\sqrt{2})}$$

The roots are $-1 \pm j\sqrt{2}$. Since both roots are in the LHP, the system is both BIBO and asymptotically stable.

Solution 4.3-17

(a) Here,

$$(D^2 + 3D + 2)y(t) = (D+3)x(t)$$

or

$$(D+1)(D+2)y(t) = (D+3)x(t).$$

The system transfer function is

$$H(s) = \frac{s+3}{(s+1)(s+2)}.$$

The characteristic roots are -1 and -2 (both in LHP). Hence the system is asymptotically and BIBO stable.

(b) In this case,

$$(D^2 + 3D + 2)y(t) = (D+1)x(t)$$

or

$$(D+1)(D+2)y(t) = (D+1)x(t).$$

The system transfer function is

$$H(s) = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}.$$

The characteristic roots are -1 and -2 (both in LHP). Due to a pole/zero cancellation, the only pole of H(s) is at -2. Hence the system is asymptotically and BIBO stable.

(c) In this case,
$$(D^2 + D - 2)y(t) = (D - 1)x(t)$$

or
$$(D - 1)(D + 2)y(t) = (D - 1)x(t)$$

The system transfer function is
$$H(s) = \frac{s - 1}{(s - 1)(s + 2)} = \frac{1}{s + 2}.$$

The system's characteristic roots at 1 and -2 makes system asymptotically unstable. Because a system zero cancels the pole at 1, the only pole of H(s) is at -2. Thus, the system is BIBO stable.

(d) Now,
$$(D^2 - 3D + 2)y(t) = (D - 1)x(t)$$

or
$$(D - 1)(D - 2)y(t) = (D - 1)x(t).$$

The system transfer function is
$$H(s) = \frac{s - 1}{(s - 1)(s - 2)} = \frac{1}{s - 2}.$$

The characteristic roots are 1 and 2. Because a system zero cancels the pole at 1, the only pole of H(s) is at 2. Hence, the system is both asymptotically and BIBO unstable.

Solution 4.4-1

The system has transfer function $H(s) = \frac{1}{1+6s}$ and differential equation
$$y(t) + 6\dot{y}(t) = x(t).$$

Taking the unilateral Laplace transform yields
$$Y(s) + 6\left[sY(s) - y(0^-)\right] = X(s).$$

(a) In this part, we note that $y(0^-) = 3$ and $X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$. Thus,
$$Y(s)\left[1 + 6s\right] = \frac{1}{s} + 18.$$

Solving for $Y(s)$ produces
$$Y(s) = \frac{1}{s(1 + 6s)} + \frac{18}{1 + 6s} = \frac{\frac{1}{6}}{s(s + \frac{1}{6})} + \frac{3}{s + \frac{1}{6}} = \frac{1}{s} + \frac{-1}{s + \frac{1}{6}} + \frac{3}{s + \frac{1}{6}} = \frac{1}{s} + \frac{2}{s + \frac{1}{6}}.$$

Inverting, we see that
$$y(t) = u(t) + 2e^{-t/6}u(t).$$

(b) In this part, $X(s) = \frac{e^{-3s}}{s}$ and we desire $y(0^-)$ so that $y(6) = 1$. From the Laplace transform of the differential equation, we see that
$$Y(s)\left[1 + 6s\right] = \frac{e^{-3s}}{s} + 6y(0^-).$$

Thus,

$$Y(s) = e^{-3s}\frac{1}{s} + e^{-3s}\frac{-1}{s + \frac{1}{6}} + \frac{y(0^-)}{s + \frac{1}{6}}.$$

Inverting yields

$$y(t) = u(t - 3) - e^{-(t-3)/6}u(t - 3) + y(0^-)e^{-t/6}u(t).$$

Evaluating at $t = 6$ yields

$$y(6) = 1 = 1 - e^{-1/2} + y(0^-)e^{-1}.$$

Thus,

$$y(0^-) = (1 - 1 + e^{-1/2})e^1 = e^{1/2} \approx 1.6487.$$

Solution 4.4-2

(a) Designate the current flowing clockwise around the loop as $i(t)$. By KVL,

$$x(t) = Ri(t) + y(t) \Rightarrow Ri(t) = x(t) - y(t).$$

The voltage $v_c(t)$ across the capacitor is

$$v_c(t) = y(t) - Ri(t) = y(t) - (x(t) - y(t)) = 2y(t) - x(t).$$

For an ideal capacitor,

$$i(t) = C\frac{dv_c(t)}{dt} = 2C\dot{y}(t) - C\dot{x}(t).$$

Substituting this expression for $i(t)$ in the KVL equation yields

$$x(t) = R\left(2C\dot{y}(t) - C\dot{x}(t)\right) + y(t).$$

Simplifying, we obtain

$$\dot{y}(t) + \frac{1}{2RC}y(t) = \frac{1}{2}\dot{x}(t) + \frac{1}{2RC}x(t).$$

(b) For $x(t) = 3e^{-t}u(t)$, it follows that $X(s) = \frac{3}{s+1}$. Letting $R = C = 1$, we see that

$$\dot{y}(t) + \frac{1}{2}y(t) = \frac{1}{2}\dot{x}(t) + \frac{1}{2}x(t).$$

Taking the unilateral Laplace transform yields

$$sY(s) - y(0^-) + \frac{1}{2}Y(s) = \frac{1}{2}sX(s) + \frac{1}{2}X(s) = \frac{(s + 1)3}{2(s + 1)} = \frac{3}{2}.$$

Since $v_c(0^-) = 5$, we see that $i(0^-) = -\frac{5}{2}$ and $y(0^-) = 5 + 1i(0^-) = \frac{5}{2}$. Substituting this result, we obtain

$$\left(s + \frac{1}{2}\right)Y(s) - \frac{5}{2} = \frac{3}{2}.$$

Thus,

$$Y(s) = \frac{4}{s + \frac{1}{2}}.$$

Inverting, we obtain

$$y(t) = 4e^{-t/2}u(t).$$

Solution 4.4-3

Figure S4.4-3 shows the transformed network. The loop equations are

$$(1 + \frac{1}{s})Y_1(s) - \frac{1}{s}Y_2(s) = \frac{1}{(s+1)^2}$$

$$-\frac{1}{s}Y_1(s) + (s+1+\frac{1}{s})Y_2(s) = 0$$

$$\begin{bmatrix} \frac{s+1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & \frac{s^2+s+1}{s} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} \\ 0 \end{bmatrix}.$$

Cramer's rule yields

$$Y_2(s) = \frac{1}{(s+1)^2(s^2+2s+2)} = \frac{1}{(s+1)^2} - \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2+1}.$$

Inverting, we obtain

$$v_0(t) = y_2(t) = (te^{-t} - e^{-t}\sin t)u(t).$$
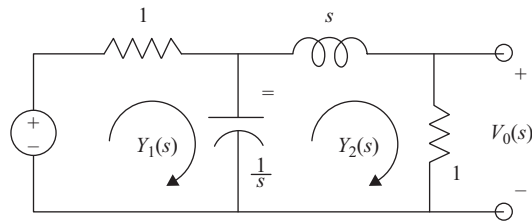
Figure S4.4-3

Solution 4.4-4

Before the switch is opened, the inductor current is 5A, that is $y(0) = 5$. Figure S4.4-4 shows the transformed circuit for $t \geq 0$ with an initial condition generator. The current $Y(s)$ is given by

$$Y(s) = \frac{(10/s) + 5}{3s + 2} = \frac{5s + 10}{s(3s + 2)} = \frac{5}{3}\left[\frac{3}{s} - \frac{2}{s + (2/3)}\right].$$

Inverting, we obtain

$$y(t) = (5 - \frac{10}{3}e^{-2t/3})u(t).$$
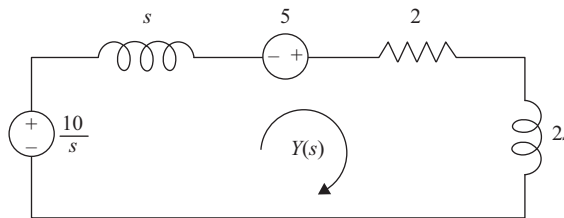
Figure S4.4-4

Solution 4.4-5

The impedance seen by the source $x(t)$ is

$$Z(s) = \frac{Ls(1/Cs)}{Ls + (1/Cs)} = \frac{Ls}{LCs^2 + 1} = \frac{Ls\omega_0^2}{s^2 + \omega_0^2}.$$

The current $Y(s)$ is given by

$$Y(s) = \frac{X(s)}{Z(s)} = \frac{s^2 + {\omega_0}^2}{Ls{\omega_0}^2}X(s).$$

(a)

$$X(s) = \frac{As}{s^2 + {\omega_0}^2}, \qquad Y(s) = \frac{A}{L{\omega_0}^2}, \quad \text{and} \quad y(t) = \frac{A}{L{\omega_0}^2}\delta(t).$$

(b)

$$X(s) = \frac{A\omega_0}{s^2 + {\omega_0}^2}, \qquad Y(s) = \frac{A}{L\omega_0 s}, \quad \text{and} \quad y(t) = \frac{A}{L\omega_0}u(t).$$

Solution 4.4-6

At $t = 0$, the steady-state values of currents y_1 and y_2 are $y_1(0) = 2$ and $y_2(0) = 1$. Figure S4.4-6 shows the transformed circuit for $t \geq 0$ with initial condition generators. The loop equations are

$$(s+2)Y_1(s) - Y_2(s) = 2 + \frac{6}{s}$$
$$-Y_1(s) + (s+2)Y_2(s) = 1.$$

Cramer's rule yields

$$Y_1(s) = \frac{2s^2 + 11s + 12}{s(s+1)(s+3)} = \frac{4}{s} - \frac{3/2}{s+1} - \frac{1/2}{s+3}$$
$$Y_2(s) = \frac{s^2 + 4s + 6}{s(s+1)(s+3)} = \frac{2}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3}.$$

Inverting, we obtain

$$y_1(t) = (4 - \tfrac{3}{2}e^{-t} - \tfrac{1}{2}e^{-3t})u(t)$$
$$y_2(t) = (2 - \tfrac{3}{2}e^{-t} + \tfrac{1}{2}e^{-3t})u(t).$$
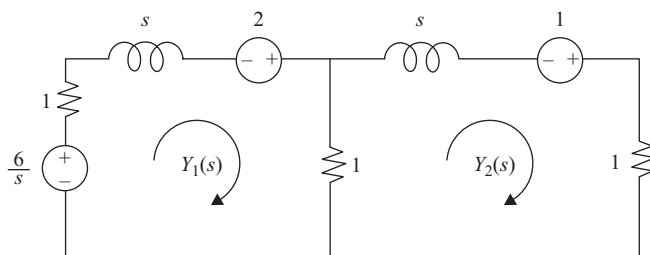
Figure S4.4-6

Solution 4.4-7

The current in the 2 H inductor at $t = 0$ is 10 A. The transformed circuit with initial condition generators is shown in Figure S4.4-7 for $t \geq 0$.

$$Y_1(s) = \frac{\frac{10}{s} + 20}{3s + \frac{1}{s} + 1} = \frac{20s + 10}{3s^2 + s + 1} = \frac{20}{3}\left[\frac{s + 0.5}{s^2 + \frac{1}{3s} + \frac{1}{3}}\right].$$

Here $A = 1$, $B = 0.5$, $a = \frac{1}{6}$, $c = \frac{1}{3}$, $b = \frac{\sqrt{11}}{6}$, $r = \sqrt{\frac{15}{11}} = 1.168$, and $\theta = \tan^{-1}(\frac{-2}{\sqrt{11}}) = -31.1°$. Thus,

$$y_1(t) = \frac{20}{3}(1.168)e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 31.1°)u(t)$$
$$= 7.787e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 31.1°)u(t).$$

The voltage $v_s(t)$ across the switch is determined according to

$$V_s(s) = \left(s + \frac{1}{s}\right)Y(s) = \left(\frac{s^2+1}{s}\right)\left(\frac{20s+10}{3s^2+s+1}\right) = \frac{20}{s}\frac{(s^2+1)(s+0.5)}{s(s^2+\frac{1}{3s}+\frac{1}{3})}$$
$$= \frac{20}{3}\left[1 + \frac{3/2}{s} + \frac{1}{6}\frac{-8s+1}{s^2+1/3s+1/3}\right]$$
$$v_s(t) = \frac{20}{3}\delta(t) + [10 + 10.05e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 152.2°)]u(t).$$
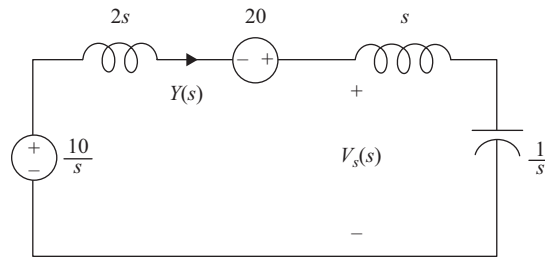
Figure S4.4-7

Solution 4.4-8

Figure S4.4-8 shows the transformed circuit with mutually coupled inductor replaced by their equivalents (see Fig. 4.14b). The loop equations are

$$(s+1)Y_1(s) - 2sY_2(s) = \frac{100}{s}$$
$$-2sY_1(s) + (4s+1)Y_2(s) = 0.$$

Cramer's rule yields

$$Y_2(s) = \frac{40}{(s+0.2)}.$$

Inverting, we obtain

$$v_0(t) = y_2(t) = 40e^{-t/5}u(t).$$
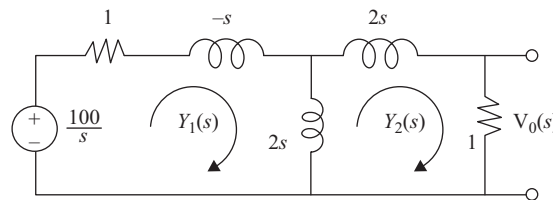
Figure S4.4-8

Solution 4.4-9

Figure S4.4-9 shows the transformed circuit with the parallel form initial condition generators. The admittance $W(s)$ seen by the source is

$$W(s) = \frac{13}{s} + s + 4 = \frac{s^2 + 4s + 13}{s}.$$

The voltage across the source terminals is

$$V(s) = \frac{I(s)}{W(s)} = \frac{\frac{1}{s} + 3}{\frac{s^2 + 4s + 13}{s}} = \frac{3s + 1}{s^2 + 4s + 13}.$$

Also

$$V_0(s) = \frac{1}{2}V(s) = \frac{3s + 1}{2(s^2 + 4s + 13)}.$$

Inverting, we obtain

$$y(t) = v_0(t) = 1.716e^{-2t}\cos(3t + 29°)u(t).$$
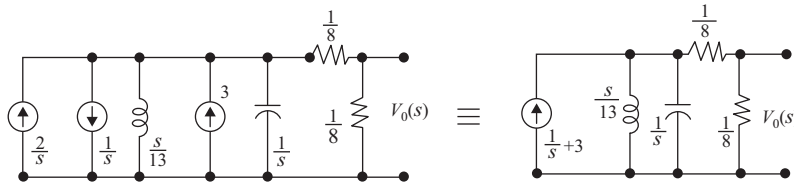
Figure S4.4-9

Solution 4.4-10

The capacitor voltage at $t = 0$ is 10 volts. The inductor current is zero. The transformed circuit with initial condition generators is shown for $t > 0$ in Fig. S4.4-10. To determine the current $Y(s)$, we determine $Z_{ab}(s)$, the impedance seen across terminals ab:

$$Z_{ab}(s) = \frac{1}{1 + \left(\frac{1}{2 + \frac{s+2}{s+3}}\right)} = \frac{3s + 8}{4s + 11}.$$

Also,

$$Y(s) = \frac{\frac{90}{s}}{\frac{5}{s} + \left(\frac{3s+8}{4s+11}\right)} = \frac{90(4s + 11)}{3s^2 + 28s + 55}$$

$$= \frac{30(4s + 11)}{s^2 + \frac{28}{3}s + \frac{55}{3}} = \frac{30(4s + 11)}{(s + 2.8)(s + 6.53)}$$

$$= -\frac{1.61}{s + 2.8} + \frac{121.61}{s + 6.53}.$$

Inverting, we obtain

$$y(t) = [121.61e^{-6.53t} - 1.61e^{-2.8t}]u(t).$$
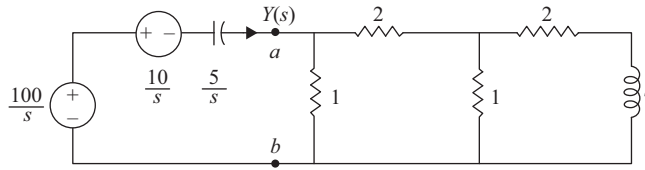
Figure S4.4-10

Solution 4.4-11

(a) Designate $i(t)$ as the clockwise loop current of the circuit. Applying KVL in the transform domain, we obtain

$$X(s) = 2RI(s) + Y(s).$$

Also,

$$Y(s) = I(s)\left[R + Ls\right].$$

Thus,

$$X(s) = \frac{2R}{R + Ls}Y(s) + Y(s) \quad \Rightarrow \quad (Ls + 3R)Y(s) = (Ls + R)X(s).$$

Solving for the transfer function, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s + \frac{R}{L}}{s + \frac{3R}{L}}.$$

(b) Since $x(t) = e^{-2}e^{-2(t-1)}u(t-1)$, we see that

$$X(s) = e^{-2}e^{-s}\frac{1}{s+2}.$$

Using $L = R = 1$, the transform of the zero-state response is

$$Y_{\text{zsr}}(s) = X(s)H(s) = e^{-2}e^{-s}\frac{1}{s+2}\left(\frac{s+1}{s+3}\right) = e^{-2}e^{-s}\left[\frac{-1}{s+2} + \frac{2}{s+3}\right].$$

Inverting, we obtain

$$y_{\text{zsr}}(t) = -e^{-2}e^{-2(t-1)}u(t-1) + 2e^{-2}e^{-3(t-1)}u(t-1)$$

or

$$y_{\text{zsr}}(t) = \left[2e^{1}e^{-3t} - e^{-2t}\right]u(t-1).$$

(c) again, $x(t) = e^{-2}e^{-2(t-1)}u(t-1)$ and

$$X(s) = e^{-2}e^{-s}\frac{1}{s+2}.$$

Using $R = 2L = 1$, the transform of the zero-state response is

$$Y_{\text{zsr}}(s) = X(s)H(s) = e^{-2}e^{-s}\frac{1}{s+2}\left(\frac{s+2}{s+6}\right) = e^{-2}e^{-s}\frac{1}{s+6}.$$

Inverting, we obtain

$$y_{\text{zsr}}(t) = e^{-2}e^{-6(t-1)}u(t-1) = e^{4}e^{-6t}u(t-1).$$

Solution 4.4-12

(a) Figure S4.4-12a shows the transformed circuit with its noninverting op amp replaced by its equivalent according to Fig. 4.16. From KVL, we know that $X(s) = I(s)R + V_c(s)$. Since $I(s) = CsV_c(s)$, we know $X(s) = (RCs + 1)V_c(s)$. Thus, $V_c(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} X(s)$. The op-amp output $V_0(s)$ is just K times the capacitor (input) voltage $V_c(s)$. That is,

$$V_0(s) = KV_c(s) = K\frac{\frac{1}{RC}}{s + \frac{1}{RC}} X(s), \quad \text{where } K = 1 + \frac{R_b}{R_a}.$$

Since $H(s) = \frac{V_0(s)}{X(s)}$, we see that

$$H(s) = \frac{Ka}{s + a}, \quad \text{where } a = \frac{1}{RC} \text{ and } K = 1 + \frac{R_b}{R_a}.$$

(b) Figure S4.4-12b shows the transformed circuit with its noninverting op amp replaced by its equivalent according to Fig. 4.16. From KVL, we know that $X(s) = I(s)\frac{1}{Cs} + V_1(s)$. Since $I(s) = \frac{1}{R}V_1(s)$, we know $X(s) = \left(\frac{1}{RCs} + 1\right)V_1(s)$. Thus, $V_1(s) = \frac{s}{s + \frac{1}{RC}}X(s)$. The op-amp output $V_0(s)$ is just K times the resistor (input) voltage $V_1(s)$. That is,

$$V_0(s) = KV_1(s) = K\frac{s}{s + \frac{1}{RC}} X(s), \quad \text{where } K = 1 + \frac{R_b}{R_a}.$$

Since $H(s) = \frac{V_0(s)}{X(s)}$, we see that

$$H(s) = \frac{Ks}{s + a}, \quad \text{where } a = \frac{1}{RC} \text{ and } K = 1 + \frac{R_b}{R_a}.$$
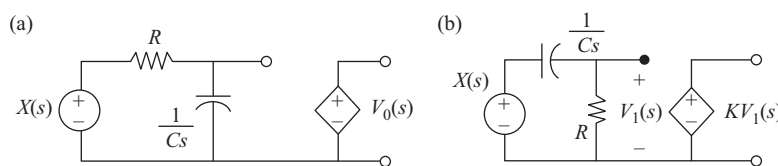
Figure S4.4-12

Solution 4.4-13

Figure S4.4-13 shows the transformed circuit. The op amp input voltage is $V_x(s) \simeq 0$. The loop equations are

$$I_1(s) + \left(\frac{6}{s} + 1\right)[I_1(s) - I_2(s)] = X(s)$$

$$-\frac{3}{s}I_2(s) + \left(\frac{6}{s} + \frac{3}{2}\right)[I_1(s) - I_2(s)] = 0.$$

Cramer's rule yields

$$I_1(s) = \frac{s(s+6)}{s^2 + 8s + 12}X(s) \quad \text{and} \quad I_2(s) = \frac{s(s+4)}{s^2 + 8s + 12}.$$

Thus,

$$Y(s) = -\frac{1}{2}[I_1(s) - I_2(s)] = \frac{-s}{s^2 + 8s + 12}X(s).$$

The transfer function is therefore

$$H(s) = \frac{-s}{s^2 + 8s + 12}.$$
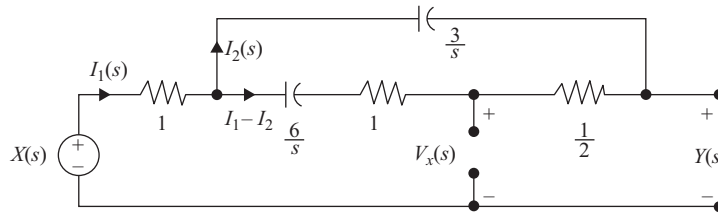
Figure S4.4-13

Solution 4.4-14

(a) Working from left to right, designate the op amp outputs as $y_1(t)$, $y_2(t)$, and $y_3(t)$. Now, $y_3(t) = -\frac{1}{3}y(t)$ and $y_1(t) = -2\int_{-\infty}^t y_3(\tau)d\tau - \frac{1}{2}\int_{-\infty}^t x(\tau)d\tau$. Thus, $y_1 = \frac{2}{3}\int_{-\infty}^t y(\tau)d\tau - \frac{1}{2}\int_{-\infty}^t x(\tau)d\tau$. Further,

$$y(t) = y_2(t) = -4x(t) - 6y_1(t) = -4x(t) - 4\int_{-\infty}^t y(\tau)d\tau + 3\int_{-\infty}^t x(\tau)d\tau.$$

or

$$y(t) + 4\int_{-\infty}^t y(\tau)d\tau = -4x(t) + 3\int_{-\infty}^t x(\tau)d\tau.$$

Taking the Laplace transform we obtain

$$Y(s) + 4\frac{1}{s}Y(s) = -4X(s) + 3\frac{1}{s}X(s).$$

Solving for $H(s)$, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{-4s + 3}{s + 4}.$$

(b) From part (a), we know that

$$y(t) + 4\int_{-\infty}^t y(\tau)d\tau = -4x(t) + 3\int_{-\infty}^t x(\tau)d\tau.$$

Differentiating, we obtain

$$\dot{y}(t) + 4y(t) = -4\dot{x}(t) + 3x(t).$$

(c) Since $x(t) = e^{-2}e^{2(t+1)}u(t+1)$, we see that

$$X(s) = e^{-2}e^s\frac{1}{s - 2}.$$

The transform of the zero-state response is

$$Y_{\text{zsr}}(s) = X(s)H(s) = e^{-2}e^s\frac{1}{s - 2}\left(\frac{-4s + 3}{s + 4}\right) = e^{-2}e^s\left[\frac{-\frac{5}{6}}{s - 2} + \frac{-\frac{19}{6}}{s + 4}\right].$$

Inverting, we obtain

$$y_{\text{zsr}}(t) = e^{-2}\left[-\frac{5}{6}e^{2(t+1)} - \frac{19}{6}e^{-4(t+1)}\right]u(t + 1)$$

or

$$y_{\text{zsr}}(t) = \left[-\frac{5}{6}e^{2t} - \frac{19}{6}e^{-6}e^{-4t}\right]u(t + 1).$$

(d) Since $y_1(0^-) = 3$, we know that $y(0^-) = 6y_1(0^-) = 18$. Setting $x(t) = 0$ in part (b), we obtain

$$\dot{y}_{\text{zir}}(t) + 4y_{\text{zir}}(t) = 0.$$

Taking the unilateral Laplace transform produces

$$sY_{\text{zir}}(s) - y(0^-) + 4Y_{\text{zir}}(s) = 0.$$

Thus,

$$Y_{\text{zir}}(s) = \frac{18}{s+4}.$$

Inverting, we obtain

$$y_{\text{zir}}(t) = 18e^{-4t}u(t).$$

Solution 4.4-15

(a) Labeling the output voltage of the first op amp as $v_{\text{c}}(t)$, we see that

$$v_{\text{c}}(t) = -\frac{1}{R_1 C} \int_{-\infty}^{t} x(\tau)d\tau - \frac{1}{R_2 C} \int_{-\infty}^{t} y(\tau)d\tau.$$

From the second op amp output, we see that

$$y(t) = -\frac{R_3}{R_2}x(t) - \frac{R_3}{R_1}v_{\text{c}}(t).$$

Combining yields

$$y(t) = -\frac{R_3}{R_2}x(t) - \frac{R_3}{R_1}\left(-\frac{1}{R_1 C}\int_{-\infty}^{t} x(\tau)d\tau - \frac{1}{R_2 C}\int_{-\infty}^{t} y(\tau)d\tau\right).$$

Differentiating and expressing in standard form, we have

$$\dot{y}(t) - \frac{R_3}{R_1 R_2 C}y(t) = -\frac{R_3}{R_2}\dot{x}(t) + \frac{R_3}{R_1^2 C}x(t).$$

The desired differential equation is $\dot{y}(t) - 1.5y(t) = -3\dot{x}(t) + 0.75x(t)$. From the $\dot{x}(t)$ coefficient we see that

$$-\frac{R_3}{R_2} = -3.$$

Using this with the $y(t)$ coefficient and $C = 100\ \mu\text{F}$ yields

$$\frac{-3}{R_1 10^{-4}} = -\frac{3}{2} \qquad \Rightarrow \qquad R_1 = 20\ \text{k}\Omega.$$

Similarly using the $x(t)$ coefficient we obtain

$$\frac{R_3}{(2\times 10^4)^2 10^{-4}} = \frac{3}{4} \qquad \Rightarrow \qquad R_3 = 30\ \text{k}\Omega.$$

Substituting the R_3 value into the $\dot{x}(t)$ coefficient we obtain

$$\frac{-30\times 10^3}{R_2} = -3 \qquad \Rightarrow \qquad R_2 = 10\ \text{k}\Omega.$$

Summarizing,

$$R_1 = 20\ \text{k}\Omega, \quad R_2 = 10\ \text{k}\Omega, \quad R_3 = 30\ \text{k}\Omega.$$

(b) From the second op amp output, we see that

$$y(t) = -\frac{R_3}{R_2}x(t) - \frac{R_3}{R_1}v_{\rm c}(t).$$

Setting $x(t) = 0$ and using $v_{\rm c}(0^-) = 2$, we see that

$$y_{\rm zir}(0^-) = -\frac{3}{2}v_{\rm c}(0^-) = -3.$$

Setting $x(t) = 0$ and taking the unilateral Laplace transform of the system's differential equation yield

$$sY_{\rm zir}(s) - y_{\rm zir}(0^-) - \frac{3}{2}Y_{\rm zir}(s) = 0.$$

Thus,

$$Y_{\rm zir}(s) = \frac{-3}{s - \frac{3}{2}}$$

and

$$y_{\rm zir}(t) = -3e^{3t/2}u(t).$$

(c) Taking the Laplace transform of the differential equation yields

$$sY(s) - \frac{3}{2}Y(s0 = -3sX(s) + \frac{3}{4}X(s).$$

Solving for the transfer function, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{-3s + \frac{3}{4}}{s - \frac{3}{2}} = -3 + \frac{-\frac{15}{4}}{s - \frac{3}{2}}.$$

Inverting, the impulse response is

$$h(t) = -3\delta(t) - \frac{15}{4}e^{3t/2}u(t).$$

(d) Since $x(t) = u(t-2)$, we see that

$$X(s) = e^{-2s}\frac{1}{s}.$$

The transform of the zero-state response is

$$Y_{\rm zsr}(s) = X(s)H(s) = e^{-2s}\frac{1}{s}\left(\frac{-3s + \frac{3}{4}}{s - \frac{3}{2}}\right) = e^{-2s}\left[\frac{-\frac{1}{2}}{s} + \frac{-\frac{5}{2}}{s - \frac{3}{2}}\right].$$

Inverting, we obtain

$$y_{\rm zsr}(t) = -\frac{1}{2}u(t-2) - \frac{5}{2}e^{3(t-2)/2}u(t-2).$$

Solution 4.4-16

(a) Referring to the circuit diagram:

$$\text{Coefficient on } \int x(t) \text{ is } -5 = -\frac{1}{R_1 10^{-4}} \quad \Rightarrow \quad R_1 = \frac{10^4}{5} = 2000 \ \Omega.$$

$$\text{Coefficient on } \int\int x(t) \text{ is } 5 = (-2)\left(-\frac{1}{R_2 10^{-4}}\right) \quad \Rightarrow \quad R_2 = 2\frac{10^4}{5} = 4000 \ \Omega.$$

$$\text{Coefficient on } \int y(t) \text{ is } -2 = -\frac{1}{R_3 10^{-4}} \quad \Rightarrow \quad R_3 = \frac{10^4}{2} = 5000 \ \Omega.$$

As a check, the coefficient on $\int\int y(t)$ is computed from the circuit as $(-1)(-2)\left(-\frac{1}{4000 \times 10^{-4}}\right) = -\frac{20}{4} = -5$, which matches the desired integral equation coefficient. Thus,

$$R_1 = 2000, \quad R_2 = 4000, \quad R_3 = 5000.$$

(b) First, we determine ICs $y(0^-)$ and $\dot{y}(0^-)$. From the second op amp output, $y(0^-) = 1$. The current through the second capacitor is

$$10^{-4}\dot{y}(0^-) = \frac{0}{R_1} + \frac{-1}{R_2} + \frac{-1}{R_3} \quad \Rightarrow \quad \dot{y}(0^-) = -10^4 \left(\frac{1}{4000} + \frac{1}{5000} \right) = -\frac{10}{4} - \frac{10}{5} = -\frac{9}{2}.$$

Setting $x(t) = 0$ and taking the unilateral Laplace transform of the differential equation yields

$$s^2 Y_{\text{zir}}(s) - sy(0^-) - \dot{y}(0^-) + 2\left(sY_{\text{zir}}(s) - y(0^-) \right) + 5Y_{\text{zir}}(s) = 0.$$

Thus,

$$\left(s^2 + 2s + 5 \right) Y_{\text{zir}}(s) = s - \frac{9}{2} + 2 = s - \frac{5}{2}.$$

Solving for $Y_{\text{zir}}(s)$ yields

$$Y_{\text{zir}}(s) = \frac{s - \frac{5}{2}}{s^2 + 2s + 5} = \frac{s - \frac{5}{2}}{(s+1)^2 + 2^2} = \frac{s+1}{(s+1)^2 + 2^2} + \frac{-\frac{7}{4}(2)}{(s+1)^2 + 2^2}.$$

Using Table 4.1 to invert, we obtain

$$y_{\text{zir}}(t) = e^{-t}\cos(2t)u(t) - \frac{7}{4}e^{-t}\sin(2t)u(t).$$

Solution 4.4-17

(a)

$$Y(s) = \frac{6s^2 + 3s + 10}{s(2s^2 + 6s + 5)}$$

$$y(0^+) = \lim_{s \to \infty} sY(s) = 3$$

$$y(\infty) = \lim_{s \to 0} sY(s) = 2$$

(b)

$$Y(s) = \frac{6s^2 + 3s + 10}{(s+1)(2s^2 + 6s + 5)}$$

$$y(0^+) = \lim_{s \to \infty} sY(s) = 3$$

$$y(\infty) = \lim_{s \to 0} sY(s) = 0$$

(c)

$$Y(s) = \frac{s^2 + 5s + 6}{s^2 + 3s + 2}$$

This $Y(s)$ is not strictly proper. We can express it as

$$Y(s) = 1 + \frac{2s + 4}{s^2 + 3s + 2}.$$

Hence,

$$y(0^+) = \lim_{s \to \infty} \frac{s(2s+4)}{s^2 + 3s + 2} = 2$$

and

$$y(\infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{s^3 + 5s^2 + 6s}{s^2 + 3s + 2} = 0.$$

(d)

$$Y(s) = \frac{s^3 + 4s^2 + 10s + 7}{s^2 + 2s + 3}$$

Because $Y(s)$ is improper, we find its strictly proper component from

$$Y(s) = (s + 2) + \frac{3s + 1}{s^2 + 2s + 3}.$$

Hence,

$$y(0^+) = \lim_{s \to \infty} s \left(\frac{3s + 1}{s^2 + 2s + 3} \right) = 3$$

and

$$y(\infty) = \lim_{s \to 0} s \left(\frac{s^3 + 4s^2 + 10s + 7}{s^2 + 2s + 3} \right) = 0.$$

Solution 4.5-1

(a) For a series connection,

$$H_s(s) = H_1(s)H_2(s) = \left(\frac{2s}{s + 1} \right) \left(\frac{1}{se^{3(s-1)}} \right) = 2e^3 e^{-3s} \frac{1}{s + 1}.$$

Inverting, the impulse response is

$$h_s(t) = 2e^3 e^{-(t-3)} u(t - 3).$$

(b) For a parallel connection,

$$H_p(s) = H_1(s) + H_2(s) = \frac{2s}{s + 1} + \frac{1}{se^{3(s-1)}} = 2 + \frac{-2}{s + 1} + e^3 e^{-3s} \frac{1}{s}.$$

Inverting, the impulse response is

$$h_p(t) = 2\delta(t) - 2e^{-t}u(t) + e^3 u(t - 3).$$

Solution 4.5-2

(a) At first glance, we are tempted to answer the question in affirmative. Let us verify the reality.

(b) The loop equations are

$$4I_1 - 2I_2 = X(s)$$
$$-2I_1 + 4I_2 = 0.$$

Cramer's rule yields

$$I_2(s) = \frac{1}{6} X(s)$$

and

$$Y(s) = I_2(s) = \frac{1}{6} X(s).$$

Therefore, $H(s) = \frac{1}{6}$ not $\frac{1}{4}$.

(c) In this case $R_3 = R_4 = 20000$ and

$$4I_1 - 2I_2 = X(s)$$
$$-2I_1 + 40002I_2 = 0.$$

Cramer's rule yields

$$I_2(s) = \frac{1}{80002}X(s)$$

$$Y(s) = 20000I_2(s) = \frac{20000}{80002}X(s) = 0.249994X(s).$$

In this case, $H(s)$ is very close to $1/4$. This is because the second ladder section causes a negligible load on the first. Let $R_3 = R_4 = R$. In this case, as $R \to \infty$, we observe that $H(s) \to 1/4$. The second ladder causes no loading in this case. The cascade rule applies only when the successive subsystems do not load the preceding subsystems.

Solution 4.5-3

The transfer function of the two paths are e^{-st} and $ae^{-s(T+\tau)}$. The two paths are in parallel. Hence the transfer function of this communication channel is

$$H(s) = e^{-sT} + ae^{-s(T+\tau)}$$
$$= e^{-sT}(1 + ae^{-s\tau}).$$

For distortionless transmission, it is adequate to undo only the term $(1 + ae^{-s\tau})$ in H(s) because e^{-sT} represents pure delay. Clearly, we need an equalizer with transfer function

$$H_{eq}(s) = \frac{1}{1 + ae^{-sT}}.$$

Comparing this form with the transfer function of the feedback system in Eq. (4.35) or Fig. 4.18d, it is immediately obvious that such an equalizer can be realized by the system of Fig. S4.5-3.
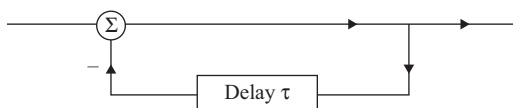
Figure S4.5-3

When this equalizer is placed in cascade with the communication channel, the effective transfer function is given by

$$H_c(s) = \frac{e^{-sT}(1 + ae^{-s\tau})}{1 + ae^{-s\tau}} = e^{-sT}.$$

The effective system represents a pure delay of T seconds, which makes it distortionless. Moreover, the equalizer is realizable.

Solution 4.5-4

The first system transfer function is

$$H(s) = \frac{\frac{1}{s-1}}{1 + \frac{2}{s-1}} = \frac{1}{s+1}.$$

With a single left halfplane pole, the system is BIBO stable.

The second system transfer function is

$$H(s) = \frac{\frac{K}{s(s+2)(s+4)}}{1 + \frac{K}{s(s+2)(s+4)}} = \frac{1}{s^3 + 6s^2 + 8s + K}.$$

We consider BIBO stability of this system for three cases: **(a)** $K = 10$, **(b)** $K = 50$, and **(c)** $K = 48$.

(a) We can verify that for $K = 10$, all the roots are in LHP and hence the system is BIBO stable.

```
>>  K = 10; poles = roots([1 6 8 K])
    poles = -4.7608 + 0.0000i
            -0.6196 + 1.3102i
            -0.6196 - 1.3102i
```

(b) For $K = 50$, we can verify that two roots are in RHP and on LHP. Hence the system is BIBO unstable.

```
>>  K = 50; poles = roots([1 6 8 K])
    poles = -6.0449 + 0.0000i
             0.0225 + 2.8759i
             0.0225 - 2.8759i
```

(c) For $K = 48$, we verify that two roots are on imaginary axis at $\pm j\sqrt{8}$ and one is in the LHP. Hence the system is BIBO unstable (but marginally stable).

```
>>  K = 48; poles = roots([1 6 8 K])
    poles = -6.0000 + 0.0000i
             0.0000 + 2.8284i
             0.0000 - 2.8284i
```

Solution 4.6-1

$$H(s) = \frac{s^2 + 2s}{s^3 + 8s^2 + 19s + 12} = \left(\frac{s}{s+1}\right)\left(\frac{s+2}{s+3}\right)\left(\frac{1}{s+4}\right) = \frac{-1/6}{s+1} - \frac{3/2}{s+3} + \frac{8/3}{s+4}$$

Figures S4.6-1a, S4.6-1b, and S4.6-1c show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.
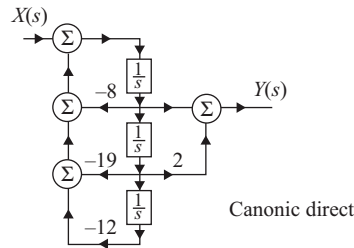
Figure S4.6-1a
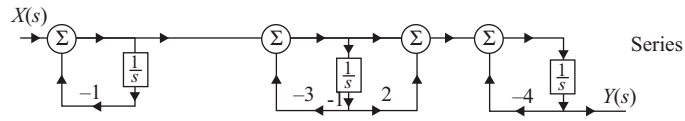
Figure S4.6-1b
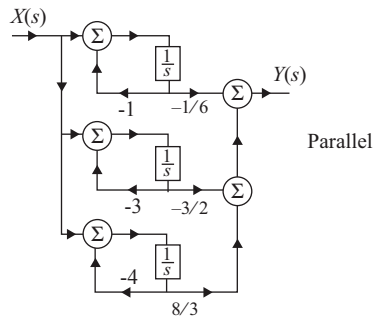
Figure S4.6-1c

Solution 4.6-2

Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-1 are shown in Figs. S4.6-2a, S4.6-2b, and S4.6-2c. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.
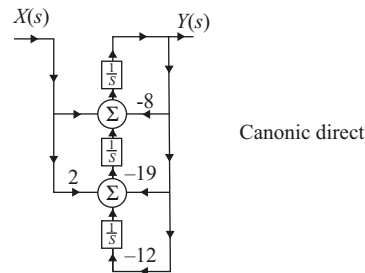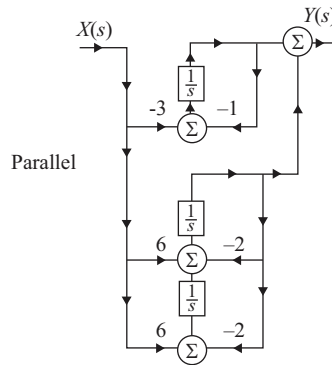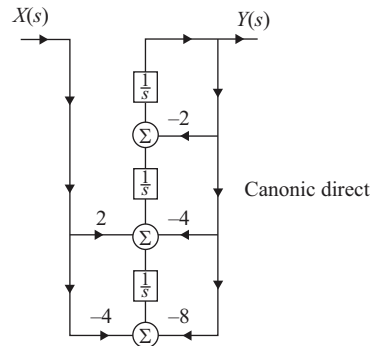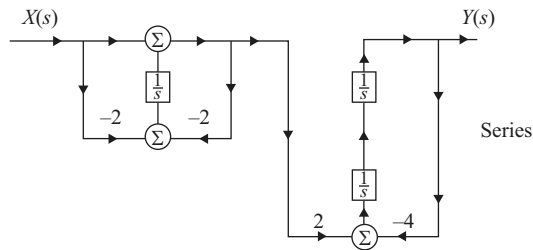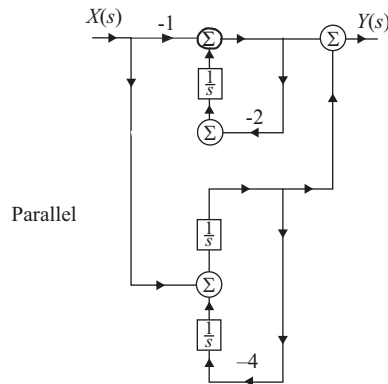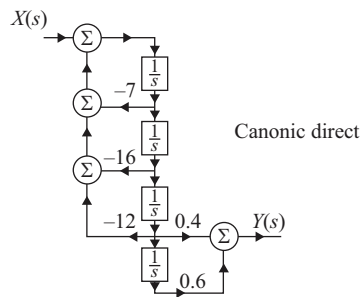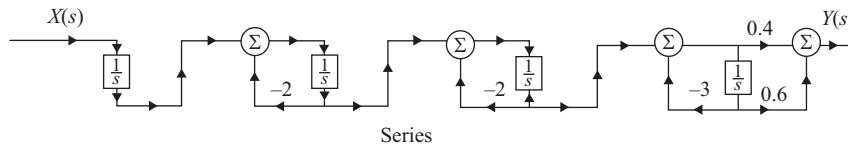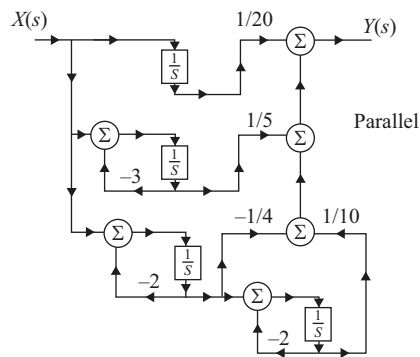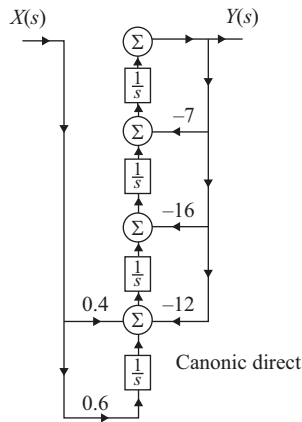

Figure S4.6-2a
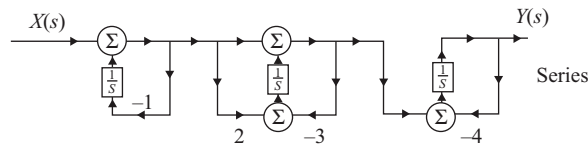
Figure S4.6-2b
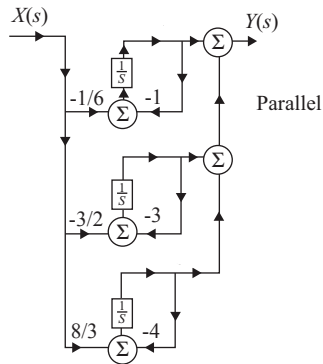
Figure S4.6-2c

Solution 4.6-3

(a)

$$H(s) = \frac{3s(s+2)}{(s+1)(s^2+2s+2)} = \frac{3s^2+6s}{s^3+3s^2+4s+2}$$

$$= \left(\frac{3s}{s+1}\right)\left(\frac{s+2}{s^2+2s+2}\right) = -\frac{3}{s+1} + \frac{6s+6}{s^2+2s+2}$$

Figures S4.6-3a, S4.6-3b, and S4.6-3c show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.
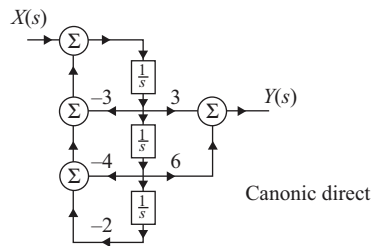
Figure S4.6-3a

Figure S4.6-3b

X(s)

Figure S4.6-3c

(b)
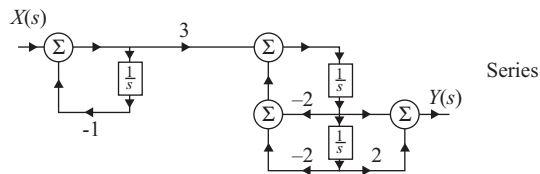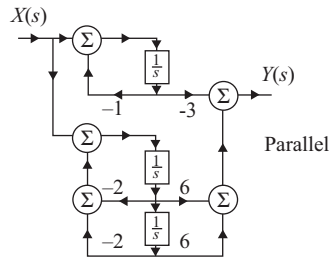
$$H(s) = \frac{2s - 4}{(s + 2)(s^2 + 4)} = \frac{2s - 4}{s^3 + 2s^2 + 4s + 8}$$

$$= \left(\frac{s - 2}{s + 2}\right)\left(\frac{2}{s^2 + 4}\right) = -\frac{1}{s + 2} + \frac{s}{s^2 + 4}$$

Figures S4.6-3d, S4.6-3e, and S4.6-3f show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.
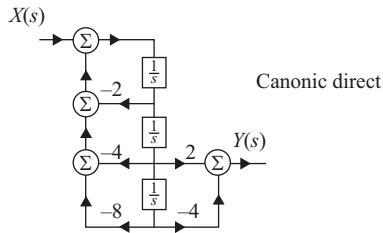
Figure S4.6-3d
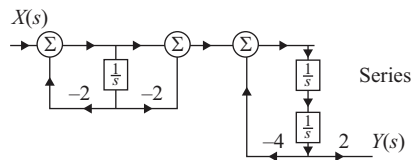
Figure S4.6-3e
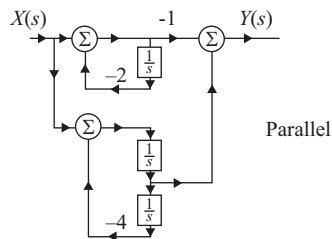
Figure S4.6-3f

Solution 4.6-4

(a) Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-3a are shown in Figs. S4.6-4a, S4.6-4b, and S4.6-4c. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.
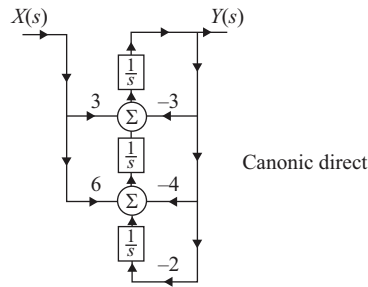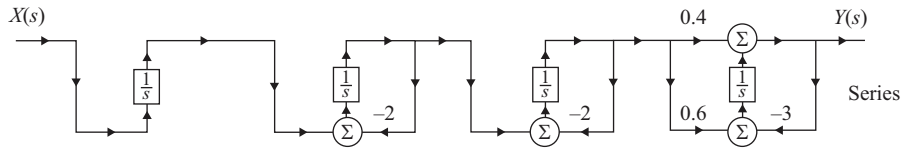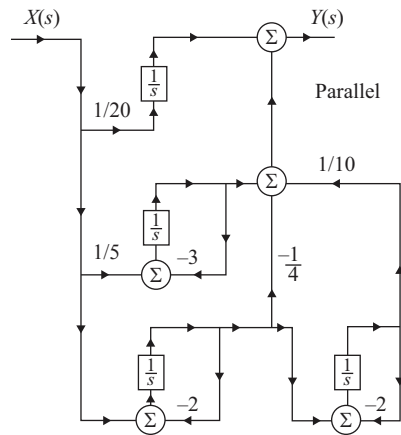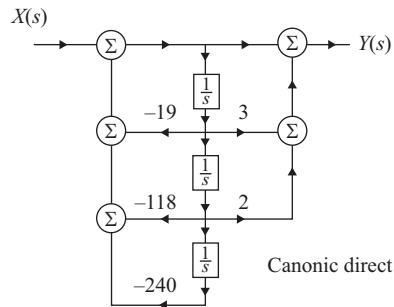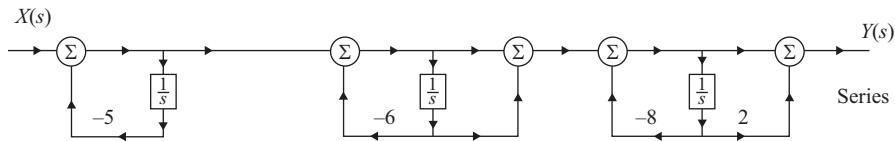

Figure S4.6-4a
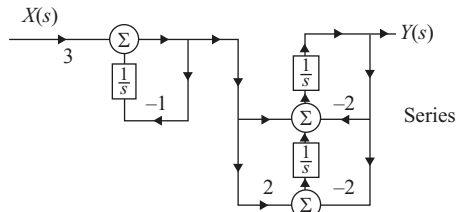
Figure S4.6-4b
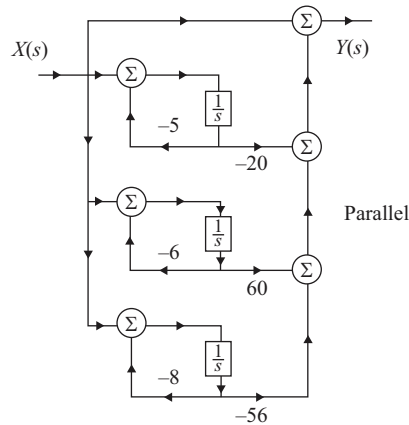

Figure S4.6-4c

(b) Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-3b are shown in Figs. S4.6-4d, S4.6-4e, and S4.6-4f. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.

Figure S4.6-4d

Figure S4.6-4e

Figure S4.6-4f

Solution 4.6-5

$$H(s) = 7\frac{2s+3}{5(s^4 + 7s^3 + 16s^2 + 12s)} = \frac{0.4s + 0.6}{s^4 + 7s^3 + 16s^2 + 12s}$$

$$= \left(\frac{1}{s}\right)\left(\frac{1}{s+2}\right)\left(\frac{1}{s+2}\right)\left(\frac{0.4s+0.6}{s+3}\right) = \frac{\frac{1}{20}}{s} - \frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{10}}{(s+2)^2} + \frac{\frac{1}{5}}{s+3}$$

Figures S4.6-5a, S4.6-5b, and S4.6-5c show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.

Figure S4.6-5a

Figure S4.6-5b

Figure S4.6-5c

Solution 4.6-6

Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-5 are shown in Figs. S4.6-6a, S4.6-6b, and S4.6-6c. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.
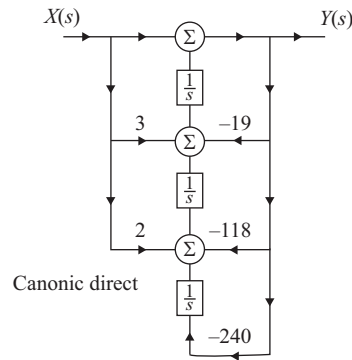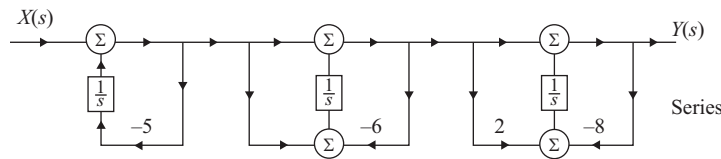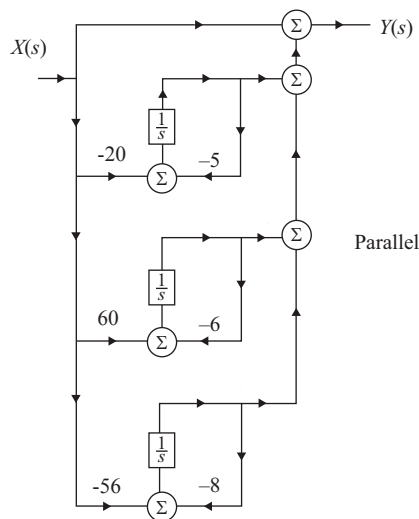

Figure S4.6-6a

Figure S4.6-6b

Figure S4.6-6c

Solution 4.6-7

$$H(s) = \frac{s(s+1)(s+2)}{(s+5)(s+6)(s+8)} = \frac{s^3 + 3s^2 + 2s}{s^3 + 19s^2 + 118s + 240} = 1 - \frac{20}{s+5} + \frac{60}{s+6} - \frac{56}{s+8}$$

Figures S4.6-7a, S4.6-7b, and S4.6-7c show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.
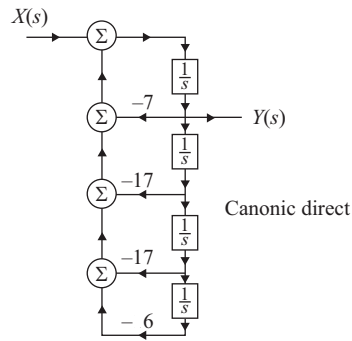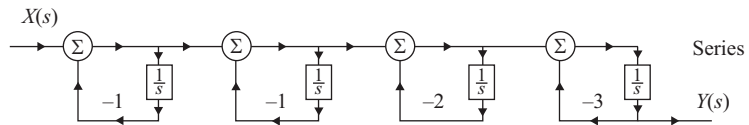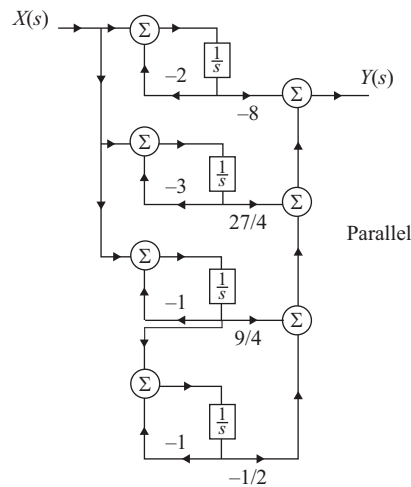

Figure S4.6-7a

Figure S4.6-7b

Figure S4.6-7c

Solution 4.6-8

Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-7 are shown in Figs. S4.6-8a, S4.6-8b, and S4.6-8c. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.

Figure S4.6-8a

Figure S4.6-8b

Figure S4.6-8c

Solution 4.6-9

$$H(s) = \frac{s^3}{(s+1)^2(s+2)(s+3)} = \frac{s^3}{s^4 + 7s^3 + 17s^2 + 17s + 6}$$

$$= \left(\frac{s}{s+1}\right)\left(\frac{s}{s+1}\right)\left(\frac{s}{s+2}\right)\left(\frac{1}{s+3}\right) = -\frac{8}{s+2} + \frac{\frac{27}{4}}{s+3} + \frac{\frac{9}{4}}{s+1} - \frac{\frac{1}{2}}{(s+1)^2}$$

Figures S4.6-9a, S4.6-9b, and S4.6-9c show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.

Figure S4.6-9a

Figure S4.6-9b

Figure S4.6-9c

Solution 4.6-10

Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-9 are shown in Figs. S4.6-10a, S4.6-10b, and S4.6-10c. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.
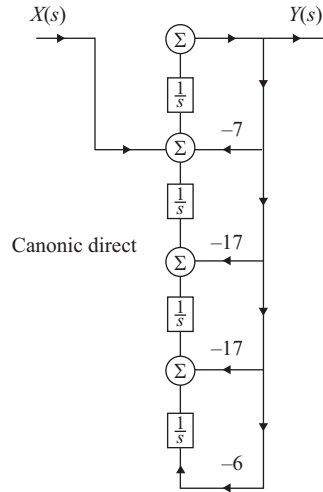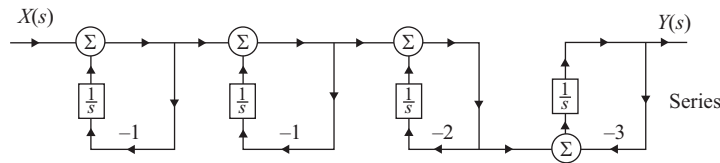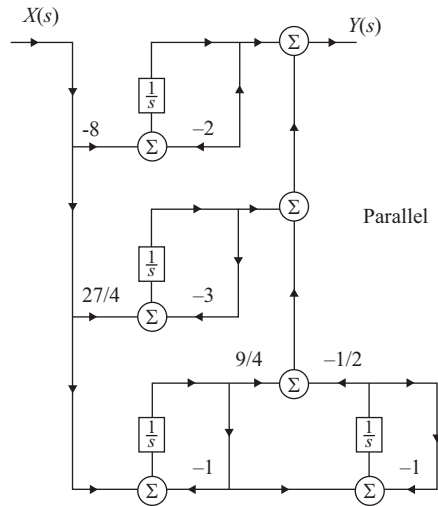

Figure S4.6-10a

Figure S4.6-10b

Figure S4.6-10c

Solution 4.6-11

$$H(s) = \frac{s^3}{(s+1)(s^2+4s+13)} = \frac{s^3}{s^3+5s^2+17s+13}$$

$$= \left(\frac{s}{s+1}\right)\left(\frac{s^2}{s^2+4s+13}\right) = -\frac{0.1}{s+1} + \frac{s^2-0.9s+1.3}{s^2+4s+13} = 1 - \frac{0.1}{s+1} - \frac{4.9s+11.7}{s^2+4s+13}$$

Figures S4.6-11a, S4.6-11b, and S4.6-11c show the canonical, series and parallel realizations. Some variations in structures are possible. For example, different series realizations occur by making different pairings of poles and zeros.
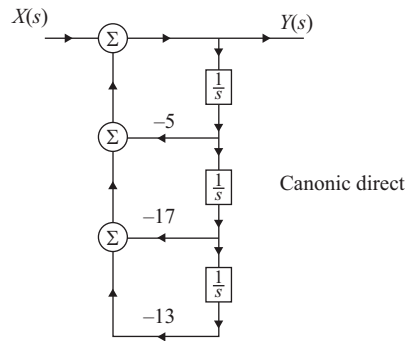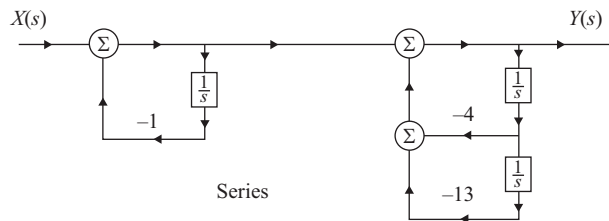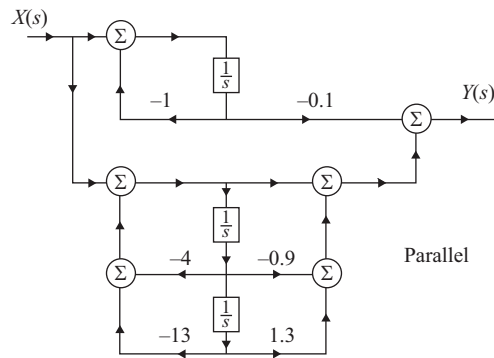

Figure S4.6-11a

Figure S4.6-11b

Figure S4.6-11c

Solution 4.6-12

Transposed versions of the canonic direct, series, and parallel realizations for the transfer function in Prob. 4.6-11 are shown in Figs. S4.6-12a, S4.6-12b, and S4.6-12c. Here, we transpose component sub block diagrams separately; it is also possible to transpose the entire block diagram, which produces slightly different (but mathematically equivalent) results.
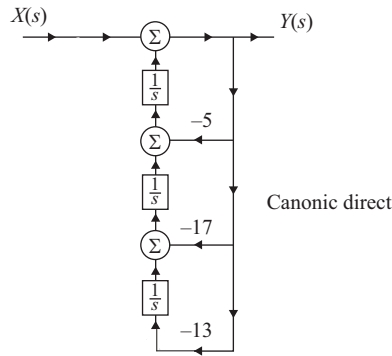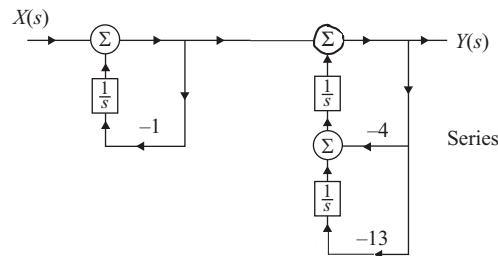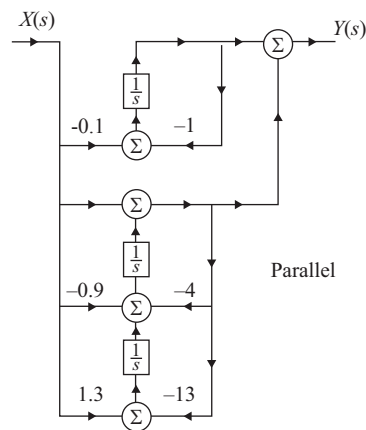

Figure S4.6-12a

Figure S4.6-12b

Figure S4.6-12c

Solution 4.6-13

Here,

$$H(s) = \frac{(s-2j)(s+2j)}{(s-j)(s+j)(s+2)} = \frac{s^2+4}{s^3+2s^2+s+2}.$$

A TDFII realization of $H(s)$ is shown in Fig. S4.6-13. Of course, the paths shown with a scalar multiplier of 0 can be eliminated completely.

There are two primary reasons that TDFII tends to be a good structure:

- TDFII is canonical, which ensures a realization with the fewest number of (expensive) integrators

- TDFII places system zeros before system poles, which helps avoid overflow/saturation errors
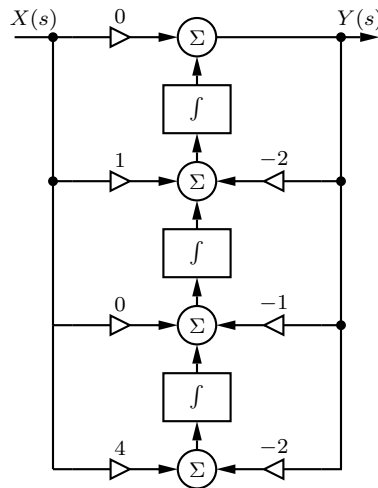

Figure S4.6-13

Solution 4.6-14

(a) To determine a single-stage structure, we express the transfer function as

$$H(s) = \frac{(s-2j)(s+2j)(s-3j)(s+3j)}{9(s+1)(s+2)(s+1-j)(s+1+j)} = \frac{\frac{1}{9}s^4 + \frac{13}{9}s^2 + 4}{s^4 + 5s^3 + 10s^2 + 10s + 4}.$$

Figure S4.6-14a shows a single fourth-order real TDFII realization of $H(s)$. While block realizations of $H(s)$ are rarely unique, Fig. S4.6-14a is essentially unique given the constraints of being single-stage and TDFII.

(b) To determine a cascade structure, we express the transfer function as

$$H(s) = \frac{(s-2j)(s+2j)(s-3j)(s+3j)}{9(s+1)(s+2)(s+1-j)(s+1+j)} = \left(\frac{s^2+4}{s^2+3s+2}\right)\left(\frac{\frac{1}{9}s^2+1}{s^2+2s+2}\right).$$

Figure S4.6-14b shows a block realization of $H(s)$ using a cascade of second-order real DFII structures. Since different pole/zero pairings and gain distributions are possible, this realization is not unique, even given the constraint that the realization be a cascade of second-order real DFII structures.
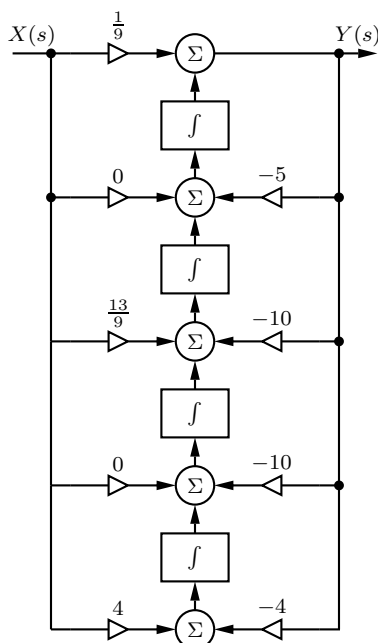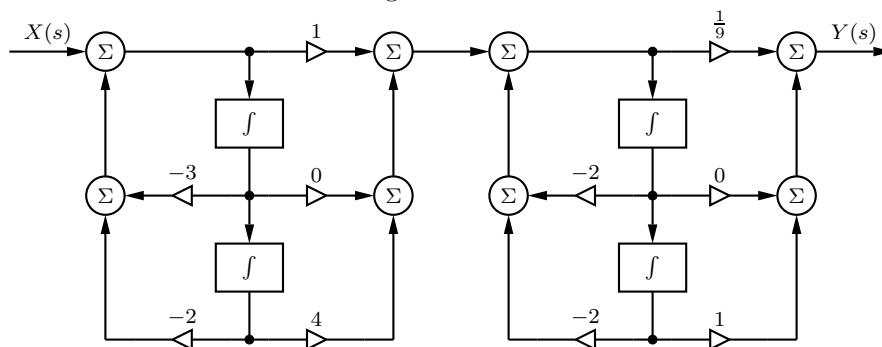
Figure S4.6-14a

Figure S4.6-14b

(c) To determine a parallel structure, we compute the partial fraction expansion of $H(s)$ as

$$H(s) = \frac{(s-2j)(s+2j)(s-3j)(s+3j)}{9(s+1)(s+2)(s+1-j)(s+1+j)} = \frac{1}{9} + \frac{-\frac{52}{9}}{s+2} + \frac{\frac{50}{9}}{s+1} + \frac{-\frac{1}{6}+\frac{29j}{18}}{s+1-j} + \frac{-\frac{1}{6}-\frac{29j}{18}}{s+1+j}.$$

Next, we combine the first three terms (which includes the $\frac{1}{9}$ direct term) into a second-order real rational function and the last two terms into another second-order real rational function. This yields

$$H(s) = \frac{\frac{1}{9}s^2 + \frac{1}{9}s + \frac{50}{9}}{s^2+3s+2} + \frac{-\frac{1}{3}s - \frac{32}{9}}{s^2+2s+2}.$$

Figure S4.6-14c shows a block realization of $H(s)$ using a parallel connection of second-order real DFI structures. We could also have combined the direct term with the last two terms and obtained a different sum of two second-order real transfer functions. Clearly, the realization of Fig. S4.6-14c is therefore not unique, even given the constraint that the realization be a parallel connection of second-order real DFI structures.
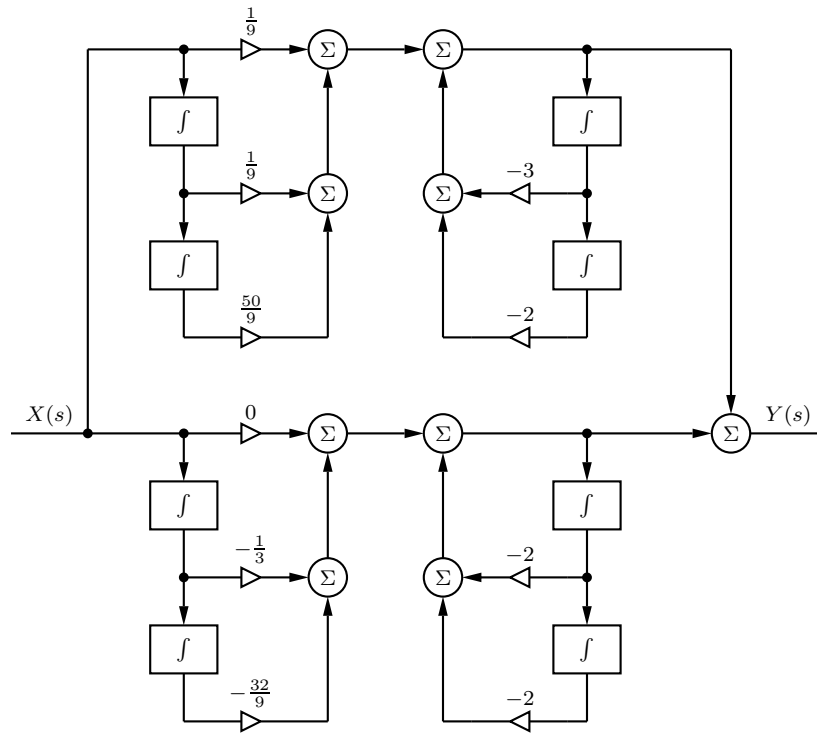
Figure S4.6-14c

Solution 4.6-15

(a) Application of Eq. (4.35) to Fig. P4.6-15a yields

$$H_{\mathrm{a}}(s) = \frac{\frac{1}{(s+a)^2}}{1 + \frac{b^2}{(s+a)^2}} = \frac{1}{(s+a)^2 + b^2}.$$

(b) Figure P4.6-15b is also a feedback loop with forward gain $G(s) = \frac{1}{s+a}$ and the loop gain $\frac{b^2}{(s+a)^2}$. Therefore,

$$H_{\mathrm{b}}(s) = \frac{\frac{1}{s+a}}{1 + \frac{b^2}{(s+a)^2}} = \frac{s+a}{(s+a)^2 + b^2}.$$

(c) The output in Fig. P4.6-15c is the same of $B - aA$ times the output of Fig. P4.6-15a and A times the output of Fig. P4.6-15b. Therefore, its transfer function is

$$H_{\mathrm{c}}(s) = (B - aA)H_1(s) + AH_2(s)$$
$$= \frac{B - aA}{(s+a)^2 + b^2} + \frac{A(s+a)}{(s+a)^2 + b^2}$$
$$= \frac{As + B}{(s+a)^2 + b^2}$$

Solution 4.6-16

These transfer functions are readily realized by using the arrangement in Fig. 4.28 by a proper choice of $Z_f(s)$ and $Z(s)$. While we present component values that produce the desired transfer functions, other choices are possible as long as the resulting gains are correct.

(a) In Fig. S4.6-16a, $Z_f(s) = \dfrac{\frac{R_f}{C_f s}}{R_f + \frac{1}{C_f s}} = \dfrac{1}{C_f(s+a)}$ where $a = \dfrac{1}{R_f C_f}$, $Z(s) = R$, and

$$H_a(s) = -\frac{Z_f(s)}{Z(s)} = -\frac{k}{s+a}, \text{ where } k = \frac{1}{RC_f} \text{ and } a = \frac{1}{R_f C_f}.$$

Choose $R = 10,000$, $R_f = 20,000$ and $C_f = 10^{-5}$. This yields $k = 10$ and $a = 5$. Therefore, the transfer function for Fig. S4.6-16a is

$$H_a(s) = \frac{-10}{s+5}.$$

(b) This is same as (a) followed by an amplifier of gain -1 as shown in Fig. S4.6-16b. Therefore, the transfer function for Fig. S4.6-16b is

$$H_b(s) = \frac{10}{s+5}.$$

(c) For the first stage in Fig. S4.6-16c (see Fig. 4.32b of Drill 4.14), $Z_f(s) = \dfrac{1}{C_f(s+a)}$, where $a = \dfrac{1}{R_f C_f}$, $Z(s) = \dfrac{1}{C(s+b)}$, where $b = \dfrac{1}{RC}$, and

$$H(s) = -\frac{Z_f(s)}{Z(s)} = -\frac{C}{C_f}\left(\frac{s+b}{s+a}\right).$$

Choose $C = C_f = 10^{-4}$, $R = 5000$, $R_f = 2000$. This yields a transfer function $H(s) = -\left(\frac{s+2}{s+5}\right)$. This is followed by an op amp of gain -1 as shown in Fig. S4.6-16c. This yields

$$H_c(s) = \frac{s+2}{s+5}$$
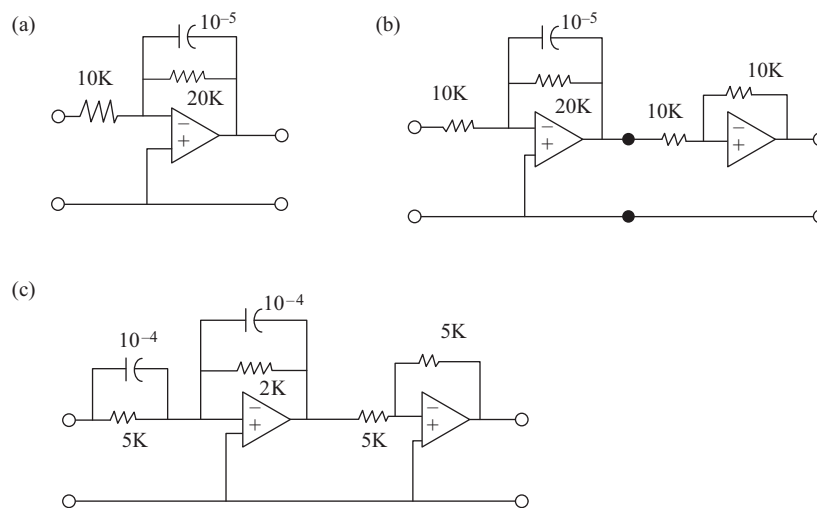
Figure S4.6-16

Solution 4.6-17

One realization, derived in the previous problem, is shown in Fig. S4.6-17a.
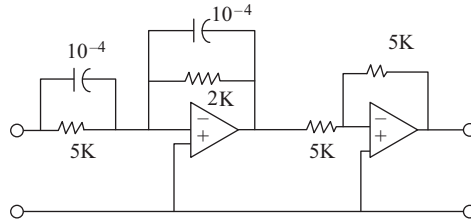
Figure S4.6-17a

For the second realization, we express $H(s)$ as

$$H(s) = \frac{s+2}{s+5} = 1 - \frac{3}{s+5}.$$

We realize $H(s)$ as a parallel combination of $H_1(s) = 1$ and $H_2(s) = -3/(s+5)$ as shown in Fig. S4.6-17b. The second stage serves as a summer for which the inputs are the input and output of the first stage. Because the summer has a gain -1, we need a third stage of gain -1 to obtain the desired transfer function.
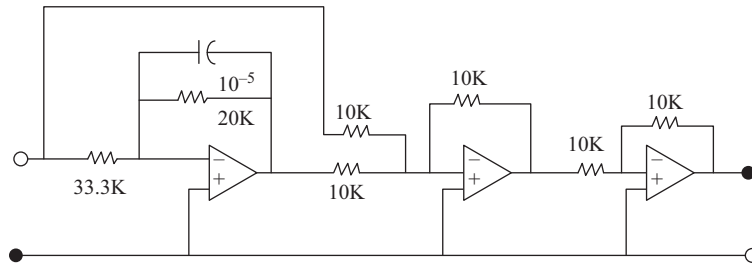
Figure S4.6-17b

Solution 4.6-18

A canonical realization of $H(s)$ is shown in Fig. S4.6-18. Observe that this is identical to $H(s)$ in Ex. 4.25 with minor differences. Hence, the op-amp circuit in Fig. 4.31c can be used for our purpose with appropriate changes in the element values. The last summer input resistors now are $\frac{100}{3}\, k\Omega$ and $\frac{100}{7}\, k\Omega$ instead of $50\, k\Omega$ and $20\, k\Omega$.

Figure S4.6-18

Solution 4.6-19

We follow the procedure in Ex. 4.25 with appropriate modifications. In this case, $a_2 = 13$, $a_1 = 4$, $b_2 = 2$, $b_1 = 5$, and $b_0 = 1$ (Ex. 4.25 has $a_2 = 10$, $a_1 = 4$, $b_2 = 5$, $b_1 = 2$, and $b_0 = 0$). Because b_0 is nonzero here, we have one more feedforward connection. Figure S4.6-19 shows the development of the suitable realization.

Figure S4.6-19

Solution 4.6-20

From Fig. 4.28, we can construct an integrator (with gain $-\frac{R}{L}$) by using an inductor L at the input and a resistor R in the feedback. With this in mind, we express the system $\frac{d}{dt}y(t) + 2y(t) = x(t) - 3\frac{d}{dt}x(t)$ in integral form as

$$y(t) + 2\int y(t) = -3x(t) + \int x(t).$$

Thus,

$$y(t) = -3x(t) - \left[-\int x(t) \right] - 2(-1)\left[-\int y(t) \right].$$

From this expression, we draw the op-amp realization shown in Fig. S4.6-20, using various (mostly) realistic component values to obtain the needed gains.
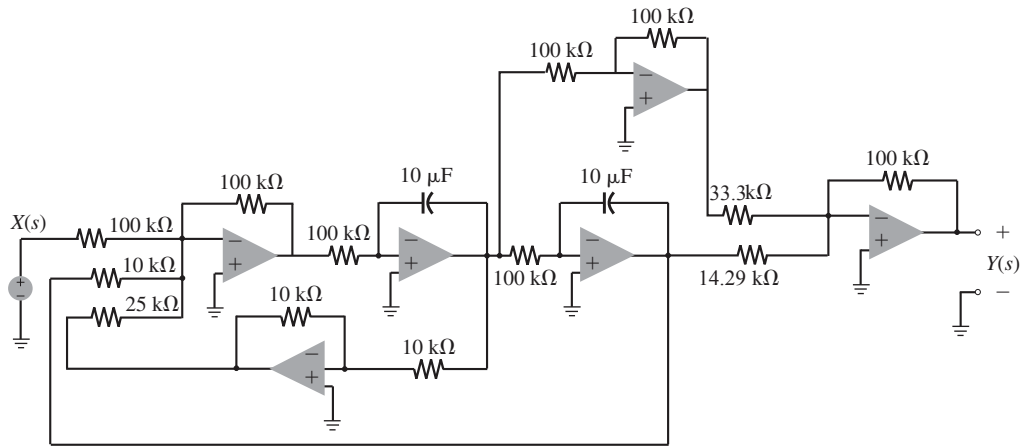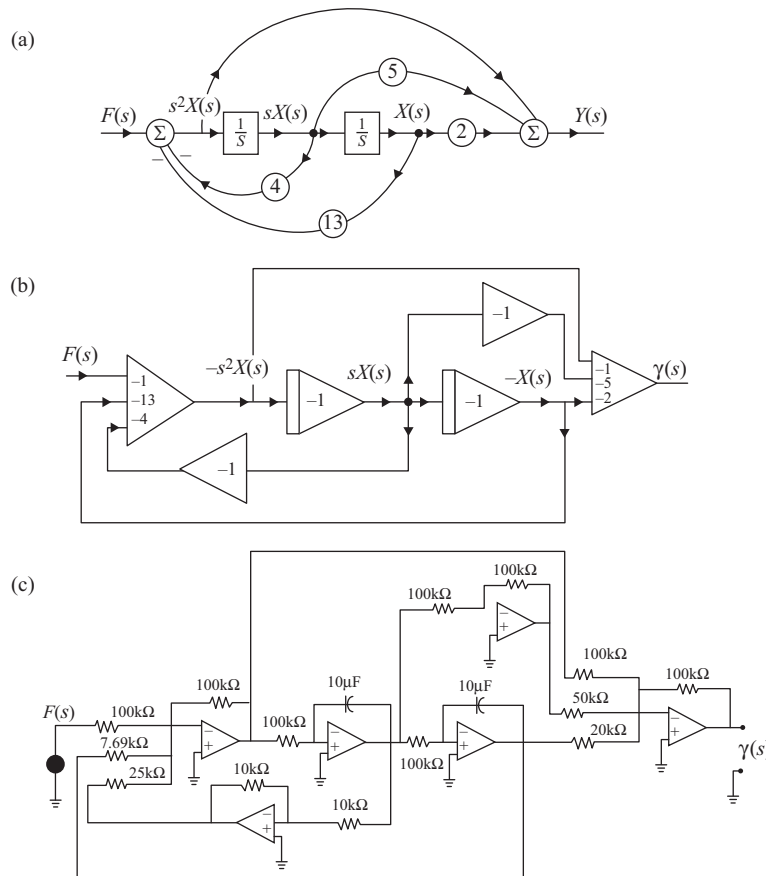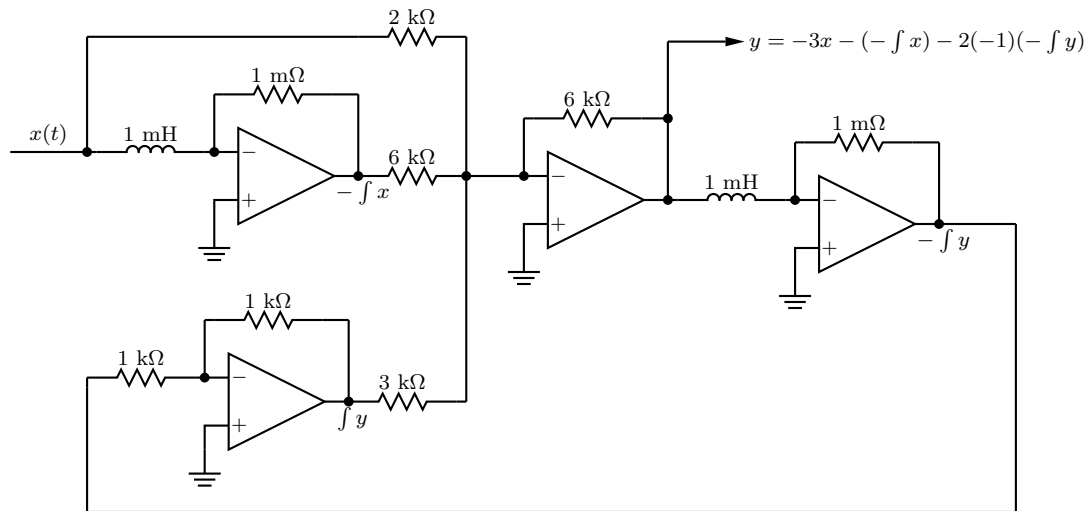

Figure S4.6-20

Using inductors rather than capacitors in a circuit can be problematic. Compared with capacitors, inductors are generally more expensive and usually have worse tolerance. Furthermore, it can be difficult to obtain gains near 1 with realistic R and L values. As seen in Fig. S4.6-20, an inductor value of 1 mH is realistic but the corresponding resistor of 1 mΩ is problematic.

Solution 4.7-1

(a) In this case,

$$H(j\omega) = \frac{\omega_c}{j\omega + \omega_c} \implies |H(j\omega)| = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}.$$

The dc gain is $H(0) = 1$ and the gain at $\omega = \omega_c$ is $1/\sqrt{2}$, which is -3 dB below the dc gain. Hence, the 3-dB bandwidth is ω_c. Also the dc gain is unity. Hence, the gain-bandwidth product is ω_c.

We could derive this result indirectly as follows. The system is a lowpass filter with a single pole at $\omega = \omega_c$. The dc gain is $H(0) = 1$ (0 dB). Because, there is a single pole at ω_c (and no zeros), there is only one asymptote starting at $\omega = \omega_c$ (at a rate -20 dB/dec.). The break point is ω_c, where there is a correction of -3 dB. Hence, the amplitude response at ω_c is 3 dB below 0 dB (the dc gain). Thus, the 3-dB bandwidth of this filter is ω_c.

(b) The transfer function of this system is

$$H(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{\omega_c}{s+\omega_c}}{1 + \frac{9\omega_c}{s+\omega_c}} = \frac{\omega_c}{s + 10\omega_c}.$$

We use the same argument as in part (a) to deduce that the dc gain is 0.1 and the 3-dB bandwidth is $10\omega_c$. Hence, the gain-bandwidth product is ω_c.

(c) The transfer function of this system is

$$H(s) = \frac{G(s)}{1 - G(s)H(s)} = \frac{\frac{\omega_c}{s+\omega_c}}{1 - \frac{0.9\omega_c}{s+\omega_c}} = \frac{\omega_c}{s + 0.1\omega_c}.$$

We use the same argument as in part (a) to deduce that the dc gain is 10 and the 3-dB bandwidth is $0.1\omega_c$. Hence, the gain-bandwidth product is ω_c.

(d) Included in previous parts.

Solution 4.8-1

In this problem, we consider a controllable, observable LTIC with transfer function $H(s) = \frac{s^2+4}{2s^2+4s+4}$.

(a) We use MATLAB to direct calculate the magnitude response at $\omega = 0, 1, 2, 3, 5, 10,$ and ∞.

```
>>  H = @(s) (s.^2+4)./(2*s.^2+4*s+4); omega = [0, 1, 2, 3, 5, 10, 10^10];
>>  Hmag = abs(H(1j*omega))
    Hmag = 1.0000    0.6708         0    0.2712    0.4187    0.4799    0.5000
```

Figure S4.8-1 uses these values to plot the magnitude response $|H(j\omega)|$ over $0 \le \omega \le 10$.
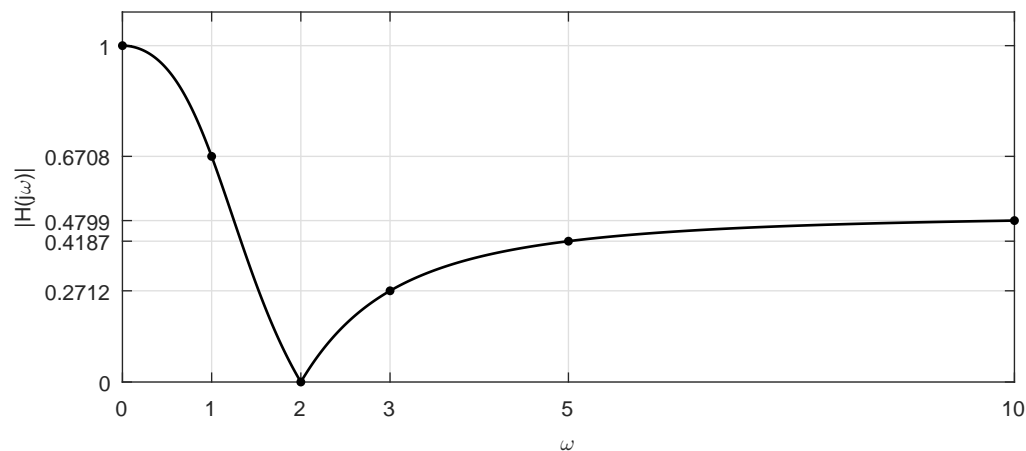
Figure S4.8-1

(b) Frequency (including both magnitude and phase) response reflects a system's sinusoidal steady state behavior. Therefore, it is appropriate for the test engineer to apply sinusoids to measure the system's magnitude response. To measure magnitude response, the engineer could apply the following simple procedure:

- Select some frequency of interest ω.
- Apply a sinusoidal input $x(t) = A_{\text{in}} \cos(\omega t + \theta_{\text{in}})$ to the system. The amplitude A_{in} should be known and fixed, but the phase θ_{in} is unimportant.

- Since the system is LTIC, the output will take the form $y(t) = A_{\text{out}} \cos(\omega t + \theta_{\text{out}})$. Using the oscilloscope, measure the peak amplitude A_{out}.

- The magnitude response at frequency ω is simply the ratio $A_{\text{out}}/A_{\text{in}}$. That is, $|H(j\omega)| = A_{\text{out}}/A_{\text{in}}$.

- Repeat the previous steps for an appropriate selection of frequencies, such as $\omega = 0, 1, 2, 3, 5,$ and 10.

(c) If the engineer accidentally constructs the inverse system $H^{-1}(s) = \frac{1}{H(s)} = \frac{2s^2+4s+4}{s^2+4}$, the measured magnitude response will be the inverse of the desired magnitude response. That is,

$$|H_{\text{measured}}(j\omega)| = \frac{1}{|H_{\text{desired}}(j\omega)|}.$$

In this case, however, tests using inputs at or near $\omega = 2$ will behave quite badly. This is because $H^{-1}(s)$ has roots on the ω-axis at $\omega = \pm 2$, causing an input at $\omega = \pm 2$ to produce unbounded output (at least until the system saturates or breaks). Considered another way, although $H(s)$ is a BIBO (and asymptotically) stable system, $H^{-1}(s)$ is a BIBO unstable (marginally stable) system. Magnitude response is always problematic (or meaningless) for BIBO unstable systems.

Solution 4.8-2

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 5j\omega + 4} = \frac{j\omega + 2}{(4 - \omega^2) + j5\omega}$$

$$|H(j\omega)| = \sqrt{\frac{\omega^2 + 4}{(4 - \omega^2)^2 + (5\omega)^2}} = \sqrt{\frac{\omega^2 + 4}{\omega^4 + 17\omega^2 + 16}}$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{5\omega}{4 - \omega^2}\right)$$

(a) For $x(t) = 5\cos(2t + 30°)$, $\omega = 2$ and

$$|H(j2)| = \sqrt{\frac{2}{25}} = \frac{\sqrt{2}}{5}$$

$$\angle H(j2) = \tan^{-1} - \tan^{-1}(\infty) = 45° - 90° = -45°.$$

Thus,

$$y(t) = 5\frac{\sqrt{2}}{5}\cos(2t + 30° - 45°) = \sqrt{2}\cos(2t - 15°).$$

(b) Using the results from part (a), input $x(t) = 10\sin(2t + 45°)$ produces output

$$y(t) = 10\left(\frac{\sqrt{2}}{5}\right)\sin(2t + 45° - 45°) = 2\sqrt{2}\sin 2t$$

(c) For $x(t) = 10\cos(3t + 40°)$, $\omega = 3$ and

$$|H(j\omega)| = \sqrt{\frac{13}{250}} = 0.228 \quad \text{and} \quad \angle H(j3) = 56.31° - 108.43° = -52.12°$$

Therefore,

$$y(t) = 10(0.228)\cos(3t + 40° - 52.12°) = 2.28\cos(3t - 12.12°).$$

Solution 4.8-3

$$H(j\omega) = \frac{j\omega + 3}{(j\omega + 2)^2}$$

$$|H(j\omega)| = \frac{\sqrt{\omega^2 + 9}}{\omega^2 + 4} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(\frac{\omega}{3}) - \tan^{-1}(\frac{\omega}{2})$$

(a) For $x(t) = 10u(t) = 10e^{j0t}u(t)$, $\omega = 0$ and $H(j0) = 0.75$. Therefore, the steady-state response is

$$y_{ss}(t) = 0.75 \times 10e^{j0t}u(t) = 7.5u(t).$$

(b) For $x(t) = \cos(2t + 60°)\,u(t)$, $\omega = 2$, and

$$|H(j2)| = \frac{\sqrt{13}}{8} \quad \text{and} \quad \angle H(j2) = 33.69° - 90° = -56.31°.$$

Therefore, the steady-state response is

$$y_{ss}(t) = \frac{\sqrt{13}}{8}\cos(2t + 60° - 56.31°)u(t) = \frac{\sqrt{13}}{8}\cos(2t + 3.69°)u(t).$$

(c) For $x(t) = \sin(3t - 45°)u(t)$, $\omega = 3$, and

$$|H(j3)| = \frac{\sqrt{18}}{13} \quad \text{and} \quad \angle H(j3) = 45° - 112.62° = -67.62°.$$

Therefore, the steady-state response is

$$y_{ss}(t) = \frac{\sqrt{18}}{13}\sin(3t - 45° - 67.62°)u(t) = \frac{\sqrt{18}}{13}\sin(3t - 112.62°)u(t).$$

(d) For $x(t) = e^{j3t}u(t)$, $\omega = 3$. Using the results of part (c), the steady-state response is

$$y_{ss}(t) = H(j3)e^{j3t} = |H(j3)|e^{j[3t+\angle H(j3)]}u(t) = \frac{\sqrt{18}}{13}e^{j[3t-67.62°]}u(t).$$

Solution 4.8-4

$$H(j\omega) = \frac{-(j\omega - 10)}{j\omega + 10} = \frac{10 - j\omega}{10 + j\omega}, \qquad |H(j\omega)| = \sqrt{\frac{\omega^2 + 100}{\omega^2 + 100}} = 1$$

$$\angle H(j\omega) = \tan^{-1}(-\frac{\omega}{10}) - \tan^{-1}(\frac{\omega}{10}) = -2\tan^{-1}(\frac{\omega}{10})$$

(a) For $x(t) = e^{j\omega t}$,

$$y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j[\omega t+\angle H(j\omega)]} = e^{j[\omega t - 2\tan^{-1}(\omega/10)]}.$$

(b) For $x(t) = \cos(\omega t + \theta)$,

$$y(t) = \cos[\omega t + \theta - 2\tan^{-1}(\frac{\omega}{10})].$$

(c) For $x(t) = \cos t$, $\omega = 1$ and

$$|H(j1)| = 1 \quad \text{and} \quad \angle H(j\omega) = -2\tan^{-1}(\frac{1}{10}) = -11.42°.$$

Thus,

$$y(t) = \cos(t - 11.42°).$$

(d) For $x(t) = \sin 2t$, $\omega = 2$ and

$$|H(j2)| = 1 \quad \text{and} \quad \angle H(j2) = -2\tan^{-1}(\frac{2}{10}) = -22.62°.$$

Thus,
$$y(t) = \sin(2t - 22.62°).$$

(e) For $x(t) = \cos 10t$, $\omega = 10$ and

$$|H(j10)| = 1 \quad \text{and} \quad \angle H(j10) = -2\tan^{-1}(\frac{10}{10}) = -90°.$$

Thus,
$$y(t) = \cos(10t - 90°) = \sin 10t.$$

(f) For $x(t) = \cos 100t$, $\omega = 100$ and

$$|H(j100)| = 1 \quad \text{and} \quad \angle H(j100) = -2\tan^{-1}(\frac{100}{10}) = -168.58°.$$

Thus,
$$y(t) = \cos(100t - 168.58°).$$

Since the magnitude response is unity for all frequencies, this is an allpass filter. Consequently, the filter only impacts the phase characteristics of the input, not the magnitude characteristics.

Solution 4.8-5

(a) From the graph, the two system zeros are at $s = \pm j1.5$. Thus, $s^2 + b_1 s + b_2 = (s + j1.5)(s - j1.5) = s^2 + 2.25$. The two system poles are at $s = -1 \pm j0.5$. Thus, $s^2 + a_1 s + a_2 = (s + 1 + j0.5)(s + 1 - j0.5) = s^2 + 2s + 1.25$. At DC, the system function is $H(j0) = -1 = k\frac{b_2}{a_2} = k\frac{2.25}{1.25} = k\frac{9}{5}$. Therefore,

$$k = -\tfrac{5}{9}, \; b_1 = 0, \; b_2 = \tfrac{9}{4}, \; a_1 = 2, \text{ and } a_2 = \tfrac{5}{4}.$$

(b) The DC gain is given as $S(j0) = -1$. Thus, the input of 4 just becomes -4. To compute the output to $\cos(t/2 + \pi/3)$, $H(j0.5)$ is required. Graphically, $|H(j0.5)| = |k|\frac{(1)(2)}{(1)(\sqrt{2})} = \frac{10}{9\sqrt{2}}$ and $\angle H(j0.5) = \pi - \pi/2 + \pi/2 - (0 + \pi/4) = 3\pi/4$. Thus, the output to $\cos(t/2 + \pi/3)$ is just $\frac{10}{9\sqrt{2}}\cos(t/2 + \pi/3 + 3\pi/4)$, and the output to $x(t) = 4 + \cos(t/2 + \pi/3)$ is

$$y(t) = -4 + \frac{10}{9\sqrt{2}}\cos(t/2 + 13\pi/12) \approx -4 + 0.7857\cos(t/2 + 3.4034).$$

Solution 4.8-6

For the CT system described by $(D+1)(D+2)\{y(t)\} = x(t-1)$, let us define $x'(t) = x(t-1)$. Thus, we can describe the system in terms of the modified input $x'(t)$ as $(D+1)(D+2)\{y(t)\} = x'(t)$ and solve it in the standard way. Using input $x'(t)$ and output $y(t)$, the system transfer function is

$$H(s) = \frac{Y(s)}{X'(s)} = \frac{1}{s^2 + 3s + 2}.$$

(a) Here, $x(t) = 1$, so $\omega = 0$ and $x(t-1) = 1 = x'(t)$. Since $H(0) = \frac{1}{2}$, we see that

$$y(t) = H(0)x'(t) = \frac{1}{2}.$$

(b) Here, $x(t) = \cos(t)$, so $\omega = 1$ and $x(t-1) = \cos(t-1) = x'(t)$. Now,

$$|H(j1)| = \frac{1}{\sqrt{10}} \quad \text{and} \quad \angle H(j1) = \tan^{-1}(-3) = -1.2490.$$

Thus,

$$y(t) = |H(j1)|\cos(t - 1 + \angle H(j1)) = \frac{1}{\sqrt{10}}\cos(t - 2.2490).$$

Solution 4.8-7

The frequency of $x(t) = \frac{1}{3}e^{j(6t+\pi/3)}u(6t + \pi/3)$ is $\omega = 6$. To determine the steady-state response, we thus need to compute $|H(6j)|$ and $\angle H(6j)$.

$$|H(6j)| = \left|\frac{4(6j)}{(6j + 1 + 6j)(6j + 1 - 6j)}\right| = 4\frac{6}{1\sqrt{12^2 + 1^2}} = \frac{24}{\sqrt{145}} \approx 1.9931$$

$$\angle H(6j) = \angle\frac{4(6j)}{(6j + 1 + 6j)(6j + 1 - 6j)} = \frac{\pi}{2} - \tan^{-1}(12) \approx 0.0831$$

Thus, the steady-state response to $x(t) = \frac{1}{3}e^{j(6t+\pi/3)}u(6t + \pi/3)$ is

$$y_{ss}(t) = |H(6j)|\frac{1}{3}e^{j(6t+\pi/3+\angle H(6j))}u(6t + \pi/3) \approx 0.6644e^{j(6t+1.1303)}u(6t + 1.0472).$$

Solution 4.9-1

Based on the problem description, we know that the system transfer function takes the form

$$H(s) = K\frac{s - s_z}{s - 2},$$

where constant K is a gain factor and constant s_z designates the location of the system zero. Since the system is real, we know that K and s_z are real. Since the system is lowpass, we know that $|s_z| > 2$. We make the further assumption that s_z is positive (resulting in a so-called minimum-phase system).

(a) We know the passband gain is one, or 0 dB. To achieve 40 dB of stopband attenuation requires

$$-40 = 20\log_{10}H(j\infty) = 20\log_{10}K \quad \Rightarrow \quad K = 10^{-2} = 0.01.$$

The maximum passband gain of unity occurs at $\omega = 0$. Thus,

$$H(0) = 1 = K\frac{s_z}{2} \quad \Rightarrow \quad s_z = 200.$$

Taken together, the transfer function is

$$H(s) = 0.01\frac{s - 200}{s - 2}.$$

Figure S4.9-1a shows the straight-line Bode approximation (solid) and true magnitude response (dashed). Since the system pole causes a -20 dB/decade drop in gain after $\omega = 2$, it is sensible that the desired -40 dB of attenuation is reached two decades later at $\omega = 200$, at which point the system zero flattens the response and holds the stopband attenuation at -40 dB.

(b) With less stopband attenuation that (a), we expect the zero to be closer to the pole than 200. To achieve 30 dB of stopband attenuation requires

$$-30 = 20\log_{10}H(j\infty) = 20\log_{10}K \quad \Rightarrow \quad K = 0.0316.$$
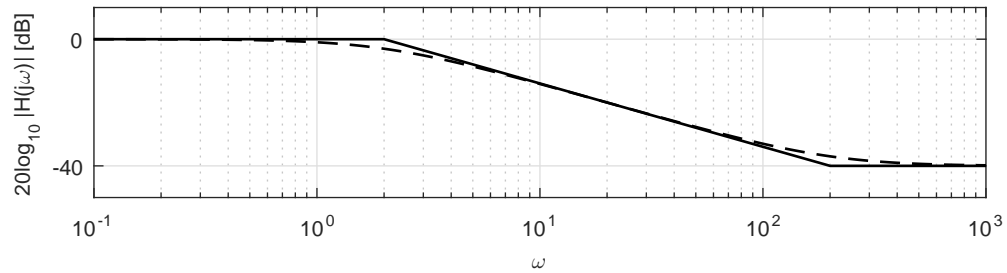
Figure S4.9-1a

The maximum passband gain of unity occurs at $\omega = 0$. Thus,

$$H(0) = 1 = K\frac{s_z}{2} \quad \Rightarrow \quad s_z = 63.2456.$$

Taken together, the transfer function is

$$H(s) = 0.0316\frac{s - 63.2456}{s - 2}.$$

Figure S4.9-1b shows the straight-line Bode approximation (solid) and true magnitude response (dashed). As in (a), the system pole causes a -20 dB/decade drop in gain after $\omega = 2$. The gain hits -30 dB at $\omega = 63.2456$, at which point the system zero flattens the response and holds the stopband attenuation at -30 dB.

Figure S4.9-1b

Solution 4.9-2

Based on the problem description, we know that the system transfer function takes the form

$$H(s) = K\frac{s - s_z}{s - 2},$$

where constant K is a gain factor and constant s_z designates the location of the system zero. Since the system is real, we know that K and s_z are real. Since the system is highpass, we know that $|s_z| < 2$. We make the further assumption that s_z is positive (resulting in a so-called minimum-phase system).

(a) To achieve a passband gain of 1 requires

$$1 = H(j\infty) = \lim)\omega \to \infty K\frac{j\omega - s_z}{j\omega - 2} \quad \Rightarrow \quad K = 1.$$

To achieve 40 dB of stopband attenuation requires

$$-40 = 20\log_{10}H(0) = 20\log_{10}\left(\frac{s_z}{2}\right) \quad \Rightarrow \quad s_z = 0.02.$$

Taken together, the transfer function is

$$H(s) = \frac{s - 0.02}{s - 2}.$$

Figure S4.9-2a shows the straight-line Bode approximation (solid) and true magnitude response (dashed). The response starts at -40 dB and, upon reaching the zero at $\omega = 0.02$ increases at 20 dB/decade. Two decades later, at $\omega = 2$, the gain has reach 0 dB, at which point the system pole flattens the response and holds the passband at 0 dB.
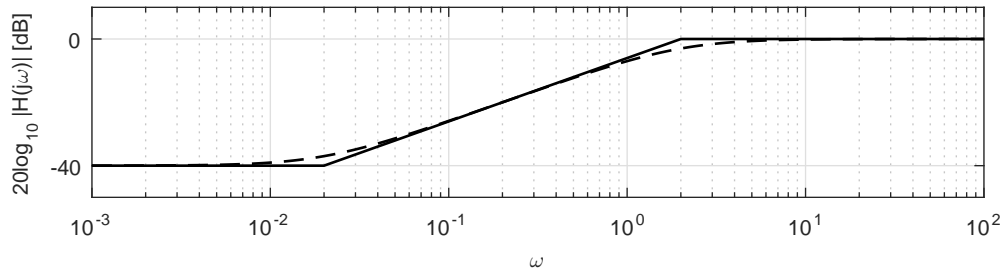
Figure S4.9-2a

(b) With less stopband attenuation that (a), we expect the zero to be closer to the pole that 0.02. To achieve a passband gain of 1 requires

$$1 = H(j\infty) = \lim)\omega \to \infty K \frac{j\omega - s_z}{j\omega - 2} \quad \Rightarrow \quad K = 1.$$

To achieve 30 dB of stopband attenuation requires

$$-30 = 20\log_{10} H(0) = 20\log_{10}\left(\frac{s_z}{2}\right) \quad \Rightarrow \quad s_z = 0.0632.$$

Taken together, the transfer function is

$$H(s) = \frac{s - 0.0632}{s - 2}.$$

Figure S4.9-2b shows the straight-line Bode approximation (solid) and true magnitude response (dashed). The response starts at -30 dB and, upon reaching the zero at $\omega = 0.0632$ increases at 20 dB/decade. At $\omega = 2$, the gain has reach 0 dB, at which point the system pole flattens the response and holds the passband at 0 dB.
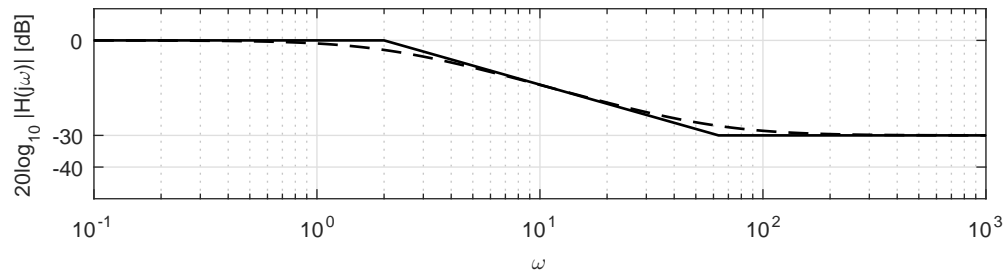

Figure S4.9-2b
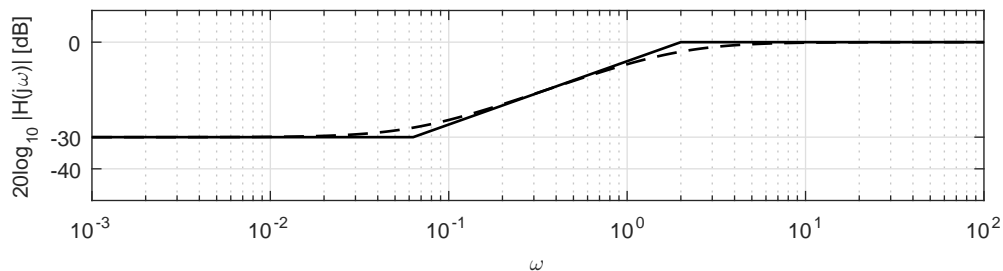
Solution 4.9-3

Based on the problem description, we know that the system transfer function takes the form

$$H(s) = K \frac{(s - s_z)^2}{(s - 2)^2},$$

where constant K is a gain factor and constant s_z designates the location of the system zero. Since the system is real, we know that K and s_z are real. Since the system is lowpass, we know that $|s_z| > 2$. We make the further assumption that s_z is positive (resulting in a so-called minimum-phase system).

(a) We know the passband gain is one, or 0 dB. To achieve 40 dB of stopband attenuation requires

$$-40 = 20\log_{10}H(j\infty) = 20\log_{10}K \quad \Rightarrow \quad K = 10^{-2} = 0.01.$$

The maximum passband gain of unity occurs at $\omega = 0$. Thus,

$$H(0) = 1 = K\frac{s_z^2}{2^2} \quad \Rightarrow \quad s_z = 20.$$

Taken together, the transfer function is

$$H(s) = 0.01\frac{(s-20)^2}{(s-2)^2}.$$

Figure S4.9-3a shows the straight-line Bode approximation (solid) and true magnitude response (dashed). Since the repeated system pole causes a -40 dB/decade drop in gain after $\omega = 2$, it is sensible that the desired -40 dB of attenuation is reached one decade later at $\omega = 20$, at which point the repeated system zero flattens the response and holds the stopband attenuation at -40 dB.
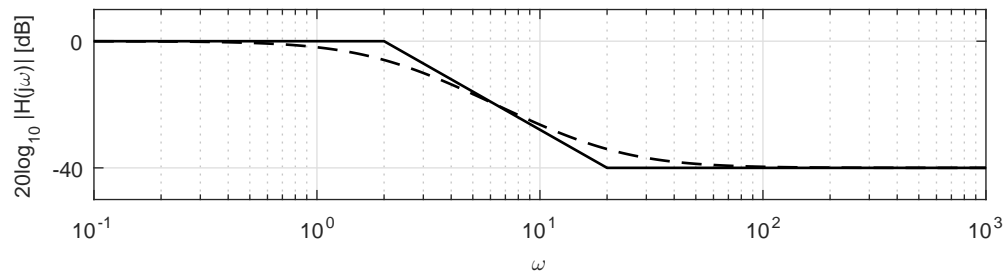
Figure S4.9-3a

(b) With less stopband attenuation that (a), we expect the zero to be closer to the pole than 20. To achieve 30 dB of stopband attenuation requires

$$-30 = 20\log_{10}H(j\infty) = 20\log_{10}K \quad \Rightarrow \quad K = 0.0316.$$

The maximum passband gain of unity occurs at $\omega = 0$. Thus,

$$H(0) = 1 = K\frac{s_z^2}{2^2} \quad \Rightarrow \quad s_z = 11.2468.$$

Taken together, the transfer function is

$$H(s) = 0.0316\frac{(s-11.2468)^2}{(s-2)^2}.$$

Figure S4.9-3b shows the straight-line Bode approximation (solid) and true magnitude response (dashed). As in (a), the repeated system pole causes a -40 dB/decade drop in gain after $\omega = 2$. The gain hits -30 dB at $\omega = 11.2468$, at which point the repeated system zero flattens the response and holds the stopband attenuation at -30 dB.
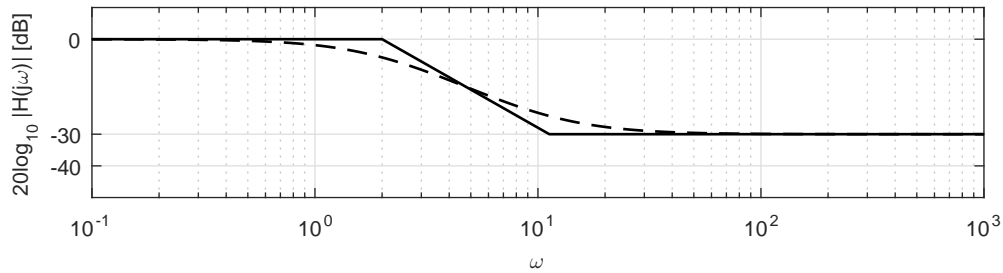
Figure S4.9-3b

Solution 4.9-4

(a) The transfer function can be expressed as

$$H(s) = \frac{100}{2 \times 20} \frac{s(\frac{s}{100} + 1)}{(\frac{s}{2} + 1)(\frac{s}{20} + 1)} = 2.5 \frac{s(\frac{s}{100} + 1)}{(\frac{s}{2} + 1)(\frac{s}{20} + 1)}.$$

The amplitude response: The zero at 0 causes a continuous 20 dB/decade gain increase, intersecting $20\log_{10} 2.5 = 7.95$ dB at $\omega = 1$. We incorporate the additional asymptotes at 2 (-20 dB/dec.), 20 (-20 dB/dec.), and 100 (20 dB/dec.) to produce the straight-line Bode approximation shown in Fig. S4.9-4a (solid line). The true magnitude response (dashed line) can be obtained by computer or by applying the corrections to the straight-line Bode approximation, as described in the text. A similar procedure is followed for the phase response, which is also shown in Fig. S4.9-4a.
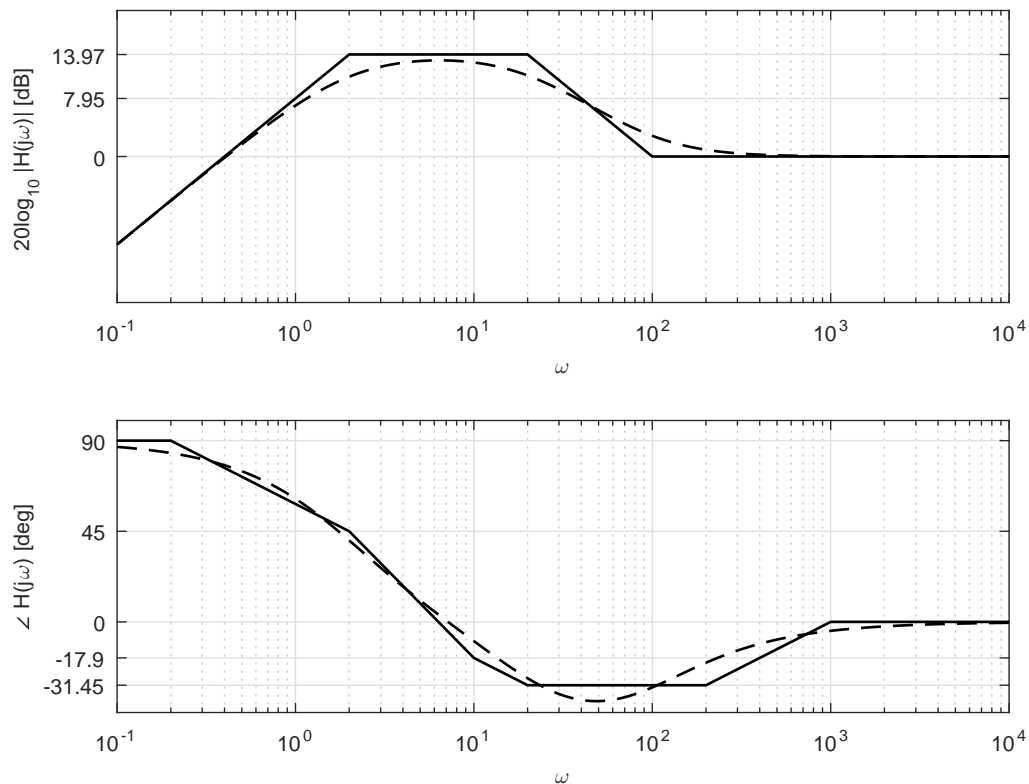
Figure S4.9-4a

(b) The transfer function can be expressed as

$$H(s) = \frac{10 \times 20}{100} \frac{(\frac{s}{10} + 1)(\frac{s}{20} + 1)}{s^2(\frac{s}{100} + 1)} = 2\frac{(\frac{s}{10} + 1)(\frac{s}{20} + 1)}{s^2(\frac{s}{100} + 1)}.$$

The amplitude response: The two poles at 0 causes a continuous -40 dB/decade gain decrease, intersecting $20\log_{10}2 = 6.02$ dB at $\omega = 1$. We incorporate the additional asymptotes at $\omega = 10$ (20 dB/dec.), 20 (20 dB/dec.), and 100 (-20 dB/dec.) to produce the straight-line Bode approximation shown in Fig. S4.9-4b (solid line). The true magnitude response (dashed line) can be obtained by computer or by applying the corrections to the straight-line Bode approximation, as described in the text. A similar procedure is followed for the phase response, which is also shown in Fig. S4.9-4b.
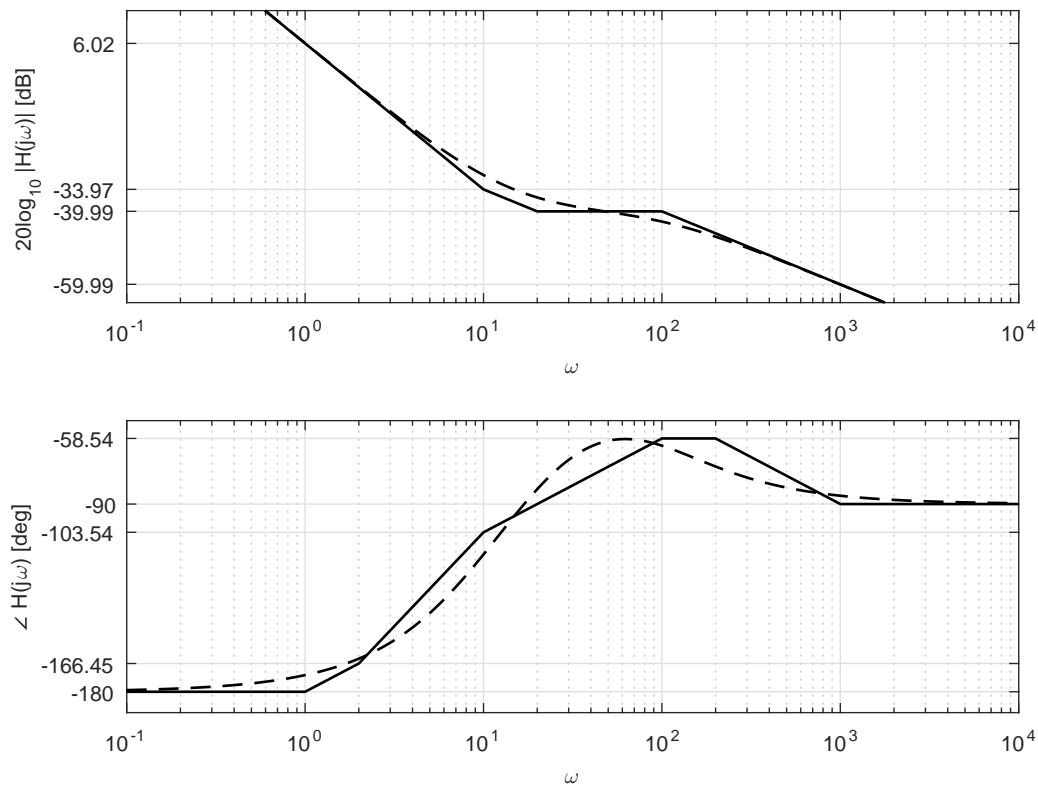
Figure S4.9-4b

(c) The transfer function can be expressed as

$$H(s) = \frac{10 \times 200}{400 \times 1000} \frac{(\frac{s}{10} + 1)(\frac{s}{200} + 1)}{(\frac{s}{20} + 1)^2(\frac{s}{1000} + 1)} = \frac{1}{200} \frac{(\frac{s}{10} + 1)(\frac{s}{200} + 1)}{(\frac{s}{20} + 1)^2(\frac{s}{1000} + 1)}.$$

The amplitude response: We use a dc gain of $20\log_{10}\frac{1}{200} = -46.02$ dB and then incorporate the asymptotes at $\omega = 10$ (20 dB/dec.), 20 (-40 dB/dec.), 200 (20 dB/dec.), and 1000 (-20 dB/dec.) to produce the straight-line Bode approximation shown in Fig. S4.9-4c (solid line). The true magnitude response (dashed line) can be obtained by computer or by applying the corrections to the straight-line Bode approximation, as described in the text. A similar procedure is followed for the phase response, which is also shown in Fig. S4.9-4c. Clearly, the interactions between nearby poles and zeros make the straight-line Bode approximation of phase only a rough estimate of the true phase response.
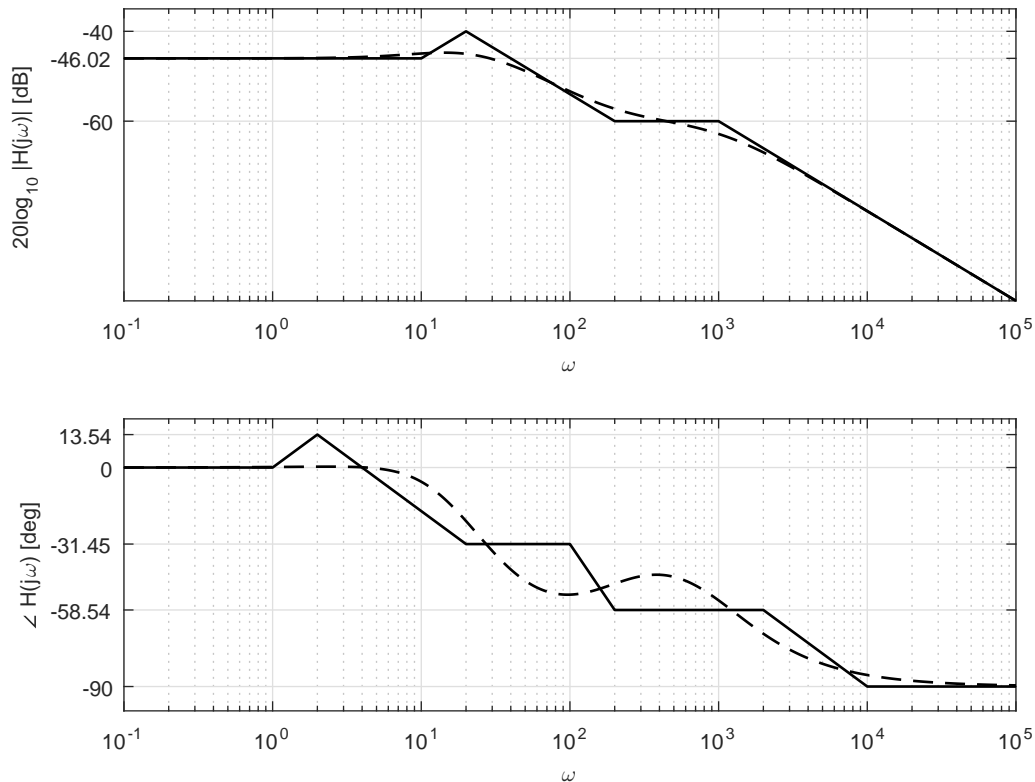
Figure S4.9-4c

Solution 4.9-5

(a) The transfer function can be expressed as

$$H(s) = \frac{1}{16} \frac{s^2}{\left(\frac{s}{1}+1\right)\left(\frac{s^2}{16}+\frac{s}{4}+1\right)}.$$

The amplitude response: The two zeros at 0 causes a continuous 40 dB/decade gain increase, intersecting $20\log_{10}\frac{1}{16} = -24.08$ dB at $\omega = 1$. We incorporate the additional asymptotes at $\omega = 1$ (-20 dB/dec.) and 4 (-40 dB/dec.) to produce the straight-line Bode approximation shown in Fig. S4.9-5a (solid line). The true magnitude response (dashed line) can be obtained by computer or by applying the corrections to the straight-line Bode approximation, as described in the text. A similar procedure is followed for the phase response, which is also shown in Fig. S4.9-5a. Error is somewhat large near $\omega = 4$ due to the nature of the straight-line phase approximation for complex-conjugate roots.

Figure S4.9-5a

(b) The transfer function can be expressed as

$$H(s) = \frac{1}{100} \frac{s}{(\frac{s}{1} + 1)(\frac{s^2}{100} + 0.1414s + 1)}.$$

The amplitude response: The zero at 0 causes a continuous 20 dB/decade gain increase, intersecting $20\log_{10}\frac{1}{100} = -40$ dB at $\omega = 1$. We incorporate the additional asymptotes at $\omega = 1$ (-20 dB/dec.) and 10 (-40 dB/dec.) to produce the straight-line Bode approximation shown in Fig. S4.9-5b (solid line). The true magnitude response (dashed line) can be obtained by computer or by applying the corrections to the straight-line Bode approximation, as described in the text. A similar procedure is followed for the phase response, which is also shown in Fig. S4.9-5b. Error is somewhat large near $\omega = 10$ due to the nature of the straight-line phase approximation for complex-conjugate roots.

(c) The transfer function can be expressed as

$$H(s) = \frac{10}{100} \frac{\frac{s}{10} + 1}{s(\frac{s^2}{100} + 0.1414s + 1)}.$$

The amplitude response: The pole at 0 causes a continuous -20 dB/decade gain decrease, intersecting $20\log_{10}\frac{1}{10} = -20$ dB at $\omega = 1$. We incorporate the additional asymptotes at $\omega = 10$ (20 dB/dec.) and 10 (-40 dB/dec.) to produce the straight-line Bode approximation shown in Fig. S4.9-5c (solid line). The true magnitude response (dashed line) can be obtained by computer or by applying the corrections to the straight-line Bode approximation, as described in the text. A similar procedure is followed for the phase response, which is also shown in Fig. S4.9-5c. Error is somewhat large near $\omega = 10$ due to the nature of the straight-line phase approximation for complex-conjugate roots.
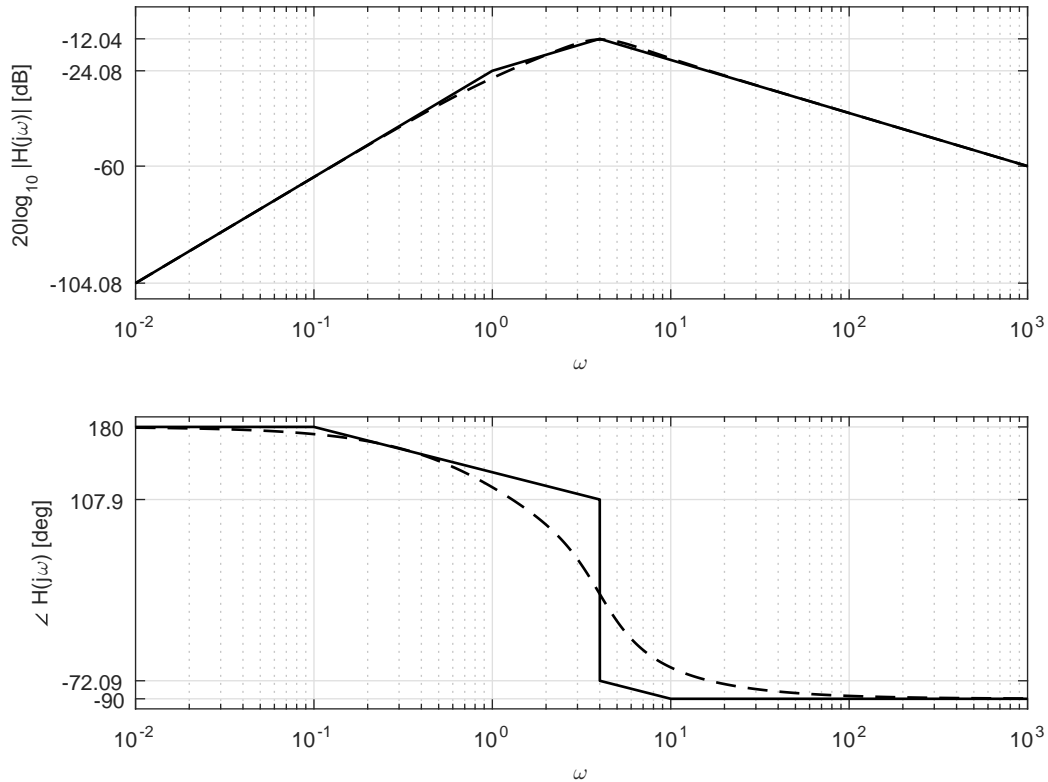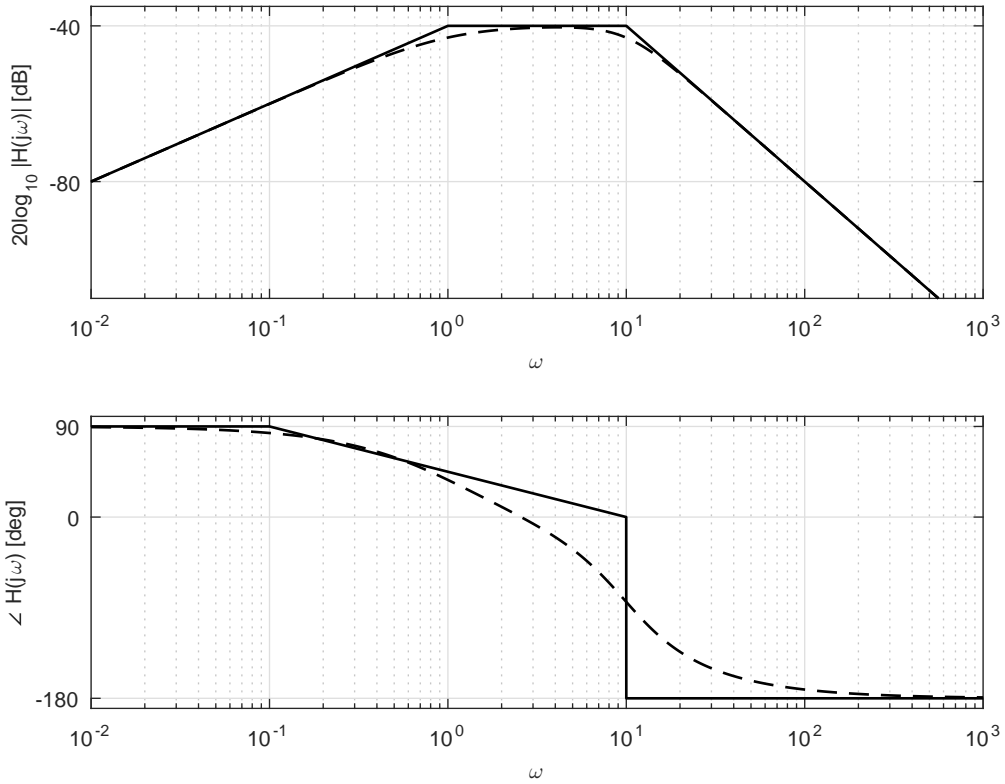
Figure S4.9-5b
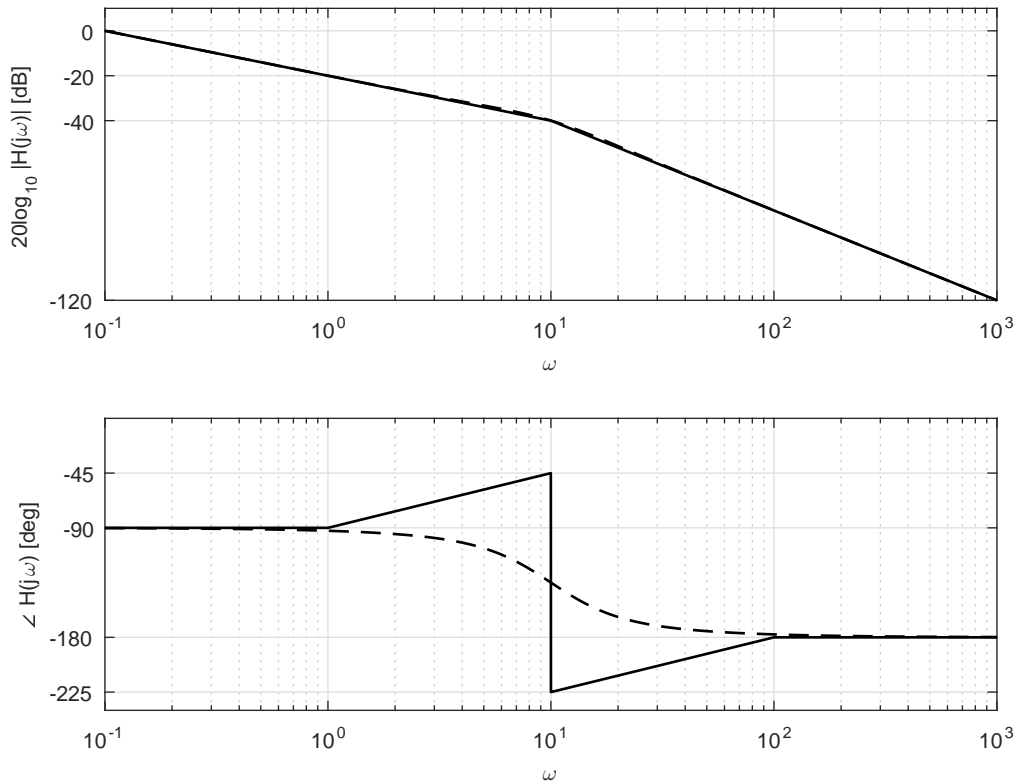
Figure S4.9-5c

Solution 4.9-6

To rise at 20 dB per decade, a zero must be present before $\omega = 0.1$. At $\omega = 30$, the magnitude response begins to fall at -20 dB per decade. This requires two poles at that frequency: one pole to counteract the previous zero and another pole to cause the -20 dB per decade slope. The magnitude response levels out at $\omega = 500$, which requires the action of a zero. Thus, a second order system should be sufficient.

$$H(s) = k \frac{s(s + 500)}{(s + 30)^2}.$$

To determine the constant k, notice that $20 \log(|H(j1)|) = 10$ or $|H(j1)| = \sqrt{10} \approx k \frac{1}{500} 30^2 = k \frac{5}{9}$. Thus, $k = \frac{9\sqrt{10}}{4}$ should work well. Combining yields

$$H(s) = \frac{9\sqrt{10}}{5} \frac{s(s + 500)}{(s + 30)^2}.$$

MATLAB is used to verify the result (solid).

```
>>  H = @(s) 9*sqrt(10)/5*s.*(s+500)./((s+30).^2);
>>  w = logspace(-1,4,10001); semilogx(w,20*log10(abs(H(1j*w))),'k-');
>>  grid on; axis([.1 1e4 -10 40]);
>>  xlabel('\omega'); ylabel('20log_{10} |H(j\omega)| [dB]');
```

The resulting plot of Fig. S4.9-6, with a straight-line Bode approximation (dashed) added for clarity, closely matches the original plot of Fig. P4.9-6, thereby confirming $H(s)$.
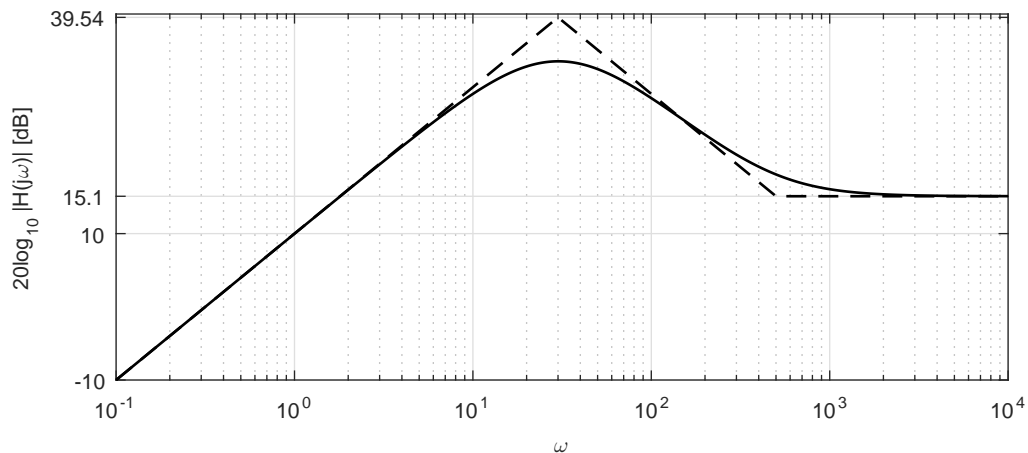
Figure S4.9-6

Solution 4.9-7

(a) Call the voltage across the first capacitor $v_{C_1}(t)$. KCL thus yields $x(t) = C_1 \dot{v}_{C_1}(t) + C_2 \dot{v}_{C_2}(t) = C_1 \dot{v}_{C_1}(t) + C_2 \dot{y}(t)$. In the transform domain, this becomes $X(s) = C_1 s V_{C_1}(s) + C_2 s Y(s)$ or $V_{C_1}(s) = \frac{X(s) - C_2 s Y(s)}{C_1 s}$. KVL yields $y(t) = R C_1 \dot{v}_{C_1}(t) + v_{C_1}(t)$. In the transform domain, this becomes $Y(S) = V_{C_1}(s)(1 + R C_1 s)$ or $V_{C_1}(s) = \frac{Y(s)}{1 + R C_1 s}$. Combining the KCL and KVL equations yields $\frac{X(s) - C_2 s Y(s)}{C_1 s} = \frac{Y(s)}{1 + R C_1 s}$. Simplifying yields $Y(s)\left(R C_1 C_2 s^2 + C_1 s + C_2 s\right) = X(s)(R C_1 s + 1)$. Thus,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{R C_1 s + 1}{R C_1 C_2 s^2 + (C_1 + C_2)s}.$$

(b) Notice, $H(s)$ has two poles, one at zero and another at a negative, real number. It also has two zeros, one at infinity and another at a negative, real number. Only plots B and D show evidence of a finite zero as well as a finite pole. Of these, only plot B can have the necessary pole at zero. Thus,

$$\text{Plot B is the only plot consistent with the system.}$$

(c) At low frequencies, $H(j\omega) \approx \frac{1}{(C_1+C_2)j\omega}$. Thus, R doesn't affect $|H(j\omega)|$ at very low frequencies.

(d) At high frequencies, $H(j\omega) \approx \frac{RC_-1j\omega}{-RC_-1C_2\omega^2} = \frac{-j}{C_2\omega}$. Thus, R doesn't affect $|H(j\omega)|$ at very high frequencies.

Solution 4.10-1

(a) Using the graphical method of Sec. 4.10.1, we compute the magnitude response $|H(j\omega)|$ for ω = 0, 2, 4, and 10.

$$|H(j0)| = 2\frac{4(4)}{\sqrt{5}\sqrt{5}} = \frac{32}{5} = 6.4$$

$$|H(j2)| = 2\frac{2(6)}{1\sqrt{17}} = \frac{24}{4.1} = 5.8$$

$$|H(j4)| = 0$$

$$|H(j10)| = 2\frac{6(14)}{8(12)} = \frac{7}{4} = 1.75$$

Since the system is real, $|H(j\omega)| = |H(-j\omega)|$. Using these calculations, Fig. S4.10-1 shows the magnitude response $|H(j\omega)|$ over $-10 \leq \omega \leq 10$.

(b) Using the graphical method of Sec. 4.10.1, we compute the phase response $\angle H(j\omega)$ for $\omega = 0$, 2, 4, and 10.

$$\angle H(j0) = 0 + (90° - 90°) - \left(\tan^{-1}(2) - \tan^{-1}(2)\right) = 0°$$

$$\angle H(j2) = 0 + (90° - 90°) - \left(\tan^{-1}(0) + \tan^{-1}(4)\right) = -76°$$

$$\angle H(j4^-) = 0 + (90° - 90°) - \left(\tan^{-1}(2) + \tan^{-1}(6)\right) = -143.9°$$

$$\angle H(j4^+) = 0 + (90° + 90°) - \left(\tan^{-1}(2) + \tan^{-1}(6)\right) = 36.1°$$

$$\angle H(j10) = 0 + (90° + 90°) - \left(\tan^{-1}(8) + \tan^{-1}(12)\right) = 11.9°$$

Since the system is real, $\angle H(j\omega) = -\angle H(-j\omega)$. Using these calculations, Fig. S4.10-1 shows the phase response $\angle H(j\omega)$ over $-10 \leq \omega \leq 10$.

(c) The input $x(t) = -1 + 2\cos(2t) - 3\sin(4t + \pi/3) + 4\cos(10t)$ is comprised of frequencies $\omega = 0$, 2, 4, and 10. Using $|H(j\omega)|$ and $\angle H(j\omega)$ from parts (a) and (b), the output is

$$y(t) = -6.4 + 11.6\cos(2t - 76°) + 7\cos(10t + 11.9°).$$
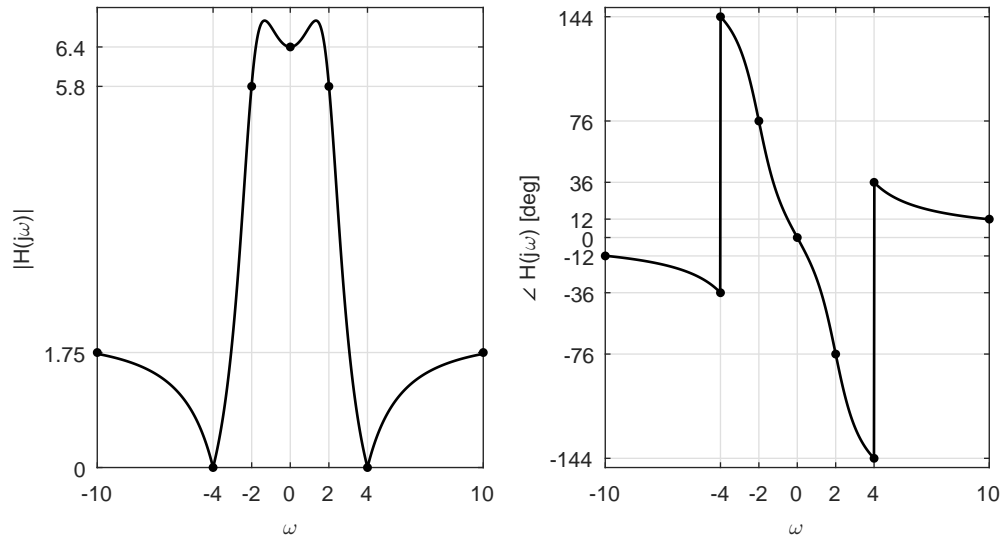
Figure S4.10-1

Solution 4.10-2

(a) With zeros at $\pm j4$, we know that, within constant k, the numerator of $H(s)$ is

$$(s + j4)(s - j4) = s^2 + 16 = s^2 + b_1 s + b_2.$$

With zeros at $-1 \pm j3$, we know that the denominator of $H(s)$ is

$$(s + 1 + j3)(s + 1 - j3) = s^2 + 2s + 10 = s^2 + a_1 s + a_2.$$

Since $H(j0) = -2 = k\frac{b_2}{a_2} = k\frac{16}{10}$, we know that $k = -\frac{20}{16} = -\frac{5}{4}$. Thus,

$$k = -\tfrac{5}{4},\ b_1 = 0,\ b_2 = 16,\ a_1 = 2,\ \text{and}\ a_2 = 10.$$

(b) Using the graphical method of Sec. 4.10.1, we compute the magnitude response $|H(j\omega)|$ for $\omega = 0, 3, 4,$ and 10.

$$|H(j0)| = |-2| = 2$$

$$|H(j3)| = \frac{5}{4}\frac{1(7)}{1\sqrt{37}} = 1.44$$

$$|H(j4)| = 0$$

$$|H(j10)| = \frac{5}{4}\frac{14(6)}{\sqrt{50}\sqrt{170}} = 1.14$$

$$|H(j\infty)| = \frac{5}{4} = 1.25$$

Since the system is real, $|H(j\omega)| = |H(-j\omega)|$. We use MATLAB to plot these points as well as $|H(j\omega)|$ over $-10 \le \omega \le 10$ (see Fig. S4.10-2).

```
>>  H = @(s) -5/4*(s.^2+16)./(s.^2+2*s+10);
>>  Xm = 10; w = linspace(-Xm,Xm,1000);
>>  subplot(121); plot(w,abs(H(1j*w)),'k-',...
>>      [-10 -4 -3 0 3 4 10],[1.14 0 1.44 2 1.44 0 1.14],'k.');
>>  xlabel('\omega'); ylabel('|H(j\omega)|'); axis([-Xm Xm 0 2.5]); grid
```

(c) Using the graphical method of Sec. 4.10.1, we compute the phase response $\angle H(j\omega)$ for $\omega = 0$, 3, 4, and 10.

$$\angle H(j0) = \pm 180 + (90° - 90°) - \left(\tan^{-1}(3) - \tan^{-1}(3)\right) = \pm 180°$$

$$\angle H(j3) = \pm 180 + (90° - 90°) - \left(0 + \tan^{-1}(6)\right) = 99.5°$$

$$\angle H(j4^-) = \pm 180 + (90° - 90°) - \left(\tan^{-1}(1) + \tan^{-1}(7)\right) = 53.1°$$

$$\angle H(j4^+) = \pm 180 + (90° + 90°) - \left(\tan^{-1}(1) + \tan^{-1}(7)\right) = -126.9°$$

$$\angle H(j10) = \pm 180 + (90° + 90°) - \left(\tan^{-1}(7) + \tan^{-1}(13)\right) = -167.5°$$

Since the system is real, $\angle H(j\omega) = -\angle H(-j\omega)$. We use MATLAB to plot these points as well as $\angle H(j\omega)$ over $-10 \le \omega \le 10$ (see Fig. S4.10-2).

```
>>  H = @(s) -5/4*(s.^2+16)./(s.^2+2*s+10);
>>  Xm = 10; w = linspace(-Xm,Xm,1000);
>>  P = angle(H(1j*w))*180/pi
>>  subplot(122); plot(w,P,'k-',[-10 -4 -4 -3 0 0 3 4 4 10],...
>>     [167.5 126.9 -53.1 -99.5 -180 180 99.5 53.1 -126.9 -167.5],'k.');
>>  xlabel('\omega'); ylabel('\angle H(j\omega) [deg]');
>>  axis([-10 10 -190 190]); grid
```

(d) The input $x(t) = -3 + \cos(3t + \pi/3) - \sin(4t - \pi/8)$ is comprised of frequencies $\omega = 0$, 3, and 4. Using $|H(j\omega)|$ and $\angle H(j\omega)$ from parts (a) and (b), the output is

$$y(t) = 6 + 1.44\cos(3t + 159.5°).$$
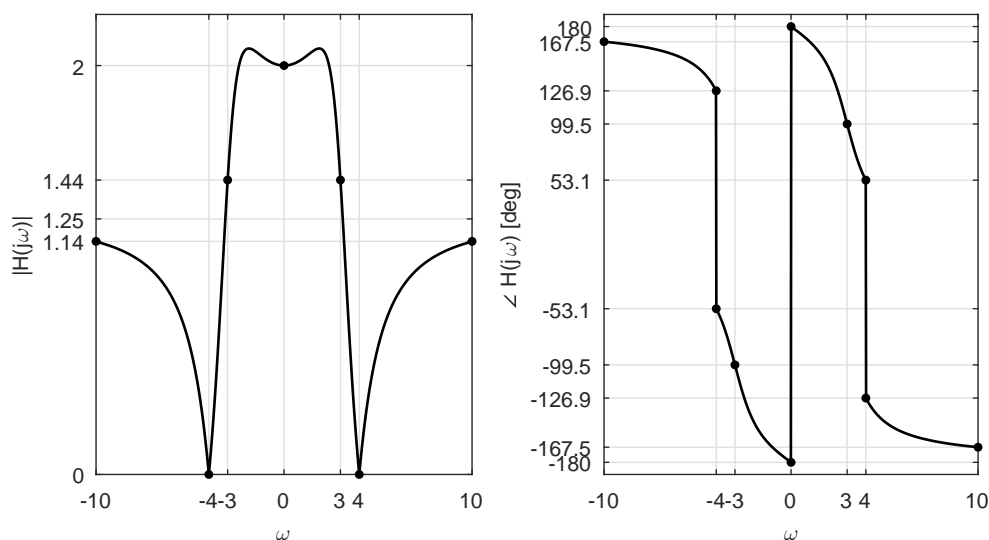
Figure S4.10-2

Solution 4.10-3

We plot the poles $-1 \pm j7$ and $1 \pm j7$ in the s-plane. To find response at some frequency ω, we connect all the poles and zeros to the point $j\omega$ (see Fig. S4.10-3). Note that the product of the distances from the zeros is equal to the product of the distances from the poles for all values of ω. Therefore $|H(j\omega)| = 1$. Graphical argument shows that $\angle H(j\omega)$ (sum of the angles from the zeros − sum of the angles from poles) starts at zero for $\omega = 0$ and then reduces continuously (becomes negative)

as ω increases, with fastest rate of decrease near $\omega = 7$. As $\omega \to \infty$, $\angle H(\omega) \to -2\pi$. Because the system is real, magnitude and phase responses have even and odd symmetry, respectively. Figure S4.10-3 shows rough sketches and accurate plots of the magnitude and phase responses. Clearly, this is an allpass filter.
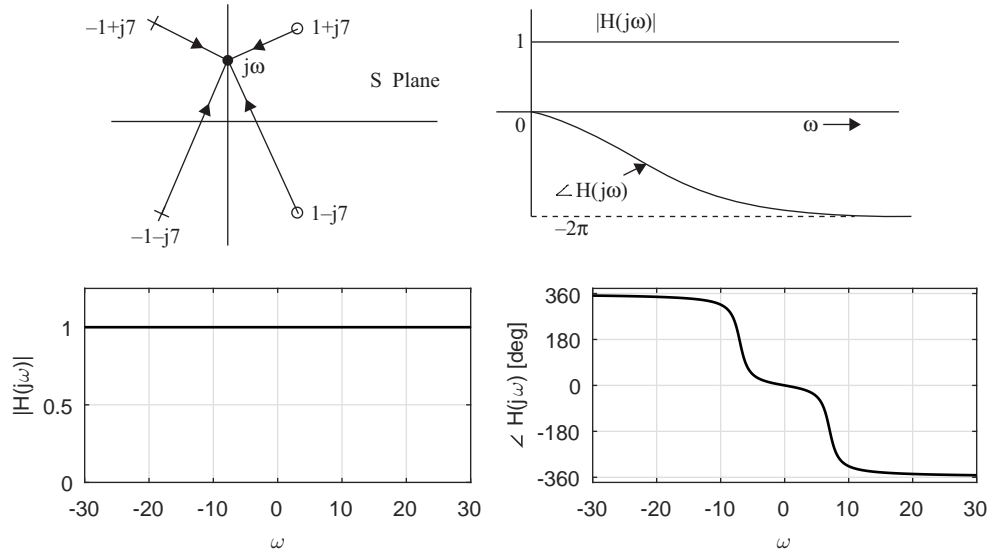
Figure S4.10-3

Solution 4.10-4

In this problem, the transfer function takes the general form $H(s) = k\frac{s-z}{s-p}$, where z designates the location of the system zero, p designates the location of the system pole, and k is a gain term. For simplicity and because it is unspecified in the problem, we take $k = 1$. Magnitude and phase response sketches are easily adjusted for cases when $k \neq 1$.

(a) If r and d are the distances of the zero and pole, respectively, from $j\omega$, then the amplitude response $|H(j\omega)|$ is the ratio r/d. This ratio is 0.5 for $\omega = 0$. Therefore, the dc gain is 0.5. The ratio $r/d = 1$ for $\omega = \infty$. Thus, the gain is unity at $\omega = \infty$. Also, the angles of the line segments connecting the zero and pole to the point $j\omega$ are both zero for $\omega = 0$, and are both $\pi/2$ for $\omega = \infty$. Therefore, $\angle H(j\omega) = 0$ at $\omega = 0$ and $\omega = \infty$. In between, the angle is positive as shown in Fig. S4.10-4a.

(b) In this case the ratio r/d is 2 for $\omega = 0$. Therefore, the dc gain is 2. Also the ratio $r/d = 1$ for $\omega = \infty$. Thus, the gain is unity at $\omega = \infty$. The angles of the line segments connecting the zero and pole to the point $j\omega$ are both zero for $\omega = 0$, and are both $\pi/2$ for $\omega = \infty$. Therefore, $\angle H(j\omega) = 0$ at $\omega = 0$ and $\omega = \infty$. In between, the angle is negative as shown in Fig. S4.10-4b.
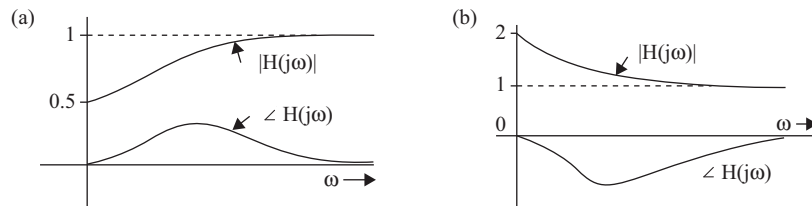
Figure S4.10-4

Solution 4.10-5

(a) Using the graphical method of Sec. 4.10.1, we compute the magnitude response $|H(j\omega)|$ for ω = 0, 1, 3, 10, 100, and ∞.

$$|H(j0)| = \frac{3(3)}{3\sqrt{5}\sqrt{5}} = \frac{3}{5} = 0.6$$

$$|H(j1)| = \frac{2(4)}{3(2)\sqrt{8}} = \frac{\sqrt{2}}{3} = 0.4714$$

$$|H(j3)| = 0$$

$$|H(j10)| = \frac{7(13)}{3\sqrt{85}\sqrt{125}} = 0.2943$$

$$|H(j100)| = \frac{97(103)}{3\sqrt{9805}\sqrt{10205}} = 0.3329$$

$$|H(j\infty)| = \frac{1}{3} = 0.3333$$

Since the system is real, $|H(j\omega)| = |H(-j\omega)|$. Using these calculations, Fig. S4.10-5 shows the magnitude response $|H(j\omega)|$ over $-10 \leq \omega \leq 10$.

(b) The input $x(t) = \cos(t) + \sin(3t + \pi/3) + \cos(100t)$ is comprised of frequencies $\omega = 1$, 3, and 100. We calculate the phase response at $\omega = 1$ and $\omega = 100$ as

$$\angle H(j) = -45° \qquad \text{and} \qquad \angle H(100j) = 2.29°.$$

Using these values and $|H(j\omega)|$ from part (a), the output to $x(t)$ is

$$y(t) = 0.4714\cos(t - 45°) + 0.3329\cos(100t + 2.29°).$$

(c) The pole/zero plot of $H_2(s) = H(-s)$ is shown in Fig. S4.10-5. Compared with system $H(s)$, system $H(-s)$ has the same zeros but the poles are swapped from the left half-plane to the right half-plane. Since $H_2(s)$ is a causal system with right half-plane poles, it is asymptotically unstable, and frequency response has no meaning. Thus, the output $y_2(t)$ of system $H_2(s)$ in response to $x(t)$ is unstable (∞, grows without bound) and can't be determined.
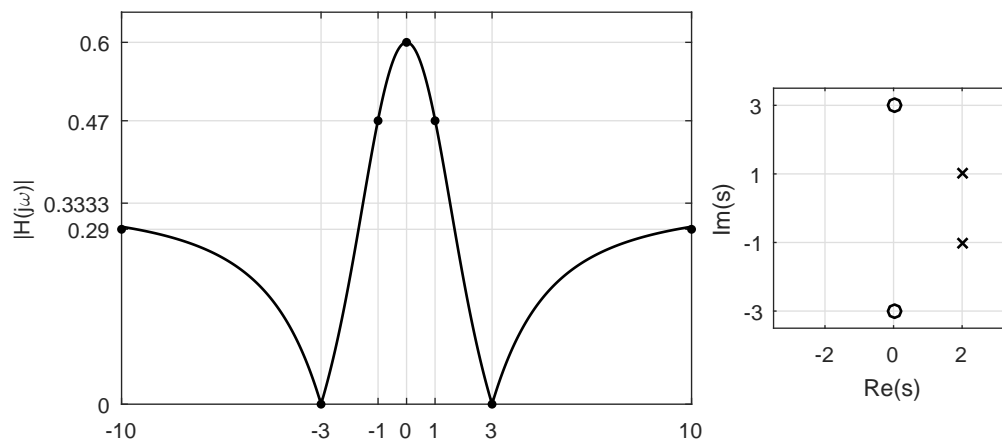
Figure S4.10-5

Solution 4.10-6

The poles are at $-a \pm j10$. Moreover zero gain at $\omega = 0$ and $\omega = \infty$ requires that there be a single zero at $s = 0$. This clearly causes the gain to be zero at $\omega = 0$. Also because there is one excess pole over zero, the gain for large values of ω is $1/\omega$, which approaches 0 as $\omega \to \infty$. Therefore, the suitable transfer function is

$$H(s) = \frac{s}{(s + a + j10)(s + a - j10)} = \frac{s}{s^2 + 2as + (100 + a^2)}.$$

The amplitude response is high in the vicinity of $\omega = 10$ provided a is small. Smaller the a, more pronounced the gain in the vicinity of $\omega = 10$. For $a = 0$, the gain at $\omega = 10$ is ∞.

Solution 4.10-7

Cynthia is correct. Although the system is all-pass and has $|H(j\omega)| = 1$, the phase response is not zero. Thus, the output generally has different phase than the input. Furthermore, the output can also include transient components that would not be present in the original input.

Solution 4.10-8

Both Amy and Jeff are correct. By definition, a zero is any value s that forces $H(s) = 0$ and a pole is any value s that forces $H(s) = \infty$. Thus, the system $H(s) = s = \frac{1}{s^{-1}}$ has both a zero at $s = 0$ and a pole at $s = \infty$. Remember, a rational system function always has the same number of poles and zeros; if $H(s) = s$ has an obvious zero at $s = 0$ there must be a matching pole somewhere, even if it is not finite. By similar argument, the system $H(s) = \frac{1}{s}$ has a pole at $s = 0$ and a zero at $s = \infty$.

Solution 4.10-9

At high frequencies, the highest powers of s dominate both the numerator and denominator of $H(s)$. That is, $\lim_{s \to \infty} H(s) = \lim_{s \to \infty} \frac{b_0 s^M}{s^N}$.

Lowpass and bandpass filters both require $\lim_{s \to \infty} H(s) = 0$, which ensures a response of zero at high frequencies. Only $H(s)$ that are strictly proper ($M < N$) yield the required $\lim_{s \to \infty} H(s) = 0$.

Highpass and bandstop filters both require $\lim_{s \to \infty} H(s) = k$, where k is some finite, non-zero constant. Only $H(s)$ that are proper ($M = N$) yield the required $\lim_{s \to \infty} H(s) = b_0 = k$.

The case $M > N$ is not considered, since such systems are not physically practical.

Solution 4.10-10

At high frequencies, the highest powers of s dominate both the numerator and denominator of $H(s)$. That is, $\lim_{s \to \infty} H(s) = \lim_{s \to \infty} \frac{b_0 s^M}{s^N}$. Thus, the log magnitude response at high frequencies is given by $\lim_{\omega \to \infty} \log |H(j\omega)| = \log(b_0) + M \log(\omega) - N \log(\omega)$. The fastest attenuation as a function of frequency requires M to be as small as possible. Thus, for a given N, the attenuation rate of an all-pole lowpass filter ($M = 0$) is faster than the attenuation rate of any filter with a finite number of zeros ($M \neq 0$).

Solution 4.10-11

No, it is not possible for such a system to function as a lowpass filter. For any choice of $([k, b_1, b_2, a_1, a_2] \in \mathcal{R})$, the system function $H(s) = k\frac{s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$ is proper. Thus, the system function always has high-frequency gain of k. For $k \neq 0$, the system cannot be lowpass. Furthermore, for $k = 0$ the system becomes a useless "nopass filter" (again, not lowpass).

Solution 4.10-12

Nick is more correct than his professor. A cascade of two identical filters, each with system response $H(j\omega)$, gives a total response of $H^2(j\omega)$. Since realizable filters, such as Butterworth filters, are not ideal, the cascade system will tend to have a faster transition band and greater stopband

attenuation. In a sense, the resulting fourth-order system really does provide "twice the filtering" of the original second-order system.

Unfortunately, there are also problems with Nick's approach. Simply cascading a designed lowpass filter twice has negative consequences. For example, the cutoff frequency shifts to a lower frequency than desired. As the cascaded RC example in Sec. 4.12 suggests, a cascade of low-order filters is inferior to a carefully designed, equivalent-order filter. In general, a fourth-order Butterworth filter performs better than a cascade of two second-order Butterworth filters.

Solution 4.10-13

(a) Using tables,

$$H(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = e^{-s/2}\,\frac{e^{s/2} - e^{-s/2}}{s}.$$

Substituting $s = j\omega$ yields the frequency response

$$H(j\omega) = e^{-j\omega/2}\,\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} = e^{-j\omega/2}\,\frac{\sin(\omega/2)}{\omega/2} = e^{-j\omega/2}\,\mathrm{sinc}\left(\frac{\omega}{2}\right).$$

The sinc type of frequency response (with a linear phase shift of $-\omega/2$) represents a lowpass system.

(b) Since $h(t)$ is finite duration, the system has no finite poles. There are, however, an infinite number of finite zeros for $\mathrm{sinc}(\omega/2)$ at $\omega = 2\pi k$ or $s = j2\pi k$, where k is any non-zero integer.

(c) In transform-domain, the inverse system is given by the reciprocal of $H_c(s)$. Thus,

$$H_c^{-1}(s) = \frac{1}{H_c(s)} = e^{s/2}\,\frac{\frac{s}{j2}}{\sin\left(\frac{s}{j2}\right)}.$$

The inverse system has no finite zeros and an infinite number of finite poles. Since poles lie on the ω-axis, the inverse system cannot be asymptotically stable.

The same approach does not work in the time-domain. That is, $h_c^{-1}(t) \neq \frac{1}{h(t)}$. The impulse response needs to be obtained from an inverse Laplace transform of $H_c^{-1}(s)$. Unfortunately, it is difficult to take the inverse Laplace transform of $H_c^{-1}(s)$; no closed form solution for $h_c^{-1}(t)$ is known to exist.

It is possible to approximate $h_c^{-1}(t)$. Consider the following idea. Replace the denominator $\sin\left(\frac{s}{j2}\right)$ with a truncated Taylor series expansion. The result is a rational approximation to $H_c^{-1}(s)$ that can be inverted using partial fraction expansion techniques. Although not perfect, the result can perform reasonably for many low-frequency inputs.

Solution 4.10-14

No, the suggested lowpass to highpass transformation $H_{\mathrm{HP}}(s) = 1 - H_{\mathrm{LP}}(s)$ does not work in general. Although it is possible to relate the ideal magnitude responses according to $|H_{\mathrm{HP}}(j\omega)| = 1 - |H_{\mathrm{LP}}(j\omega)|$, the phase information contained in $H(s)$ generally makes $H_{\mathrm{HP}}(s) \neq 1 - H_{\mathrm{LP}}(s)$.

As an example, consider an ideal lowpass filter described by

$$H_{\mathrm{LP}}(j\omega) = \begin{cases} -1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

The transformation $1 - H_{\mathrm{LP}}(s)$ is clearly not highpass.

$$1 - H_{\mathrm{LP}}(s) = \begin{cases} 2 & |\omega| \leq \omega_c \\ 1 & |\omega| > \omega_c \end{cases}.$$

Solution 4.10-15

(a) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)u(t)$ in response to $x(t) = \cos(100\pi t)u(t)$. Noting $Y(s) = \frac{100\pi}{s^2+(100\pi)^2}$ and $X(s) = \frac{s}{s^2+(100\pi)^2}$, one way to obtain $y(t)$ from $x(t)$ is using the system $H(s) = Y(s)/X(s) = \frac{100\pi}{s^2+(100\pi)^2}\frac{s^2+(100\pi)^2}{s} = \frac{100\pi}{s}$.

(b) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)u(t)$ in response to $x(t) = \sin(50\pi t)u(t)$. Noting $Y(s) = \frac{100\pi}{s^2+(100\pi)^2}$ and $X(s) = \frac{s}{s^2+(50\pi)^2}$, one way to obtain $y(t)$ from $x(t)$ is using the system $H(s) = Y(s)/X(s) = \frac{100\pi}{s^2+(100\pi)^2}\frac{s^2+(50\pi)^2}{s} = \frac{100\pi\left(s^2+(50\pi)^2\right)}{s(s^2+(100\pi)^2)}$.

(c) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)$ in response to $x(t) = \cos(100\pi t)$. To do this, the system must have $H(j100\pi) = e^{-j\pi/2}$. That is, the magnitude response at $\omega = 100\pi$ must be unity, and the phase response at $\omega = 100\pi$ must be $-\pi/2$.

(d) No, it is not possible for the system to output $y(t) = \sin(100\pi t)$ in response to $x(t) = \sin(50\pi t)$. In an LTI system, an everlasting sinusoidal input of frequency 50π cannot produce a different frequency output.

Solution 4.11-1

(a) Let $x_1(t) = x(t)u(t) = e^t u(t)$ and $x_2(t) = x(t)u(-t) = u(-t)$. Then $X_1(s)$ has a region of convergence $\mathrm{Re}(s) > 1$. And $X_2(s)$ has a region $\mathrm{Re}(s) < 0$. Hence, there is no common region of convergence for $X(s) = X_1(s) + X_2(s)$.

(b) $x_1(t) = e^{-t}u(t)$, and $X_1(s) = \frac{1}{s+1}$ converges for $\mathrm{Re}(s) > -1$. Also, $x_2(t) = u(-t)$, and $X_2(s) = -\frac{1}{s}$ converges for $\mathrm{Re}(s) < 0$. Therefore, the region of convergence is the strip

$$-1 < \mathrm{Re}(s) < 0$$

(c)

$$\frac{1}{t^2+1}e^{-st} \to 0 \; \} \quad \begin{matrix} \text{as } t \to \infty \text{ if } \mathrm{Re}(s) \geq 0 \\ \text{as } t \to -\infty \text{ if } \mathrm{Re}(s) \leq 0 \end{matrix} \; .$$

Thus, convergence occurs at $\mathrm{Re}(s) = 0$ (ω-axis)

(d)

$$x(t) = \frac{1}{1+e^t}$$

$$\frac{1}{1+e^t}e^{-st} \to 0 \; \} \quad \begin{matrix} \text{as } t \to \infty \text{ if } \mathrm{Re}(s) > -1 \\ \text{as } t \to -\infty \text{ if } \mathrm{Re}(s) < 0 \end{matrix} \; .$$

Hence the region of convergence is $-1 < \mathrm{Re}(s) < 0$.

(e)

$$x(t) = e^{-kt^2}$$

$$e^{-kt^2}e^{-st} \to 0 \; \} \quad \begin{matrix} \text{as } t \to \infty \text{ for any value of } s \\ \text{as } t \to -\infty \text{ for any value of } s \end{matrix}$$

Hence the region of convergence is the entire s-plane.

Solution 4.11-2

(a)

$$X_a(s) = \int_{-\infty}^{\infty} e^{(-1-j)t} u(1-t) e^{-st}\, dt = \int_{-\infty}^{1} e^{t(-s-1-j)}\, dt$$

$$= \left. \frac{e^{t(-s-1-j)}}{-s-1-j} \right|_{-\infty}^{1}$$

For $-\mathrm{Re}(s) - 1 > 0$, $X_a(s)$ converges as

$$X_a(s) = \frac{e^{-s-1-j}}{-s-1-j}, \qquad \text{ROC: } \mathrm{Re}(s) < -1.$$

(b)

$$X_b(s) = \int_{-\infty}^{\infty} e^{j(t+1)\pi/2} u(-t-1) e^{-st}\, dt = e^{j\pi/2} \int_{-\infty}^{-1} e^{t(-s+j\pi/2)}\, dt$$

$$= e^{j\pi/2} \left. \frac{e^{t(-s+j\pi/2)}}{-s+j\frac{\pi}{2}} \right|_{-\infty}^{-1}$$

For $\mathrm{Re}(s) < 0$, $X_b(s)$ converges as

$$X_b(s) = \frac{-e^s}{s - j\frac{\pi}{2}}, \qquad \text{ROC: } \mathrm{Re}(s) < 0.$$

(c)

$$X_c(s) = \int_{-\infty}^{\infty} e^{j\pi/3} u(2-t) e^{-st}\, dt + \int_{-\infty}^{\infty} j\delta(t-5) e^{-st}\, dt = e^{j\pi/3} \int_{-\infty}^{2} e^{-st}\, dt + je^{-5s}$$

$$= e^{j\pi/3} \left. \frac{e^{-st}}{-s} \right|_{-\infty}^{2} + je^{-5s}$$

For $\mathrm{Re}(s) < 0$, $X_c(s)$ converges as

$$X_c(s) = -e^{j\pi/2} e^{-2s} \frac{1}{s} + je^{-5s}, \qquad \text{ROC: } \mathrm{Re}(s) < 0.$$

(d) Here we note that $x_d(t) = 1 + 1 = 2 = 2u(t) + 2u(-t)$. The bilateral Laplace transform of $2u(t)$ is $\frac{2}{s}$ with ROC $\mathrm{Re}(s) > 0$. The bilaterial Lapalce transform of $2u(-t)$ is $-\frac{2}{s}$ with ROC $\mathrm{Re}(s) < 0$. Since the two ROCs do not overlap,

the bilateral Laplace transform of $x_d(t) = 2$ does not exist.

(e)

$$X_e(s) = \int_{-\infty}^{\infty} 3u(-t) e^{-st}\, dt + \int_{-\infty}^{\infty} e^{-2t} [u(t) - u(t-10)] e^{-st}\, dt$$

$$= \int_{-\infty}^{0} 3e^{-st}\, dt + \int_{0}^{10} e^{-t(s+2)}\, dt$$

$$= \left. \frac{3e^{-st}}{-s} \right|_{-\infty}^{0} + \left. \frac{e^{-t(s+2)}}{-(s+2)} \right|_{0}^{10}$$

For $\mathrm{Re}(s) < 0$, $X_{\mathrm{e}}(s)$ converges as

$$X_{\mathrm{e}}(s) = \frac{1 - e^{-20-10s}}{s+2} - \frac{3}{s}, \qquad \text{ROC: } \mathrm{Re}(s) < 0.$$

(f) To begin, let us express $x_{\mathrm{f}}(t)$ as

$$x_{\mathrm{f}}(t) = \underbrace{e^{t-2}u(1-t)}_{x_1(t)} + \underbrace{e^{-2t}u(t+1)}_{x_2(t)}.$$

Due to linearity, the Laplace transform of $x_{\mathrm{f}}(t)$ is just the sum of the Laplace transform of $x_1(t)$ and $x_2(t)$ (assuming there is a common ROC). Now, for $x_1(t)$, we see that

$$X_1(s) = \int_{-\infty}^{1} e^{t-2} e^{-st}\, dt = e^{-2} \int_{-\infty}^{1} e^{t(1-s)}\, dt = e^{-2} \left.\frac{e^{t(1-s)}}{1-s}\right|_{-\infty}^{1} = e^{-2}\frac{e^{1-s}}{1-s}, \qquad \text{ROC: } \mathrm{Re}(s) < 1.$$

For $x_2(t)$, we see that

$$X_2(s) = \int_{-1}^{\infty} e^{-2t} e^{-st}\, dt = \int_{-1}^{\infty} e^{-t(s+2)}\, dt = \left.\frac{e^{-t(s+2)}}{-(s+2)}\right|_{-1}^{\infty} = -\frac{e^{s+2}}{-(s+2)}, \qquad \text{ROC: } \mathrm{Re}(s) > -2.$$

Combining these results, we see that

$$X_{\mathrm{f}}(s) = X_1(s) + X_2(s) = \frac{e^{-s-1}}{-s+1} + \frac{e^{s+2}}{s+2}, \qquad \text{ROC: } -2 < \mathrm{Re}(s) < 1.$$

Solution 4.11-3

From Table 4.1, we know that

$$e^{-t}u(t) \Rightarrow \frac{1}{s+1}, \qquad \text{ROC: } \mathrm{Re}(s) > -1$$

$$\cos(2t)u(t) \Rightarrow \frac{s}{s^2+4}, \qquad \text{ROC: } \mathrm{Re}(s) > 0$$

Using the time-reversal and frequency differentiation properties of the bilateral Laplace transform, we see that

$$e^{t}u(-t) \Rightarrow \frac{-1}{s-1}, \qquad \text{ROC: } \mathrm{Re}(s) < 1$$

$$t\cos(2t)u(t) \Rightarrow -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\left(\frac{1}{s^2+4} - \frac{s(2s)}{(s^2+4)^2}\right) = -\frac{s^2+4-2s^2}{(s^2+4)^2}, \qquad \text{ROC: } \mathrm{Re}(s) > 0$$

Using the time-convolution property, we see that

$$X(s) = \left(-\frac{1}{s-1}\right)\left(-\frac{-s^2+4}{s^4+8s^2+16}\right), \text{ROC: } 0 < \mathrm{Re}(s) < 1.$$

Simplifying, we obtain

$$X(s) = -\frac{-s^2+4}{s^5 - s^4 + 8s^3 - 8s^2 + 16s - 16}, \text{ROC: } 0 < \mathrm{Re}(s) < 1.$$

Solution 4.11-4

The roots of $X(s)$ are 2 and 3, corresponding to modes e^{2t} and e^{3t}. There are three possible ROC options for $X(s)$: $\mathrm{Re}(s) < 2$, $2 < \mathrm{Re}(s) < 3$, and $\mathrm{Re}(s) > 3$. Only ROC $\mathrm{Re}(s) < 2$ ensures that both modes are decaying and bounded. Thus,

the ROC $\mathrm{Re}(s) < 2$ results in the smallest maximum amplitude of $x(t)$.

Solution 4.11-5

(a)

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^t u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{1}{s+1} \qquad \text{Re}(s) > -1$$

$$x_2(-t) = e^{-t}u(t) \quad \text{and} \quad X_2(-s) = \frac{1}{s+1}$$

$$\text{and} \qquad X_2(s) = \frac{1}{-s+1} \qquad \text{Re}(s) < 1$$

$$\text{Hence,} \quad X(s) = X_1(s) + X_2(s) = \frac{1}{s+1} + \frac{1}{-s+1} = \frac{-2}{s^2-1} \qquad -1 < \text{Re}(s) < 1$$

(b)

$$x(t) = e^{-|t|}\cos t = e^{-t}\cos t\, u(t) + e^t \cos t\, u(-t) = x_1(t) + x_2(t)$$

$$\text{Hence,} \quad X_1(s) = \frac{s+1}{(s+1)^2+1} \quad \text{and} \quad X_2(-s) = \frac{s+1}{(s+1)^2+1} \quad \text{Re}(s) < 1$$

$$X(s) = X_1(s) + X_2(s) = \frac{s+1}{(s+1)^2+1} - \frac{s-1}{(s-1)^2+1} = \frac{4-2s^2}{s^4-4} \qquad -1 < \text{Re}(s) < 1$$

(c)

$$x(t) = e^t u(t) + e^{2t}u(-t); \quad X_1(s) = \frac{1}{s-1} \quad \text{Re}(s) > 1 \quad \text{and} \quad X_2(-s) = \frac{1}{s+2}$$

$$X_2(s) = \frac{1}{-s+2} \qquad \text{Re}(s) < 2.$$

$$\text{Hence,} \qquad X(s) = X_1(s) + X_2(s) = \frac{-1}{(s-1)(s-2)} \qquad 1 < \text{Re}(s) < 2$$

(d)

$$x(t) = e^{-tu(t)} = \begin{cases} e^{-t} & \text{for} \quad t > 0 \\ 1 & \text{for} \quad t < 0 \end{cases}$$

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = u(-t). \quad \text{Hence,} \quad X_1(s) = \frac{1}{s+1} \quad \text{Re}(s) > -1$$

$$\text{and} \quad X_2(-s) = \frac{1}{s}, \quad X_2(s) = \frac{-1}{s} \quad \text{Re}(s) < 0$$

$$\text{and hence:} \qquad X(s) = \frac{1}{s+1} - \frac{1}{s} = \frac{-1}{s(s+1)} \qquad -1 < \text{Re}(s) < 0$$

(e)

$$x(t) = e^{tu(-t)} = \begin{cases} x_1(t) = 1 & \text{for} \quad t > 0 \\ x_2(t) = e^t & \text{for} \quad t < 0 \end{cases}$$

$$X_1(s) = \frac{1}{s} \qquad \text{Re}(s) > 0$$

$$X_2(-s) = \frac{1}{s+1} \quad X_2(s) = \frac{1}{-s+1} \quad \text{Re}(s) < 1$$

$$\text{and hence:} \qquad X(s) = \frac{1}{s} - \frac{1}{s-1} = \frac{-1}{s(s-1)} \qquad 0 < \text{Re}(s) < 1$$

(f)

$$x(t) = \cos\omega_0 t\, u(t) + e^t u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{s}{s^2 + \omega_0^2} \qquad \mathrm{Re}(s) > 0$$

and $\quad X_2(-s) = \dfrac{1}{s+1}, \qquad X_2(s) = \dfrac{1}{1-s} \qquad \mathrm{Re}(s) < 1$

$$X(s) = X_1(s) + X_2(s) = \frac{-(s+\omega_0^2)}{(s-1)(s^2+\omega_0^2)} \qquad 0 < \mathrm{Re}(s) < 1$$

Solution 4.11-6

(a)

$$X(s) = \frac{2s+5}{(s+2)(s+3)} \qquad -3 < \mathrm{Re}(s) < -2$$

$$= \frac{1}{s+2} + \frac{1}{s+3} \qquad -3 < \mathrm{Re}(s) < -2$$

The pole -2 lies to the right, and the pole -3 lies to the left of the region of convergence; hence the first term represents causal and the second term represents anticausal signal:

$$x(t) = e^{-3t}u(t) - e^{-2t}u(-t)$$

(b)

$$X(s) = \frac{2s-5}{(s-2)(s-3)} \qquad 2 < \mathrm{Re}(s) < 3$$

$$= \frac{1}{s-2} + \frac{1}{s-3} \qquad 2 < \mathrm{Re}(s) < 3$$

The pole at -2 lies to the left and that at 3 lies to the right of the region of convergence; hence

$$x(t) = e^{2t}u(t) - e^{3t}u(-t)$$

(c)

$$X(s) = \frac{2s+3}{(s+1)(s+2)} \qquad \mathrm{Re}(s) > -1$$

$$= \frac{1}{s+1} + \frac{1}{s+2} \qquad \mathrm{Re}(s) > -1$$

Both poles lie to the left of the region of convergence, and

$$x(t) = (e^{-t} + e^{-2t})u(t)$$

(d)

$$X(s) = \frac{2s+3}{(s+1)(s+2)} \qquad \mathrm{Re}(s) < -2$$

$$= \frac{1}{s+1} + \frac{1}{s+2} \qquad \mathrm{Re}(s) < -2$$

Both poles lie to the right of the region of convergence. Hence,

$$x(t) = -(e^{-t} + e^{-2t})u(-t)$$

(e)

$$X(s) = \frac{3s^2 - 2s - 17}{(s+1)(s+3)(s-5)} \qquad -1 < \mathrm{Re}(s) < 5$$

$$= \frac{1}{s+1} + \frac{1}{s+3} + \frac{1}{s-5}$$

The poles -1 and -3 lie to the left of the region of convergence, whereas the pole 5 lies to the right:

$$x(t) = (e^{-t} + e^{-3t})u(t) - e^{5t}u(-t)$$

Solution 4.11-7

$$\frac{2s^2 - 2s - 6}{(s+1)(s-1)(s+2)} = \frac{1}{s+1} - \frac{1}{s-1} + \frac{2}{s+2}$$

(a) $\mathrm{Re}(s) > 1$: All poles to the left of the region of convergence. Therefore

$$x(t) = (e^{-t} - e^t + 2e^{-2t})u(t).$$

(b) $\mathrm{Re}(s) < -2$: All poles to the right of the region of convergence. Therefore

$$x(t) = (-e^{-t} + e^t - 2e^{-2t})u(-t).$$

(c) $-1 < \mathrm{Re}(s) < 1$: Poles -1 and -2 to the left and pole 1 to the right of the region of convergence. Therefore

$$x(t) = (e^{-t} + 2e^{-2t})u(t) + e^t u(-t).$$

(d) $-2 < \mathrm{Re}(s) < -1$: Poles -1 and 1 are to the right and pole -2 is to the left of the region of convergence. Therefore

$$x(t) = 2e^{-2t}u(t) + [-e^{-t} + e^t]u(-t).$$

Solution 4.11-8

(a)

$$x(t) = e^{-\frac{|t|}{2}}, \qquad H(s) = \frac{1}{s+1} \qquad \mathrm{Re}(s) > -1$$

$$\text{and} \qquad X(s) = \frac{1}{s+0.5} - \frac{1}{s-0.5} \qquad -\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2}$$

$$\text{Hence,} \qquad Y(s) = H(s)X(s) = \frac{1}{s+1}\left[\frac{1}{s+0.5} - \frac{1}{s-0.5}\right] \qquad -\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2}$$

$$Y(s) = \frac{-2}{s+1} + \frac{2}{s+0.5} + \frac{\frac{2}{3}}{s+1} - \frac{\frac{2}{3}}{s-0.5}$$

$$= \frac{-\frac{4}{3}}{s+1} + \frac{2}{s+0.5} - \frac{\frac{2}{3}}{s-0.5} \qquad -\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2}$$

The poles -1 and -0.5, which are to the left of the strip of convergence, yield the causal signal, and the pole 0.5, which is to the right of the strip of convergence, yields the anticausal signal. Hence,

$$y(t) = \left(-\frac{4}{3}e^{-t} + 2e^{-t/2}\right)u(t) + \frac{2}{3}e^{t/2}u(-t)$$

(b)

$$x(t) = e^t u(t) + e^{2t} u(-t)$$

$$X(s) = \frac{1}{s-1} - \frac{1}{s-2} \qquad 1 < \text{Re}(s) < 2$$

$$= \frac{-1}{(s-1)(s-2)}$$

$$\text{and} \qquad H(s) = \frac{1}{s+1} \qquad \text{Re}(s) > -1$$

Hence, $\qquad Y(s) = H(s)X(s) = \dfrac{-1}{(s+1)(s-1)(s-2)} \qquad 1 < \text{Re}(s) < 2$

$$Y(s) = \frac{-1/6}{s+1} + \frac{1/2}{s-1} - \frac{1/3}{s-2} \qquad 1 < \text{Re}(s) < 2$$

Hence, $\qquad y(t) = \left(-\dfrac{1}{6} e^{-t} + \dfrac{1}{2} e^t \right) u(t) + \dfrac{1}{3} e^{2t} u(-t)$

(c)

$$x(t) = e^{-t/2} u(t) + e^{-t/4} u(-t)$$

$$X(s) = \frac{1}{s+0.5} - \frac{1}{s+0.25} = \frac{-\frac{1}{4}}{(s+0.5)(s+0.25)} \qquad -\frac{1}{2} < \text{Re}(s) < \frac{1}{4}$$

$$\text{Also} \qquad H(s) = \frac{1}{s+1} \qquad \text{Re}(s) > -1$$

Hence, $\qquad Y(s) = H(s)X(s) = \dfrac{-\frac{1}{4}}{(s+1)(s+0.5)(s+0.25)} \qquad -\dfrac{1}{2} < \text{Re}(s) < \dfrac{1}{4}$

$$= \frac{-\frac{2}{3}}{s+1} + \frac{2}{s+0.5} - \frac{\frac{4}{3}}{s+0.25} \qquad -\frac{1}{2} < \text{Re}(s) < \frac{1}{4}$$

$$\text{and} \qquad y(t) = \left(-\frac{2}{3} e^{-t} + 2e^{-\frac{t}{2}} \right) u(t) + \frac{4}{3} e^{-\frac{t}{4}} u(-t)$$

(d)

$$x(t) = e^{2t} u(t) + e^t u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{1}{s-2} \qquad \text{Re}(s) > 2$$

$$X_2(s) = \frac{-1}{s-1} \qquad \text{Re}(s) < 1$$

$$\text{and} \qquad H(s) = \frac{1}{s+1} \qquad \text{Re}(s) > -1$$

In this case, there is no region of convergence that is common to $X_1(s)$ and $X_2(s)$. However, each of $X_1(s)$ and $X_2(s)$ have a region of convergence that is common to $H(s)$. Hence the output can be computed by finding the system response to $x_1(t)$ and $x_2(t)$ separately, and then adding these two components. This means we need not worry about the common region of convergence for $X_1(s)$ and $X_2(s)$. Thus,

$$Y(s) = Y_1(s) + Y_2(s) \qquad \text{where}$$

$$Y_1(s) = X_1(s)H(s) = \frac{1}{(s+1)(s-2)} \qquad \mathrm{Re}(s) > 2$$

$$= \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} \qquad \mathrm{Re}(s) > 2$$

Observe that both the poles (-1 and 2) are to the left of the region of convergence, hence both terms are causal, and

$$y_1(t) = \left(-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$$

$$Y_2(s) = X_2(s)H(s) = \frac{-1}{(s+1)(s-1)} \qquad -1 < \mathrm{Re}(s) < 1$$

$$= \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s-1} \qquad -1 < \mathrm{Re}(s) < 1$$

The poles -1 and 1 are to the left and the right, respectively, of the strip of convergence. Hence the first term yields causal signal and the second yields anticausal signal. Hence,

$$y_2(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^t u(-t)$$

$$\text{Therefore,} \quad y(t) = y_1(t) + y_2(t) = \left(\frac{1}{6}e^{-t} + \frac{1}{3}e^{2t}\right)u(t) + \frac{1}{2}e^t u(-t)$$

(e)

$$x(t) = e^{-\frac{t}{4}}u(t) + e^{-\frac{t}{2}}u(-t) = x_1(t) + x_2(t)$$

$$X(s) = X_1(s) + X_2(s)$$

$$\text{where} \quad X_1(s) = \frac{1}{s+0.25} \qquad \mathrm{Re}(s) > -\frac{1}{4}$$

$$X_2(s) = \frac{-1}{s+0.5} \qquad \mathrm{Re}(s) < -\frac{1}{2}$$

$$H(s) = \frac{1}{s+1} \qquad \mathrm{Re}(s) > -1$$

Here also, we have no common region of convergence, for $X_1(s)$ and $X_2(s)$ as in part d. Let $Y(s) = Y_1(s) + Y_2(s)$ where

$$Y_1(s) = \frac{1}{(s+1)(s+0.25)} \qquad \mathrm{Re}(s) > -\frac{1}{4}$$

$$= \frac{-\frac{4}{3}}{s+1} + \frac{\frac{4}{3}}{s+0.25} \qquad \mathrm{Re}(s) > -\frac{1}{4}$$

$$y_1(t) = \left(-\frac{4}{3}e^{-t} + \frac{4}{3}e^{-\frac{t}{4}}\right)u(t)$$

$$Y_2(s) = \frac{-1}{(s+1)(s+0.5)} \qquad -1 < \mathrm{Re}(s) < -\frac{1}{2}$$

$$= \frac{2}{s+1} - \frac{2}{s+0.5} \qquad -1 < \mathrm{Re}(s) < -\frac{1}{2}$$

$$\text{and} \quad y_2(t) = 2e^{-t}u(t) + 2e^{-\frac{t}{2}}u(-t)$$

$$\text{Hence,} \quad y(t) = y_1(t) + y_2(t) = \left(\frac{2}{3}e^{-t} + \frac{4}{3}e^{-\frac{t}{4}}\right)u(t) + 2e^{-\frac{t}{2}}u(-t)$$

(f)

$$x(t) = e^{-3t}u(t) + e^{-2t}u(-t) = x_1(t) + x_2(t)$$
$$X(s) = X_1(s) + X_2(s)$$

where $\qquad X_1(s) = \dfrac{1}{s+3} \qquad \operatorname{Re}(s) > -3$

$$X_2(s) = \dfrac{-1}{s+2} \qquad \operatorname{Re}(s) < -2$$

$$H(s) = \dfrac{1}{s+1} \qquad \operatorname{Re}(s) > -1$$

In this case, there is a common region of convergence for $X_1(s)$ and $H(s)$, but there is no region of convergence common to $X_2(s)$ and $H(s)$. Hence the output $y_1(t)$ will be finite but $y_2(t)$ will be ∞.

Solution 4.11-9

$$
\begin{aligned}
\mathcal{L}\left(r_{xx}(t)\right) &= \int_{-\infty}^{\infty} r_{xx}(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)x(\tau + t)d\tau \right) e^{-st}dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} x(\tau + t)e^{-st}dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{s\tau}X(s)d\tau \\
&= X(s)\int_{-\infty}^{\infty} x(\tau)e^{-\tau(-s)}d\tau \\
R_{xx}(s) &= X(s)X(-s)
\end{aligned}
$$

Solution 4.11-10

For $\operatorname{Re}(s) < 0$, we know that $\mathcal{L}^{-1}\left[\frac{2}{s}\right] = -2u(-t)$. Additionally, $\mathcal{L}^{-1}\left[1/2\right] = \delta/2$. Using properties, $\mathcal{L}^{-1}\left[s(1/2)\right] = \frac{d}{dt}\left(\delta(t)/2\right)$. Thus,

$$x(t) = -2u(-t) + \frac{d}{dt}\left(\delta(t)/2\right).$$

The function $\frac{d}{dt}\left(\delta(t)/2\right)$ is called the "unit doublet". Like $\delta(t)$, the unit doublet is not a physically realizable signal. It is a mathematical construction that is useful, among other things, in finding function derivatives. Refer to the topic of generalized derivatives.

Solution 4.11-11

(a) Yes, $x(t)$ can be left-sided. To be left-sided and absolutely integrable, the signal's region of convergence must: 1) be left-sided, 2) include the ω-axis, and 3) not include any poles. With a pole at $s = \pi$, it is possible to achieve all three necessary conditions. For example, $x(t) = e^{\pi t}u(-t)$ has a pole at $s = \pi$, is absolutely integrable, and is left-sided.

(b) No, $x(t)$ cannot be right-sided. To be right-sided and absolutely integrable, the signal's region of convergence must: 1) be right-sided, 2) include the ω-axis, and 3) not include any poles. The ω-axis cannot be included in a region of convergence that is to the right of the known pole at $s = \pi$.

(c) Yes, $x(t)$ can be two-sided. To be two-sided and absolutely integrable, the signal must: 1) have at least one pole in the right-half plane, 2) have at least one pole in the left-half plane, and 3) have a region of convergence that includes the ω-axis. With a pole at $s = \pi$, these conditions are possible. For example, $x(t) = e^{\pi t}u(-t) + e^{-\pi t}u(t)$ has a pole at $s = \pi$ (and another at $s = -\pi$), is absolutely integrable, and is two-sided.

(d) No, $x(t)$ cannot be finite duration. To be finite duration, the signal's region of convergence must include all finite values of s. However, since a pole is present at $s = \pi$, this point cannot be included in the region of convergence. Thought of another way, a pole at $s = \pi$ implies a signal component of either $e^{\pi t}u(t)$ or $e^{\pi t}u(-t)$, both of which are infinite in duration.

Solution 4.11-12

Using properties, we establish that

$$\frac{1}{s} \iff -u(-t) \qquad \text{(starting fact, } \mathrm{Re}(s) < 0)$$

$$e^{-2s}\frac{1}{s} \iff -u(-(t-2)) = -u(-t+2) \qquad \text{(time shift property)}$$

$$\frac{d}{ds}\left(e^{-2s}\frac{1}{s}\right) \iff -t[-u(-t+2)] = tu(-t+2) \qquad \text{(frequency differentiation property)}$$

$$s\left[\frac{d}{ds}\left(e^{-2s}\frac{1}{s}\right)\right] \iff \frac{d}{dt}\{tu(-t+2)\} \qquad \text{(time differentiation property)}$$

Since $\frac{d}{dt}\{tu(-t+2)\} = u(-t+2) + t[\delta(-t+2)](-1)$, we conclude that

$$x(t) = u(-t+2) - 2\delta(-t+2).$$

Solution 4.11-13

To begin, we note that

$$X(s) = \frac{2}{e^s} + \frac{1}{s}\left[e^s\frac{4}{\frac{s}{3}+2}\right] = 2e^{-s} + e^s\left[\frac{12}{s(s+6)}\right] = 2e^{-s} + e^s\left[\frac{2}{s} + \frac{-2}{s+6}\right].$$

Using the time shift property and the fact that $\delta(t) \iff 1$, we see that

$$2e^{-s} \iff 2\delta(t-1).$$

Since $\mathrm{Re}(s) < 0$, we see that

$$\frac{2}{s} \iff -2u(-t) \quad \text{and} \quad e^s\frac{2}{s} \iff -2u(-(t+1)).$$

Since $\mathrm{Re}(s) > -6$, we see that

$$\frac{-2}{s+6} \iff -2e^{-6t}u(t) \quad \text{and} \quad e^s\frac{-2}{s+6} \iff -2e^{-6(t+1)}u(t+1).$$

Combining, we obtain

$$x(t) = 2\delta(t-1) - 2u(-t-1) - 2e^{-6t-6}u(t+1).$$

Solution 4.11-14

In this case, we see that

$$X(s) = \frac{d^7}{ds^7}\left[\frac{e^{-4s}}{(s+2)(s+3)}\right] = \frac{d^7}{ds^7}\left[e^{-4s}\left(\frac{1}{s+2} + \frac{-1}{s+3}\right)\right].$$

Since $\mathrm{Re}(s) < -2$, the $\frac{1}{s+2}$ term inverts to a left-sided signal. Since $\mathrm{Re}(s) > -3$, the $\frac{-1}{s+3}$ term inverts to a right-sided signal. Combined with the time-shift and s-domain differentiation properties, we see that

$$x(t) = -t^7\left[-e^{-2(t-4)}u(-(t-4)) - e^{-3(t-4)}u(t-4)\right]$$

or

$$x(t) = t^7\left[e^{8-2t}u(4-t) + e^{12-3t}u(t-4)\right].$$

Solution 4.11-15

(a)

$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{\infty} (j + e^{jt})u(t)e^{-st}dt$$

$$= \int_0^{\infty} je^{-st} + e^{t(j-s)}dt = \left(\frac{j}{-s}e^{-st} + \frac{e^{t(j-s)}}{j-s} \right)\Big|_{t=0}^{\infty}.$$

For Re(s) > 0, this simplifies to $X_1(s) = \frac{j}{-s}(0 - e^0) + \frac{0-e^0}{j-s}$. Thus,

$$X_1(s) = \frac{j}{s} + \frac{1}{s-j} \text{ for Re}(s) > 0.$$

(b)

$$X_2(s) = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt = \int_{-\infty}^{\infty} j\cosh(t)u(-t)e^{-st}dt$$

$$= \int_{-\infty}^0 j\frac{e^t + e^{-t}}{2}e^{-st}dt = \int_{-\infty}^0 j\frac{e^{t(1-s)} + e^{t(-1-s)}}{2}dt = \left(\frac{je^{t(1-s)}}{2(1-s)} + \frac{je^{t(-1-s)}}{2(-1-s)} \right)\Big|_{t=-\infty}^0.$$

For Re(s) < −1, this simplifies to $X_2(s) = \frac{j(e^0-0)}{2(1-s)} + \frac{j(e^0-0)}{2(-1-s)} = \frac{-0.5j}{s-1} + \frac{-0.5j}{s+1}$. Thus,

$$X_2(s) = \frac{-js}{s^2 - 1} \text{ for Re}(s) < -1.$$

(c)

$$X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(e^{j(\frac{\pi}{4})}u(-t+1) + j\delta(t-5) \right)e^{-st}dt$$

$$= \int_{-\infty}^1 e^{j(\frac{\pi}{4})}e^{-st}dt + \int_{-\infty}^{\infty} j\delta(t-5)e^{-st}dt = e^{j(\frac{\pi}{4})}\frac{e^{-st}}{-s}\Big|_{t=-\infty}^1 + je^{-5s}.$$

For Re(s) < 0, this simplifies to $X_3(s) = e^{j(\frac{\pi}{4})}\frac{e^{-s}-0}{-s} + je^{-5s}$. Thus,

$$X_3(s) = -e^{j(\frac{\pi}{4})}\frac{e^{-s}}{s} + je^{-5s} \text{ for Re}(s) < 0.$$

(d)

$$X_4(s) = \int_{-\infty}^{\infty} x_4(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(j^t u(-t) + \delta(t-\pi) \right)e^{-st}dt$$

$$= \int_{-\infty}^0 e^{tj\pi/2}e^{-st}dt + \int_{-\infty}^{\infty} \delta(t-\pi)e^{-st}dt = \frac{e^{t(j\pi/2-s)}}{j\pi/2 - s}\Big|_{t=-\infty}^0 + je^{-s\pi}.$$

For Re(s) < 0, this simplifies to $X_4(s) = \frac{1-0}{j\pi/2-s} + e^{-s\pi}$. Thus,

$$X_4(s) = e^{-s\pi} - \frac{1}{s - j\pi/2} \text{ for Re}(s) < 0.$$

Solution 4.11-16

(a) To be bounded amplitude, the region of convergence must include the ω-axis. The transfer function has two poles, at $s = \pm 1$, that must be excluded from the region of convergence. Thus, the region of convergence must be

$$-1 < \mathrm{Re}(s) < 1.$$

(b) Rewrite $H(s)$ as $2^s \frac{s}{(s-1)(s+1)} = e^{\ln(s)} \left(\frac{0.5}{s-1} + \frac{0.5}{s+1} \right)$. Using $-1 < \mathrm{Re}(s) < 1$, the time-shifting property, and a table of Laplace transform pairs, the inverse transform is found to be

$$h(t) = 0.5e^{-(t+\ln(2))} u(t + \ln(2)) - 0.5e^{t+\ln(2)} u(-(t + \ln(2))).$$

Solution 4.12-1

Using program CH4MP3:

```
>>  CH4MP3(20)
    ans =   524288       0  -2621440        0   5570560        0
          -6553600       0   4659200        0  -2050048        0
            549120       0    -84480        0      6600        0
              -200       0         1
```

Thus,

$$C_{20}(x) \;=\; \begin{aligned}[t] &524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} + \\ &-2050048x^{10} + 549120x^8 - 84480x^6 + 6600x^4 - 200x^2 + 1 \end{aligned}$$

Solution 4.12-2

(a), (b) MATLAB makes it easy to compute magnitude and phase response plots, as well as highlight behavior at the frequencies of interest $\omega = [-2, 0, 2]$.

```
>>  H = @(s) 1./(s.^3+4*s.^2+8*s+8);
>>  Xm = 10; w = linspace(-Xm,Xm,1000); wpts = [-2 0 2];
>>  subplot(121); plot(w,abs(H(j*w)),'k-',...
>>      wpts,abs(H(j*wpts)),'k.');
>>  xlabel('\omega'); ylabel('|H(j\omega)|'); axis([-Xm Xm 0 1/7]); grid
>>  set(gca,'ytick',unique([0,abs(H(1j*wpts))]),'xtick',[-Xm wpts Xm]);
>>  P = angle(H(j*w))*180/pi;
>>  subplot(122); plot(w,P,'k-',wpts,angle(H(1j*wpts))*180/pi,'k.');
>>  xlabel('\omega'); ylabel('\angle H(j\omega) [deg]');
>>  axis([-Xm Xm -190 190]); grid
>>  set(gca,'ytick',unique([-180,0,180,angle(H(1j*wpts))*180/pi]),...
>>      'xtick',[-Xm wpts Xm]);
```

Figures S4.12-2a and S4.12-2b show the resulting magnitude and phase response plots.

(c) The input $x(t) = 2 - \sin(2t + \pi/3)$ is comprised of frequencies $\omega = [-2, 0, 2]$. Using $|H(j\omega)|$ and $\angle H(j\omega)$ from parts (a) and (b), the output is

$$y(t) = 2\,(0.125) - 0.0884 \sin(2t + \pi/3 - 3\pi/4).$$

The maximum value of this output signal is

$$y_{\mathrm{max}} = 0.25 + 0.0884 = 0.3384.$$
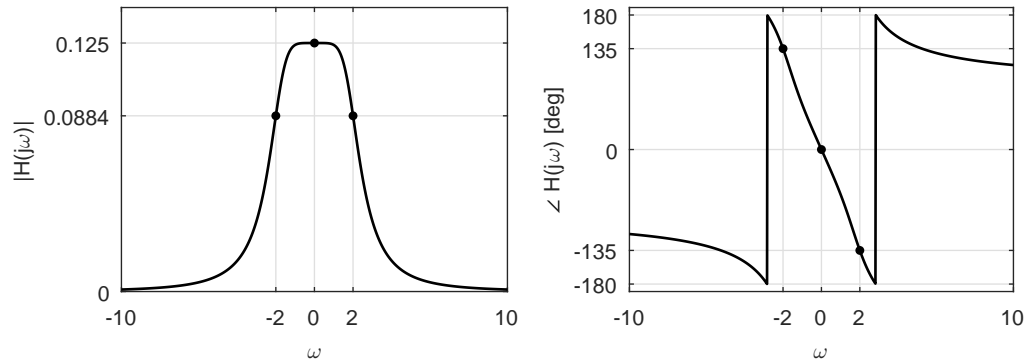
Figures S4.12-2a and S4.12-2b

(d) The transfer function of this system is

$$H(s) = \frac{1}{(s^2 + 2s + 4)(s + 2)} = \frac{0.25}{s + 2} + \frac{-.25s}{s^2 + 2s + 4}.$$

A corresponding parallel representation of the system using real DFI structures is shown in Fig. S4.12-2d.
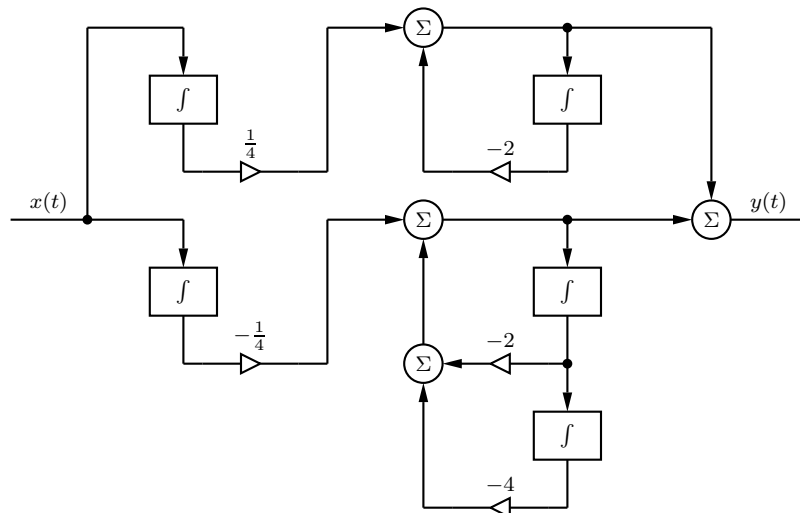
Figure S4.12-2d

Solution 4.12-3

(a) By observation and following signal paths, we see that

$$Y(s) = (-1)\left(-\frac{2}{RC_2 s}\right)(-Y(s)) + (-1)\left(-\frac{10}{RC_2 s}\right)X(s) +$$

$$(-1)\left(-\frac{1}{RC_2 s}\right)\left(-\frac{1}{RC_1 s}\right)X(s) + (-1)\left(-\frac{1}{RC_2 s}\right)\left(-\frac{26}{RC_1 s}\right)Y(s).$$

Rearranging, we obtain

$$\left(1 + \frac{2}{RC_2 s} + \frac{26}{RC_1 RC_2 s^2}\right)Y(s) = \left(\frac{10}{RC_2 s} - \frac{1}{RC_1 RC_2 s^2}\right)X(s).$$

or

$$\left(s^2 + \frac{2}{RC_2}s + \frac{26}{RC_1RC_2}\right)Y(s) = \left(\frac{10}{RC_2}s - \frac{1}{RC_1RC_2}\right)X(s).$$

Thus, the transfer function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{10}{RC_2}s - \frac{1}{RC_1RC_2}}{s^2 + \frac{2}{RC_2}s + \frac{26}{RC_1RC_2}}.$$

For RC perfectly equal to unity, we obtain

$$H(s) = \frac{10s - 1}{s^2 + 2s + 26}.$$

(b) Next, we use MATLAB to accurately plot $|H(j\omega)|$ over $-10 \le \omega \le 10$.

```
>>  H = @(s,RC1,RC2) (10*s/RC2-1/(RC1*RC2))./(s.^2+2/RC2*s+26/(RC1*RC2));
>>  wMx = 10; dw = .005; w = -wMx:dw:wMx; Hm = abs(H(1j*w,1,1));
>>  plot(w,Hm,'k-'); xlabel('\omega'); ylabel('|H(j\omega)|');
>>  set(gca,'ytick',[0 abs(H(10j,1,1)) 5],'xtick',[-wMx:2*wMx/4:wMx]);
>>  grid on; axis([-wMx wMx 0 5.5]);
```

Figures S4.12-3 shows the resulting magnitude response plot.
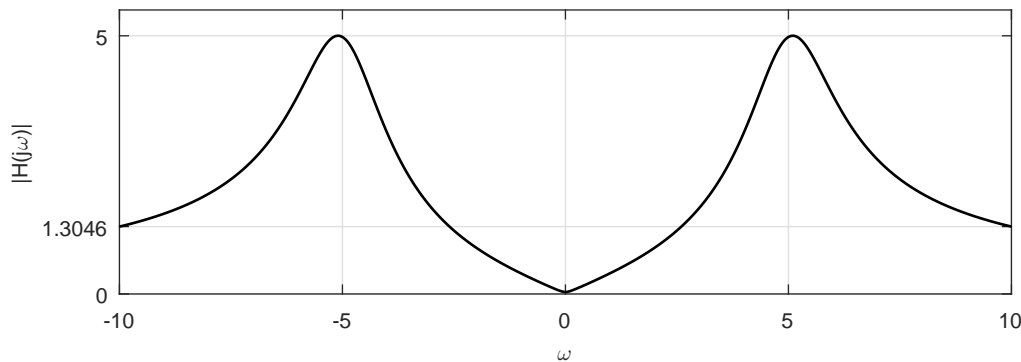
Figure S4.12-3

(c) To determine the response to $x(t) = \cos(10t) - 1$, we compute the appropriate magnitude and phase values.

```
>>  w = [0 10]; Hm = abs(H(1j*w,1,1))
    Hm = 0.0385   1.3046
>>  Ha = angle(H(1j*w,1,1))
    Ha = 3.1416   -1.2968
```

Thus, the output is

$$y(t) = 1.3046\cos(10t - 1.2968) - 0.0385e^{j3.1416} = 1.3046\cos(10t - 1.2968) + 0.0385.$$

(d) We can place the 10% capacitor first (and the 25% capacitor second) or we can place the 25% capacitor first (and the 10% capacitor second). To determine which order is best, we run many ($N = 10^4$) MATLAB simulations for each case and see which, on average, provides lower mean squared error (MSE) in $|H(j\omega)|$ over a suitable frequency range ($0 \le \omega \le 50$) that should comfortably contain the system passband. To simulate a 10% tolerance component, we add a normal random number with a standard deviation of $0.1/2 = 0.05$ to the nominal unity value; this produces a component that is 1 ± 0.1 roughly 95% of the time. A 25% tolerance component is obtained in the same way but using a standard deviation of $0.25/2 = 0.125$.

```
>>  wMx = 50; dw = .005; w = 0:dw:wMx; Hm = abs(H(1j*w,1,1));
>>  Ntrials = 10^4; rng(0); E10First = 0; E25First = 0;
>>  for trial = 1:Ntrials;
>>      RC10 = 1+randn(1)*(.1/2); RC25 = 1+randn(1)*(.25/2);
>>      E10First(trial) = sum((Hm-abs(H(1j*w,RC10,RC25))).^2);
>>      E25First(trial) = sum((Hm-abs(H(1j*w,RC25,RC10))).^2);
>>  end
>>  sum(E10First)/sum(E25First)
    ans = 1.2802
```

From this simulation, we see that putting the 10% capacitor first produces nearly 30% greater MSE than putting the 25% capacitor first. Thus, to preserve the desired $|H(j\omega)|$

it is best to place the 25% capacitor first and the 10% capacitor second.

Solution 4.12-4

(a) MATLAB is used for the design. To evaluate filter performance, the magnitude response is plotted over the frequency range $(0 \le f \le 10\text{kHz})$.

```
>>  N = 12; omega_c = 2*pi*5000;
>>  poles = roots([(j*omega_c)^(-2*N),zeros(1,2*N-1),1]);
>>  B_poles = poles(find(real(poles)<0));
>>  subplot(1,3,1), plot(real(B_poles),imag(B_poles),'xk');
>>  xlabel('Real'); ylabel('Imag');
>>  axis([-4e4 0 -4e4 4e4]); axis equal;
>>  A = poly(B_poles); A = A/A(end); B = 1;
>>  f = linspace(0,10000,1001);
>>  Hmag_B = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>>  subplot(1,3,[2,3]), plot(f,Hmag_B,'k');
>>  xlabel('f [Hz]'); ylabel('|H_{LP}(j2\pi f)|');
```
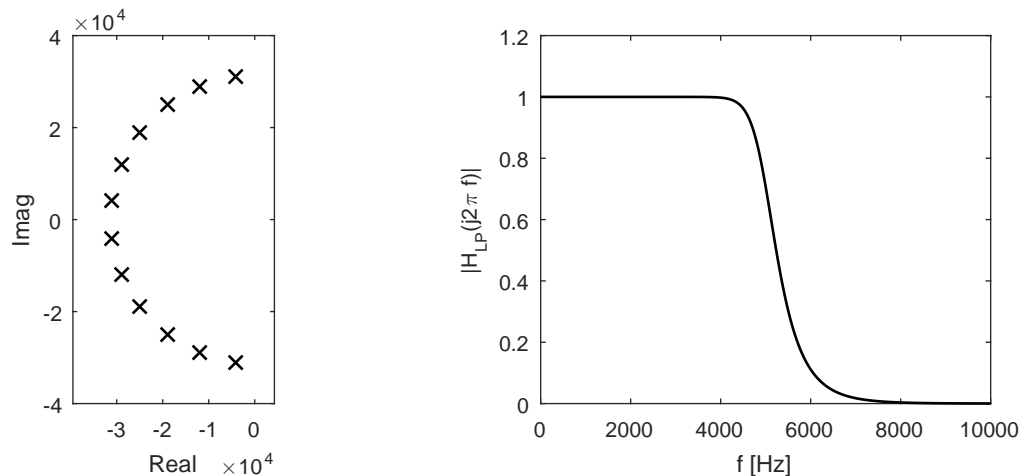

Figure S4.12-4a

The resulting figures are consistent with a Butterworth design; the poles lie on a semicircle in the left-half s-plane, and the magnitude response exhibits smooth monotonic roll-off.

(b) Modifying program `CH4MP2`, the Sallen-Key component values and magnitude response plots are easily found.

```
>>  omega_0 = 5000*2*pi; f = linspace(0,10000,200);
>>  psi = [7.5:15:90]*pi/180; Hmag_SK = zeros(6,200);
>>  for stage = 1:6,
>>     Q = 1/(2*cos(psi(stage)));
>>     disp(['Stage ',num2str(stage),...
>>       ' (Q = ',num2str(Q),...
>>       '):  R1 = R2 = ',num2str(100000)]);
>>     disp(['         C1 = ',num2str(2*Q/(omega_0*100000)),...
>>       ', C2 = ',num2str(1/(2*Q*omega_0*100000))]);
>>     B = omega_0^2; A = [1 omega_0/Q omega_0^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>>  end
>>  plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:')
>>  xlabel('f [Hz]'); ylabel('Magnitude Responses')
    Stage 1 (Q = 0.50431):  R1 = R2 = 100000
            C1 = 3.2106e-10, C2 = 3.1559e-10
    Stage 2 (Q = 0.5412):  R1 = R2 = 100000
            C1 = 3.4454e-10, C2 = 2.9408e-10
    Stage 3 (Q = 0.63024):  R1 = R2 = 100000
            C1 = 4.0122e-10, C2 = 2.5253e-10
    Stage 4 (Q = 0.82134):  R1 = R2 = 100000
            C1 = 5.2288e-10, C2 = 1.9377e-10
    Stage 5 (Q = 1.3066):  R1 = R2 = 100000
            C1 = 8.3178e-10, C2 = 1.2181e-10
    Stage 6 (Q = 3.8306):  R1 = R2 = 100000
            C1 = 2.4387e-09, C2 = 4.1548e-11
```

The resulting resistor and capacitor values are realistic.
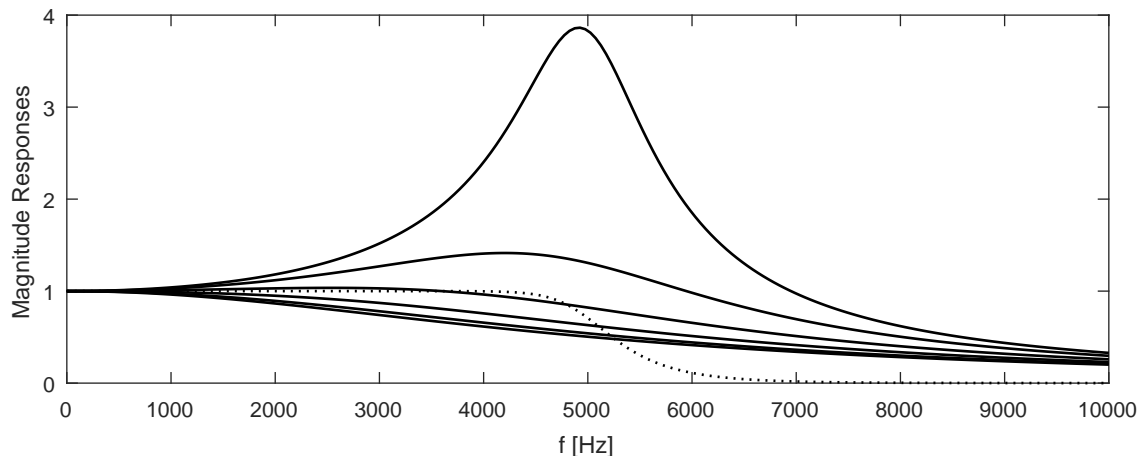
Figure S4.12-4b

Each Sallen-Key stage implements a complex-conjugate pair of poles. The flattest magnitude response corresponds to the pair of poles that are furthest from the ω-axis, or Stage 1. The most peaked magnitude response corresponds to the pair of poles that are closest to the ω-axis, or Stage 6. The remaining stages are ordered in between. The dashed curve is the total magnitude response, and it is exactly the same as the one shown in Fig. S4.12-4a.

Solution 4.12-5

(a) MATLAB is used for the design. To evaluate filter performance, the magnitude response is plotted over the frequency range ($0 \leq f \leq 10\text{kHz}$).

```
>>  omega_c = 2*pi*5000; R = 3; N = 12;
>>  epsilon = sqrt(10^(R/10)-1);
>>  k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>>  C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>>  subplot(121), plot(real(C_poles),imag(C_poles),'xk');
>>  xlabel('Real'); ylabel('Imag');
>>  axis([-4e4 0 -4e4 4e4]); axis equal;
>>  A = poly(C_poles);
>>  B = A(end)/sqrt(1+epsilon^2);
>>  f = linspace(0,10000,2001);
>>  Hmag_C = abs(polyval(B,1j*2*pi*f)./polyval(A,1j*2*pi*f));
>>  subplot(122); plot(omega/2/pi,abs(Hmag_C),'k'); grid
>>  xlabel('f [Hz]'); ylabel('|H_{LP}(j2\pi f)|');
```
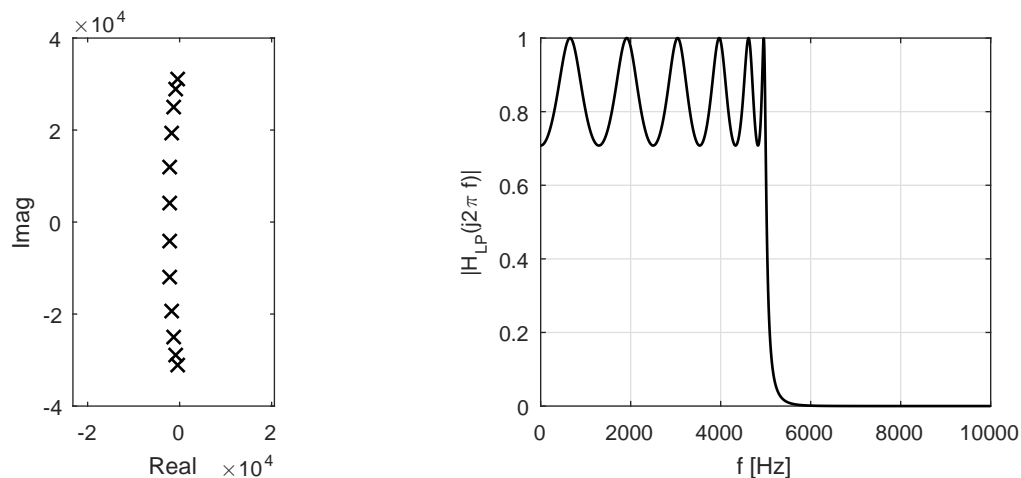

Figure S4.12-5a

The resulting figures are consistent with a Chebyshev design; the poles lie on an ellipse in the left-half s-plane, passband ripples are equal in height and never exceed $R = 3$dB, there are a total of $N = 12$ maxima and minima in the passband, and the gain rapidly and monotonically decreases after the cutoff frequency of $f_c = 5$kHz.

(b) Modifying program `CH4MP2`, the Sallen-Key component values and magnitude response plots are easily found. Due to the extreme peakedness and high gain of some stages, magnitude responses are shown using a dB scale.

```
>>  omega_c = 2*pi*5000; R = 3; N = 12;
>>  epsilon = sqrt(10^(R/10)-1);
>>  k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>>  C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>>  C_poles = C_poles(find(imag(C_poles)>0)); % Quadrant 2 poles
>>  f = linspace(0,10000,501); Hmag_SK = zeros(6,501);
>>  for stage = 1:6,
>>     omega_0 = abs(C_poles(stage));
>>     psi = pi-angle(C_poles(stage));
>>     Q = 1/(2*cos(psi));
>>     disp(['Stage ',num2str(stage),...
>>       ' (Q = ',num2str(Q),...
>>       '):  R1 = R2 = ',num2str(100000)]);
>>     disp(['        C1 = ',num2str(2*Q/(omega_0*100000)),...
>>       ', C2 = ',num2str(1/(2*Q*omega_0*100000))]);
```

```
>>     B = omega_0^2; A = [1 omega_0/Q omega_0^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>>  end
>>  plot(f,20*log10(Hmag_SK),'k',f,20*log10(prod(Hmag_SK)),'k:')
>>  xlabel('f [Hz]'); ylabel('Magnitude Responses [dB]')
>>  axis([0 10000 -40 40]);
    Stage 1 (Q = 51.7057):  R1 = R2 = 100000
            C1 = 3.311e-08, C2 = 3.0961e-12
    Stage 2 (Q = 16.4408):  R1 = R2 = 100000
            C1 = 1.1293e-08, C2 = 1.0445e-11
    Stage 3 (Q = 8.885):  R1 = R2 = 100000
            C1 = 7.0991e-09, C2 = 2.2482e-11
    Stage 4 (Q = 5.247):  R1 = R2 = 100000
            C1 = 5.4474e-09, C2 = 4.9466e-11
    Stage 5 (Q = 2.8635):  R1 = R2 = 100000
            C1 = 4.6778e-09, C2 = 1.4262e-10
    Stage 6 (Q = 1.0262):  R1 = R2 = 100000
            C1 = 4.359e-09, C2 = 1.0348e-09
```

The resulting resistor and capacitor values are realistic.
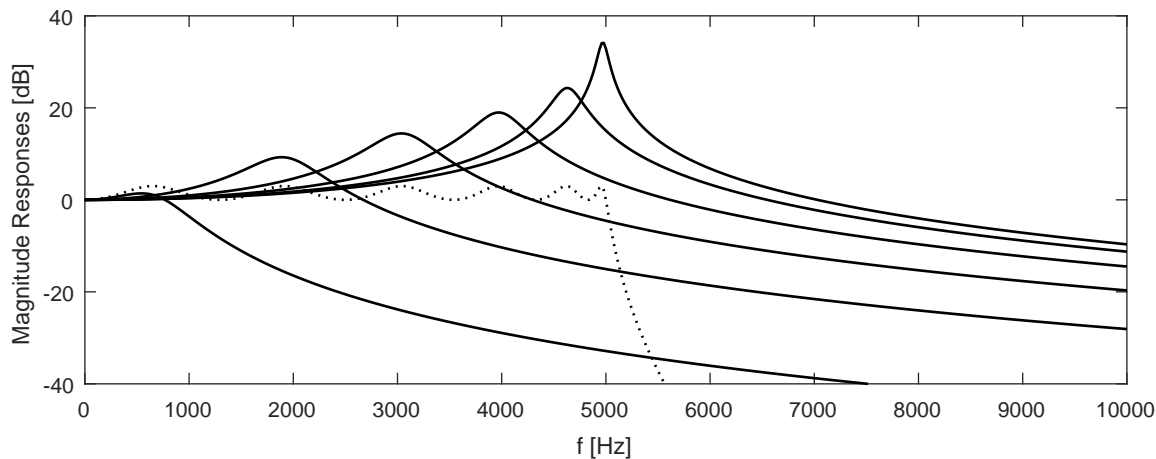
Figure S4.12-5b

Each Sallen-Key stage implements a complex-conjugate pair of poles. The most peaked magnitude response corresponds to the pair of poles that are closest to the ω-axis, or Stage 1. The least peaked magnitude response corresponds to the pair of poles that are furthest from the ω-axis, or Stage 6. The remaining stages are ordered in between. The dashed curve is the total magnitude response, and within a gain error of 3dB is exactly the same as the one shown in Fig. S4.12-5a. The gain error occurs since the Salley-Key stages are constrained to unity gain at dc, yet the Chebyshev filter requires gain $\frac{1}{\sqrt{1+\epsilon^2}}$ at dc. This error is easily corrected by adding a gain stage to the circuit.

Solution 4.12-6

(a) Using MATLAB, the Sallen-Key component values are easily found.

```
>>  omega_0 = 4000*2*pi; NS = 4;
>>  psi = [90/(2*NS):90/NS:90]*pi/180;
>>  Q = 1./(2*cos(psi));
```

```
>>  R1 = 1e9/omega_0*ones(1,NS); R2 = R1;
>>  C1 = 2*Q./(omega_0*R1); C2 = 1./(2*omega_0*Q.*R2);
>>  for stage = 1:NS,
>>    disp(['Stage ',num2str(stage),...
>>      ' (Q = ',num2str(Q(stage)),...
>>      '):  R1 = R2 = ',num2str(R1(stage))]);
>>    disp(['         C1 = ',num2str(C1(stage)),...
>>      ', C2 = ',num2str(C2(stage))]);
>>  end
    Stage 1 (Q = 0.5098):  R1 = R2 = 39788.7358
            C1 = 1.0196e-09, C2 = 9.8079e-10
    Stage 2 (Q = 0.60134):  R1 = R2 = 39788.7358
            C1 = 1.2027e-09, C2 = 8.3147e-10
    Stage 3 (Q = 0.89998):  R1 = R2 = 39788.7358
            C1 = 1.8e-09, C2 = 5.5557e-10
    Stage 4 (Q = 2.5629):  R1 = R2 = 39788.7358
            C1 = 5.1258e-09, C2 = 1.9509e-10
```

The resulting resistor and capacitor values are realistic.

(b) The transformed Sallen-Key circuit is shown in Fig. S4.12-6b. Name the node between capac-
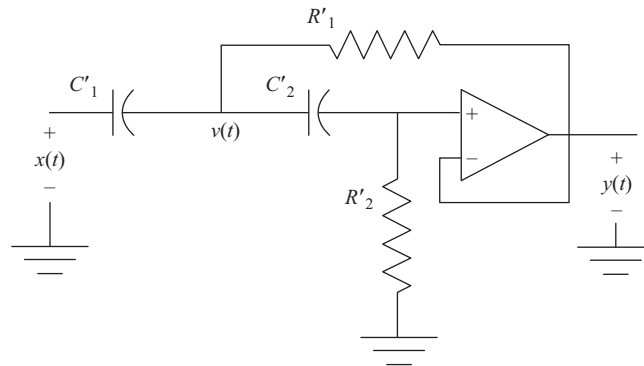
Figure S4.12-6b

itors $v(t)$. In transform domain, KCL at the positive terminal of the op-amp yields

$$\frac{Y(s) - V(s)}{\frac{1}{sC'_2}} = -\frac{Y(s) - 0}{R'_2}.$$

Solving for $V(s)$ yields

$$V(s) = Y(s)\frac{1 + R'_2 C'_2 s}{R'_2 C'_2 s}.$$

KCL at node $V(s)$ yields

$$\frac{X(s) - V(s)}{\frac{1}{C'_1 s}} + \frac{Y(s) - V(s)}{R'_1} + \frac{Y(s) - V(s)}{\frac{1}{C'_2 s}} = 0.$$

Rearranging yields

$$V(s)\left[C'_1 s + 1/R'_1 + C'_2 s\right] = C'_1 s X(s) + Y(s)\left[C'_2 s + 1/R'_1\right].$$

Substituting the previous expression for $V(s)$ yields

$$Y(s)\frac{1 + R'_2 C'_2 s}{R'_2 C'_2 s}\left[C'_1 s + 1/R'_1 + C'_2 s\right] = C'_1 s X(s) + Y(s)\left[C'_2 s + 1/R'_1\right].$$

Rearranging yields

$$Y(s)\left[\frac{1+R_2'C_2's}{R_2'C_2's}\frac{1+R_1'C_1's+R_1'C_2's}{R_1'}-\frac{1+R_1'C_2's}{R_1'}\right]=X(s)\left[C_1's\right].$$

Following simplification, we get

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s^2}{s^2+s\left(\frac{1}{R_2'C_2'}+\frac{1}{R_2'C_1'}\right)+\frac{1}{R_1'R_2'C_1'C_2'}}.$$

(c) MATLAB is used to transform the Butterworth LPF from part (a).

```
>>  R1p = 1./(C1*omega_0); R2p = 1./(C2*omega_0);
>>  C1p = 1./(R1*omega_0); C2p = 1./(R2*omega_0);
>>  for stage = 1:NS,
>>    disp(['Stage ',num2str(stage),...
>>      ' (Q = ',num2str(Q(stage)),...
>>      '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>    disp(['         R1'' = ',num2str(R1p(stage)),...
>>      ', R2'' = ',num2str(R2p(stage))]);
>>  end
    Stage 1 (Q = 0.5098):  C1' = C2' = 1e-09
            R1' = 39024.2064, R2' = 40568.2432
    Stage 2 (Q = 0.60134):  C1' = C2' = 1e-09
            R1' = 33083.1247, R2' = 47853.5056
    Stage 3 (Q = 0.89998):  C1' = C2' = 1e-09
            R1' = 22105.4372, R2' = 71617.8323
    Stage 4 (Q = 2.5629):  C1' = C2' = 1e-09
            R1' = 7762.3973, R2' = 203950.3311
```

The resulting resistor and capacitor values are realistic.

MATLAB also conveniently computes magnitude responses and pole locations. From $H(s)$, it is clear that all zeros are at zero.

```
>>  Hmag_SK = zeros(NS,200); Poles = zeros(NS,2);
>>  f = linspace(0,omega_0/pi,200);
>>  for stage = 1:NS,
>>    B = [1 0 0];
>>    A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>    1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>    Poles(stage,:) = (roots(A)).';
>>    Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>>  end
>>  subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>>  axis(omega_0*[-1.1, .1 -1.1 1.1]); axis equal;
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  subplot(122), plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:');
>>  xlabel('f [Hz]'); ylabel('Magnitude Responses');
```

The overall magnitude response plot looks like a highpass Butterworth filter; the cutoff is correctly located at ω_c and the response is smooth and monotonic. Interestingly, the Butterworth HPF poles look identical to the Butterworth LPF poles. The only difference is seen in the zeros; all the zeros of the LPF are infinite, and all the zeros of the HPF are located at $s = 0$.
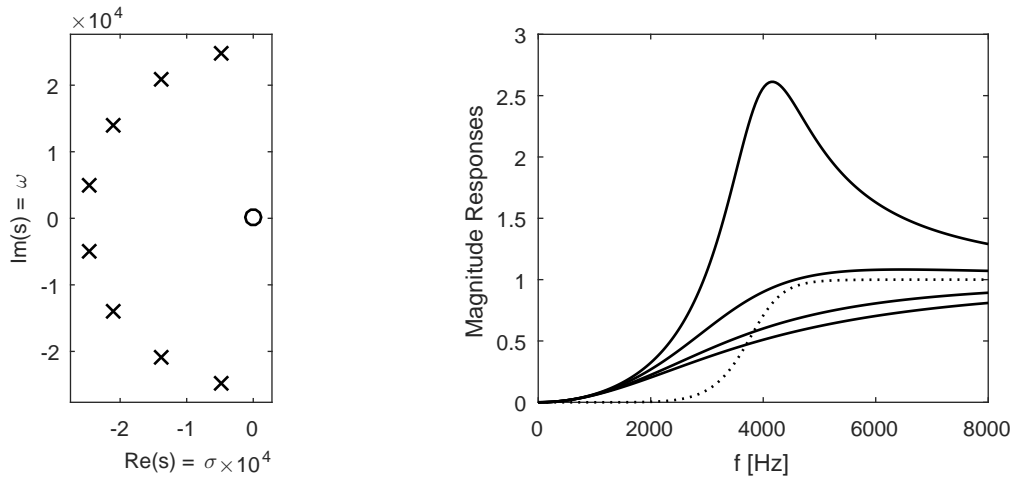
Figure S4.12-6c

Solution 4.12-7

(a) Using MATLAB, the Sallen-Key component values are easily found.

```
>>  omega_0 = 1500*2*pi; NS = 8; psi = [90/(2*NS):90/NS:90]*pi/180;
>>  Q = 1./(2*cos(psi)); R1 = 1e9/omega_0*ones(1,NS); R2 = R1;
>>  C1 = 2*Q./(omega_0*R1); C2 = 1./(2*omega_0*Q.*R2);
>>  for stage = 1:NS,
>>    disp(['Stage ',num2str(stage),...
>>      ' (Q = ',num2str(Q(stage)),...
>>      '):  R1 = R2 = ',num2str(R1(stage))]);
>>    disp(['         C1 = ',num2str(C1(stage)),...
>>      ', C2 = ',num2str(C2(stage))]);
>>  end
    Stage 1 (Q = 0.50242):  R1 = R2 = 106103.2954
            C1 = 1.0048e-09, C2 = 9.9518e-10
    Stage 2 (Q = 0.5225):  R1 = R2 = 106103.2954
            C1 = 1.045e-09, C2 = 9.5694e-10
    Stage 3 (Q = 0.56694):  R1 = R2 = 106103.2954
            C1 = 1.1339e-09, C2 = 8.8192e-10
    Stage 4 (Q = 0.64682):  R1 = R2 = 106103.2954
            C1 = 1.2936e-09, C2 = 7.7301e-10
    Stage 5 (Q = 0.78815):  R1 = R2 = 106103.2954
            C1 = 1.5763e-09, C2 = 6.3439e-10
    Stage 6 (Q = 1.0607):  R1 = R2 = 106103.2954
            C1 = 2.1214e-09, C2 = 4.714e-10
    Stage 7 (Q = 1.7224):  R1 = R2 = 106103.2954
            C1 = 3.4449e-09, C2 = 2.9028e-10
    Stage 8 (Q = 5.1011):  R1 = R2 = 106103.2954
            C1 = 1.0202e-08, C2 = 9.8017e-11
```

The resulting resistor and capacitor values are realistic.

(b) The transformed Sallen-Key circuit is shown in Fig. S4.12-7b. Name the node between capacitors $v(t)$. In transform domain, KCL at the positive terminal of the op-amp yields

$$\frac{Y(s) - V(s)}{\frac{1}{sC_2'}} = -\frac{Y(s) - 0}{R_2'}.$$
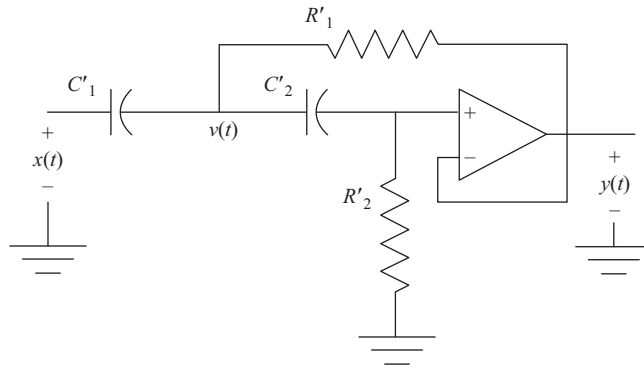
Figure S4.12-7b

Solving for $V(s)$ yields

$$V(s) = Y(s)\frac{1 + R_2'C_2's}{R_2'C_2's}.$$

KCL at node $V(s)$ yields

$$\frac{X(s) - V(s)}{\frac{1}{C_1's}} + \frac{Y(s) - V(s)}{R_1'} + \frac{Y(s) - V(s)}{\frac{1}{C_2's}} = 0.$$

Rearranging yields

$$V(s)\left[C_1's + 1/R_1' + C_2's\right] = C_1'sX(s) + Y(s)\left[C_2's + 1/R_1'\right].$$

Substituting the previous expression for $V(s)$ yields

$$Y(s)\frac{1 + R_2'C_2's}{R_2'C_2's}\left[C_1's + 1/R_1' + C_2's\right] = C_1'sX(s) + Y(s)\left[C_2's + 1/R_1'\right].$$

Rearranging yields

$$Y(s)\left[\frac{1 + R_2'C_2's}{R_2'C_2's}\frac{1 + R_1'C_1's + R_1'C_2's}{R_1'} - \frac{1 + R_1'C_2's}{R_1'}\right] = X(s)\left[C_1's\right].$$

Following simplification, we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2}{s^2 + s\left(\frac{1}{R_2'C_2'} + \frac{1}{R_2'C_1'}\right) + \frac{1}{R_1'R_2'C_1'C_2'}}.$$

(c) MATLAB is used to transform the Butterworth LPF from part (a).

```
>>  R1p = 1./(C1*omega_0); R2p = 1./(C2*omega_0);
>>  C1p = 1./(R1*omega_0); C2p = 1./(R2*omega_0);
>>  for stage = 1:NS,
>>    disp(['Stage ',num2str(stage),...
>>      ' (Q = ',num2str(Q(stage)),...
>>      '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>    disp(['        R1'' = ',num2str(R1p(stage)),...
>>      ', R2'' = ',num2str(R2p(stage))]);
>>  end
    Stage 1 (Q = 0.50242):  C1' = C2' = 1e-09
```

```
                  R1' = 105592.379, R2' = 106616.6839
      Stage 2 (Q = 0.5225):  C1' = C2' = 1e-09
                  R1' = 101534.5231, R2' = 110877.6498
      Stage 3 (Q = 0.56694):  C1' = C2' = 1e-09
                  R1' = 93574.7524, R2' = 120309.2608
      Stage 4 (Q = 0.64682):  C1' = C2' = 1e-09
                  R1' = 82018.9565, R2' = 137259.8455
      Stage 5 (Q = 0.78815):  C1' = C2' = 1e-09
                  R1' = 67311.218, R2' = 167251.6057
      Stage 6 (Q = 1.0607):   C1' = C2' = 1e-09
                  R1' = 50016.7472, R2' = 225082.7957
      Stage 7 (Q = 1.7224):   C1' = C2' = 1e-09
                  R1' = 30800.1609, R2' = 365514.6265
      Stage 8 (Q = 5.1011):   C1' = C2' = 1e-09
                  R1' = 10399.9416, R2' = 1082497.3575
```

The resulting resistor and capacitor values are reasonably realistic.

MATLAB also conveniently computes magnitude responses and pole locations. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(NS,200); Poles = zeros(NS,2);
>> f = linspace(0,omega_0/pi,200);
>> for stage = 1:NS,
>>    B = [1 0 0];
>>    A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>    1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>    Poles(stage,:) = (roots(A)).';
>>    Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis(omega_0*[-1.1, .1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122), plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:');
>> xlabel('f [Hz]'); ylabel('Magnitude Responses');
```
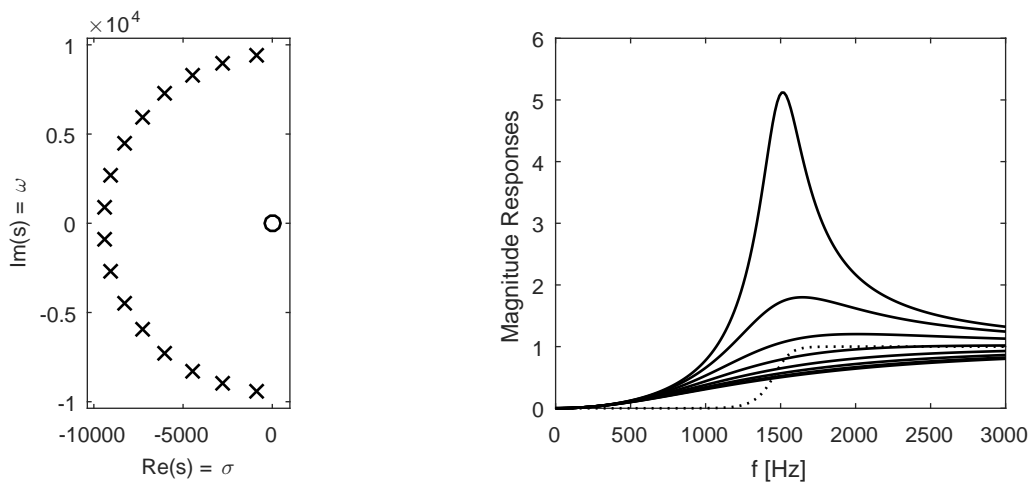

Figure S4.12-7c

The overall magnitude response plot looks like a highpass Butterworth filter; the cutoff is correctly located at ω_c and the response is smooth and monotonic. Interestingly, the Butterworth

HPF poles look identical to the Butterworth LPF poles. The only difference is seen in the zeros; all the zeros of the LPF are infinite, and all the zeros of the HPF are located at $s = 0$.

Solution 4.12-8

(a) Using MATLAB, the Sallen-Key component values are easily found.

```
>>  omega_c = 2*pi*4000; R = 3; N = 8;
>>  epsilon = sqrt(10^(R/10)-1);
>>  k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>>  C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>>  C_poles = C_poles(find(imag(C_poles)>0)); % Quadrant 2 poles
>>  f = linspace(0,10000,501); Hmag_SK = zeros(6,501);
>>  R1 = zeros(N/2,1); R2 = R1; C1 = R1; C2 = R1; Q = R1; omega_0 = R1;
>>  for stage = 1:N/2,
>>    omega_0(stage) = abs(C_poles(stage));
>>    psi = pi-angle(C_poles(stage));
>>    Q(stage) = 1/(2*cos(psi));
>>    R1(stage) = 1e9/omega_c; R2(stage) = R1(stage);
>>    C1(stage) = 2*Q(stage)./(omega_0(stage)*R1(stage));
>>    C2(stage) = 1./(2*omega_0(stage)*Q(stage).*R2(stage));
>>    disp(['Stage ',num2str(stage),...
>>      ' (Q = ',num2str(Q(stage)),...
>>      '):  R1 = R2 = ',num2str(R1(stage))]);
>>    disp(['       C1 = ',num2str(C1(stage)),...
>>      ', C2 = ',num2str(C2(stage))]);
>>    B = omega_0(stage)^2; A = [1 omega_0(stage)/Q omega_0(stage)^2];
>>    Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>>  end
   Stage 1 (Q = 22.8704):  R1 = R2 = 39788.7358
           C1 = 4.6343e-08, C2 = 2.215e-11
   Stage 2 (Q = 6.8251):  R1 = R2 = 39788.7358
           C1 = 1.6274e-08, C2 = 8.7339e-11
   Stage 3 (Q = 3.0798):  R1 = R2 = 39788.7358
           C1 = 1.0874e-08, C2 = 2.8659e-10
   Stage 4 (Q = 1.0337):  R1 = R2 = 39788.7358
           C1 = 9.2182e-09, C2 = 2.1569e-09
```

The resulting resistor and capacitor values are realistic.

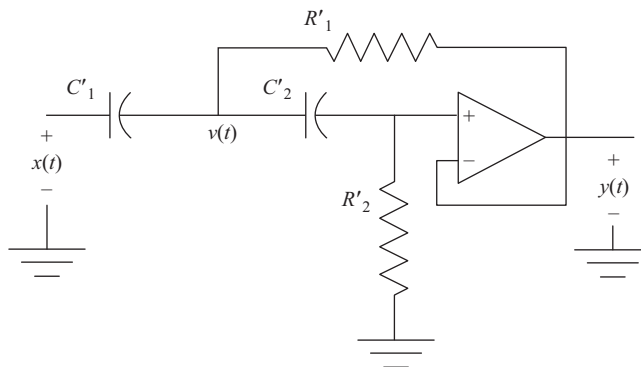**(b)** The transformed Sallen-Key circuit is shown in Fig. S4.12-8b. Name the node between capac-

Figure S4.12-8b

itors $v(t)$. In transform domain, KCL at the positive terminal of the op-amp yields

$$\frac{Y(s) - V(s)}{\frac{1}{sC_2'}} = -\frac{Y(s) - 0}{R_2'}.$$

Solving for $V(s)$ yields

$$V(s) = Y(s)\frac{1 + R_2'C_2's}{R_2'C_2's}.$$

KCL at node $V(s)$ yields

$$\frac{X(s) - V(s)}{\frac{1}{C_1's}} + \frac{Y(s) - V(s)}{R_1'} + \frac{Y(s) - V(s)}{\frac{1}{C_2's}} = 0.$$

Rearranging yields

$$V(s)\left[C_1's + 1/R_1' + C_2's\right] = C_1'sX(s) + Y(s)\left[C_2's + 1/R_1'\right].$$

Substituting the previous expression for $V(s)$ yields

$$Y(s)\frac{1 + R_2'C_2's}{R_2'C_2's}\left[C_1's + 1/R_1' + C_2's\right] = C_1'sX(s) + Y(s)\left[C_2's + 1/R_1'\right].$$

Rearranging yields

$$Y(s)\left[\frac{1 + R_2'C_2's}{R_2'C_2's}\frac{1 + R_1'C_1's + R_1'C_2's}{R_1'} - \frac{1 + R_1'C_2's}{R_1'}\right] = X(s)\left[C_1's\right].$$

Following simplification, we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2}{s^2 + s\left(\frac{1}{R_2'C_2'} + \frac{1}{R_2'C_1'}\right) + \frac{1}{R_1'R_2'C_1'C_2'}}.$$

(c) MATLAB is used to transform the Chebyshev LPF from part (a).

```
>>  R1p = 1./(C1*omega_c); R2p = 1./(C2*omega_c);
>>  C1p = 1./(R1*omega_c); C2p = 1./(R2*omega_c);
>>  for stage = 1:N/2,
>>    disp(['Stage ',num2str(stage),...
>>      ' (Q = ',num2str(Q(stage)),...
>>      '):   C1'' = C2'' = ',num2str(C1p(stage))]);
>>    disp(['          R1'' = ',num2str(R1p(stage)),...
>>      ', R2'' = ',num2str(R2p(stage))]);
>>  end
    Stage 1 (Q = 22.8704):   C1' = C2' = 1e-09
            R1' = 858.5676, R2' = 1796313.2499
    Stage 2 (Q = 6.8251):   C1' = C2' = 1e-09
            R1' = 2444.9937, R2' = 455567.9755
    Stage 3 (Q = 3.0798):   C1' = C2' = 1e-09
            R1' = 3659.1916, R2' = 138833.4048
    Stage 4 (Q = 1.0337):   C1' = C2' = 1e-09
            R1' = 4316.3108, R2' = 18446.8849
```

The resulting resistor and capacitor values possibly realistic; however, there is a fairly large dynamic range between the largest and smallest resistors.

MATLAB also conveniently computes magnitude responses and pole locations. By necessity, the transformation really stretches out the passband; it is therefore important to plot the magnitude response over a broad range of frequencies. To facilitate a reasonable plot, the magnitude response is plotted using both log-magnitude and log-frequency scales. From $H(s)$, it is clear that all zeros are at zero.

```
>>  Hmag_SK = zeros(N/2,5001); Poles = zeros(N/2,2);
>>  f = logspace(2,5,5001);
>>  for stage = 1:N/2,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>     1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>>  end
>>  subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>>  axis equal; ax = axis; axis([1.1*ax]);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  subplot(122),
>>  semilogx(f,20*log10(Hmag_SK),'k',f,20*log10(prod(Hmag_SK)),'k:')
>>  xlabel('f [Hz]'); ylabel('Magnitude Responses [dB]'); axis tight
>>  axis([100 1e5 -40 40]);
```
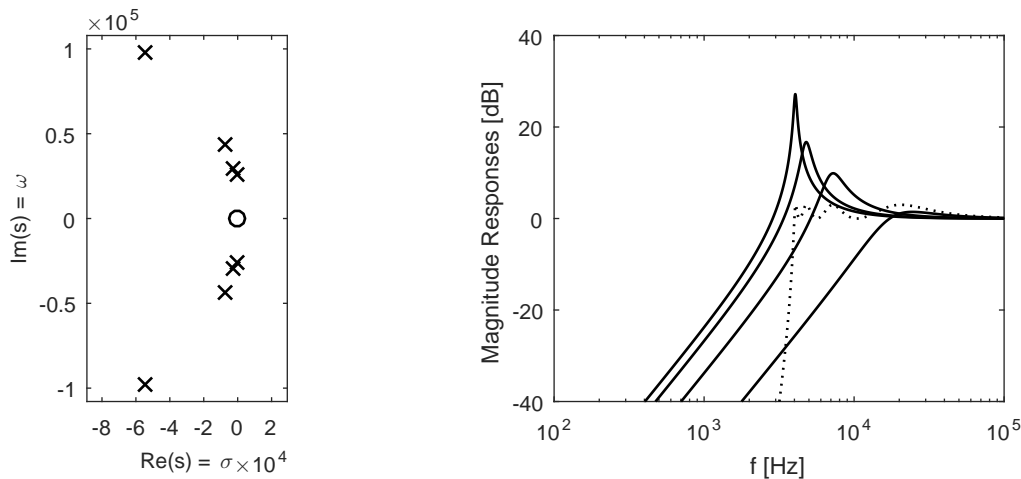

Figure S4.12-8c

The pole locations of the transformed Chebyshev filter are dramatically different than the pole locations of the original LPF. The zeros, as expected, are all concentrated at $s = 0$. The overall magnitude response plot looks like a highpass Chebyshev filter; passband ripples are equal in height and never exceed $R = 3$dB, there are a total of $N = 8$ maxima and minima in the passband, and the cutoff is correctly located at $\omega_c = 2\pi 4000$.

Solution 4.12-9

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Butterworth LPF with $\omega_c = 2\pi 3500$.

```
>>  omega_c = 2*pi*3500;
>>  [z,p,k] = butter(6,omega_c,'s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis(omega_c*[-1.1 0.1 -1.1 1.1]); axis equal;
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = butter(6,omega_c,'s');
>>  HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
```

```
>>  subplot(122),plot(f,20*log10(abs(HLP))),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```


Figure S4.12-9a

(b) Order-6 Butterworth HPF with $\omega_c = 2\pi 3500$.

```
>>  omega_c = 2*pi*3500;
>>  [z,p,k] = butter(6,omega_c,'high','s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis(omega_c*[-1.1 0.1 -1.1 1.1]); axis equal;
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = butter(6,omega_c,'high','s');
>>  HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HHP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```


Figure S4.12-9b

(c) Order-6 Butterworth BPF with passband between 2kHz and 4kHz. Notice that the command `butter` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>>  omega_c = [2*pi*2000,2*pi*4000];
>>  [z,p,k] = butter(3,omega_c,'s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis(omega_c(2)*[-1.1 0.1 -1.1 1.1]); axis equal;
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = butter(3,omega_c,'s');
>>  HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```
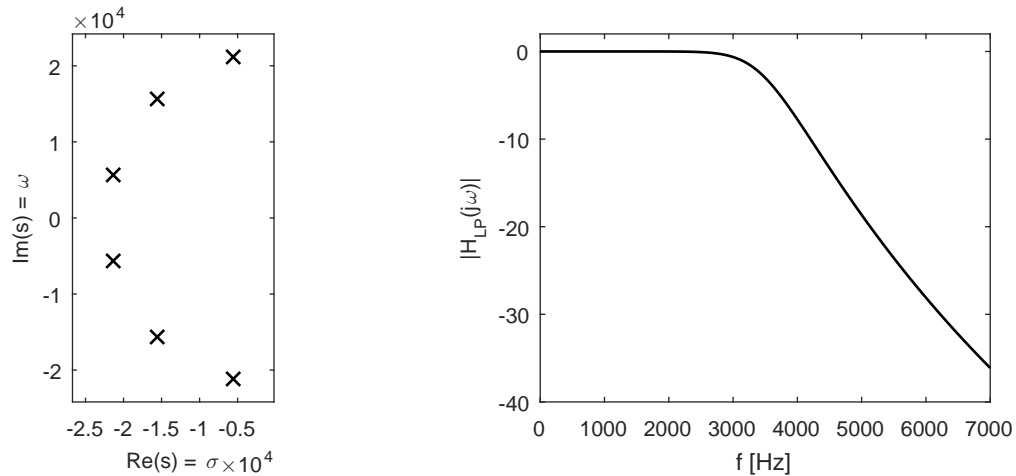

Figure S4.12-9c

(d) Order-6 Butterworth BSF with stopband between 2kHz and 4kHz. Notice that the command `butter` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>>  omega_c = [2*pi*2000,2*pi*4000];
>>  [z,p,k] = butter(3,omega_c,'stop','s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis(omega_c(2)*[-1.1 0.1 -1.1 1.1]); axis equal;
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = butter(3,omega_c,'stop','s');
>>  HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBS)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```
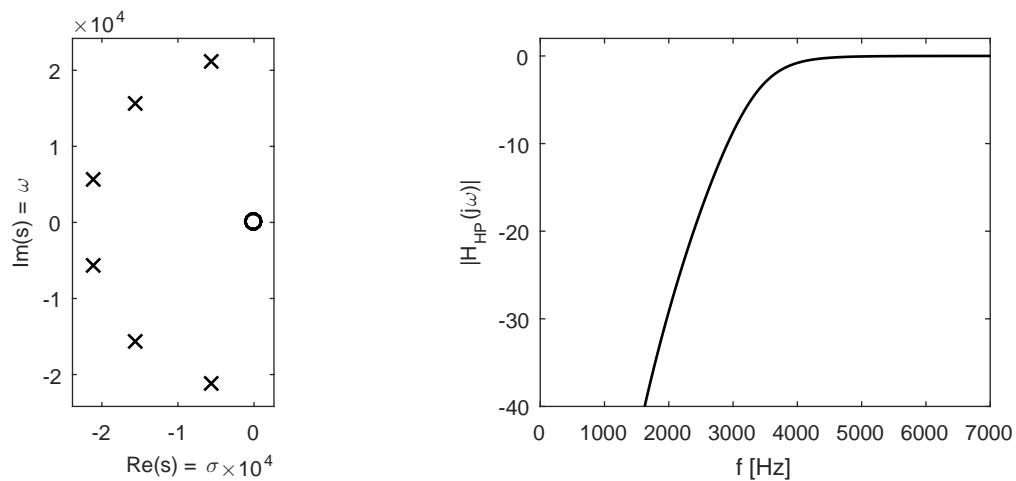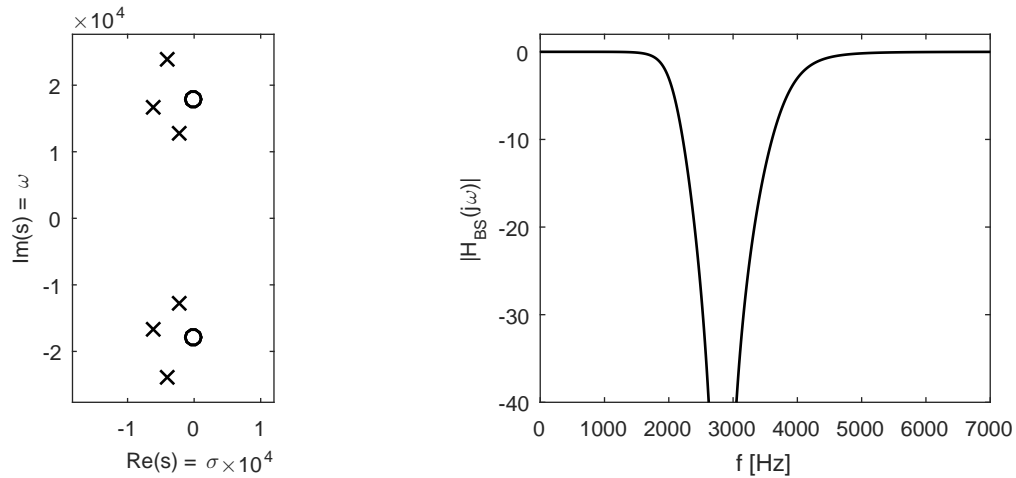
Figure S4.12-9d

Solution 4.12-10

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Chebyshev Type I LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby1(6,3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(6,3,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```
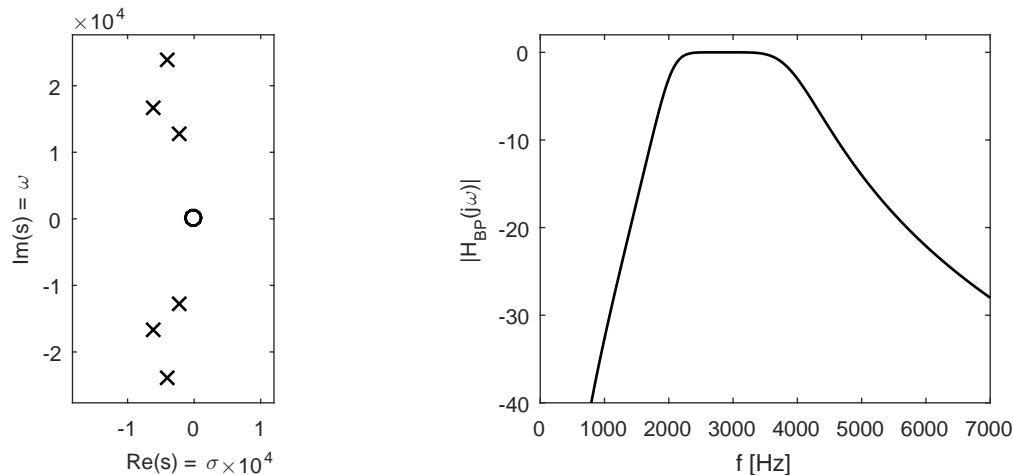

Figure S4.12-10a

(b) Order-6 Chebyshev Type I HPF with $\omega_c = 2\pi 3500$.

```
>>  omega_c = 2*pi*3500;
>>  [z,p,k] = cheby1(6,3,omega_c,'high','s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = cheby1(6,3,omega_c,'high','s');
>>  HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HHP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```


Figure S4.12-10b

(c) Order-6 Chebyshev Type I BPF with passband between 2kHz and 4kHz. Notice that the command `cheby1` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>>  omega_c = [2*pi*2000,2*pi*4000];
>>  [z,p,k] = cheby1(3,3,omega_c,'s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = cheby1(3,3,omega_c,'s');
>>  HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

(d) Order-6 Chebyshev Type I BSF with stopband between 2kHz and 4kHz. Notice that the command `cheby1` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>>  omega_c = [2*pi*2000,2*pi*4000];
>>  [z,p,k] = cheby1(3,3,omega_c,'stop','s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
```
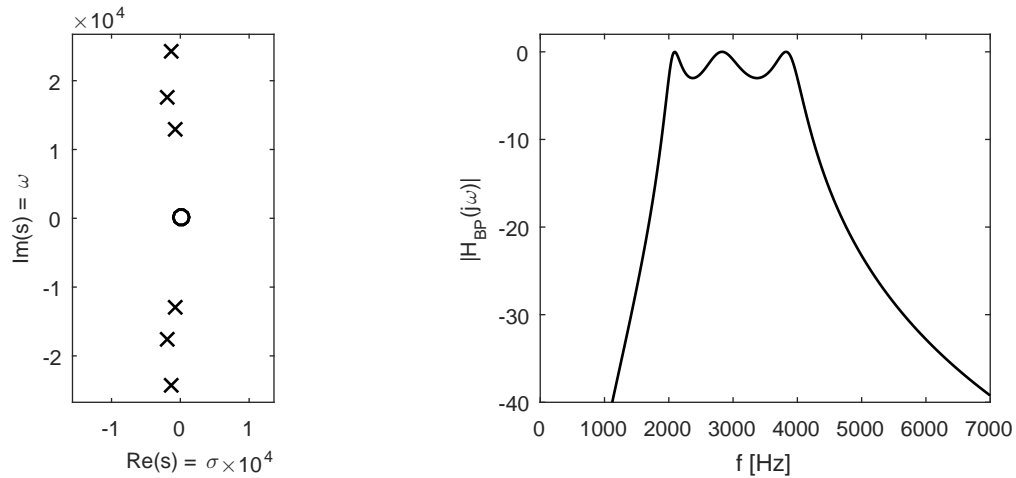
Figure S4.12-10c

```
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = cheby1(3,3,omega_c,'stop','s');
>>  HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBS)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```
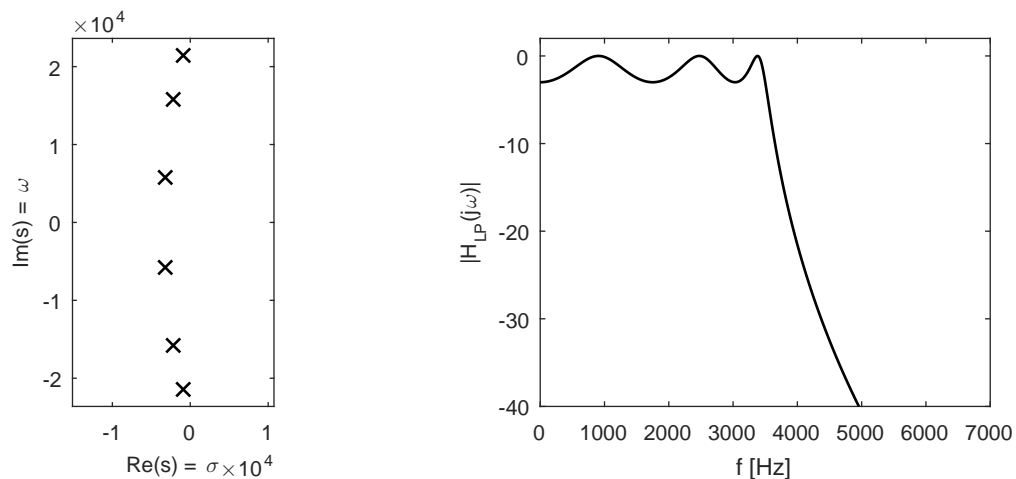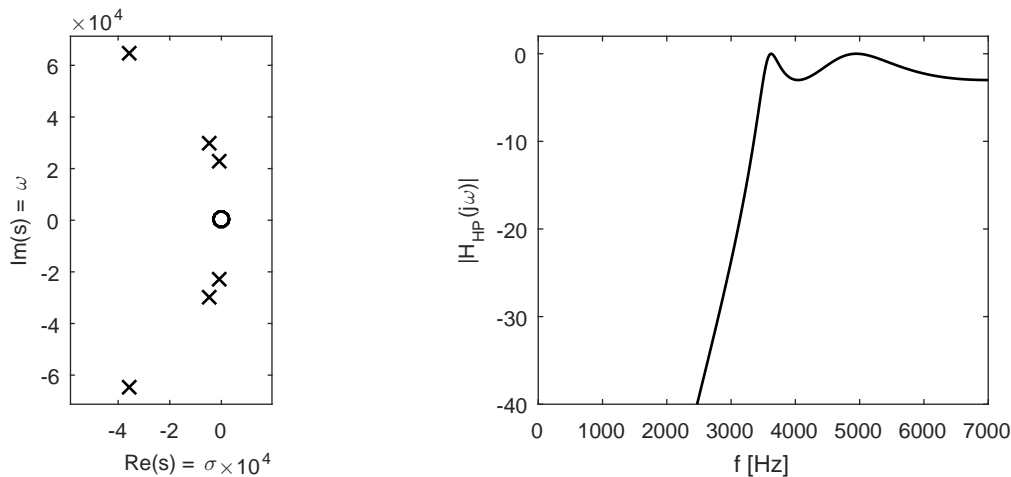

Figure S4.12-10d

(e) To demonstrate the effect of decreasing the passband ripple, consider magnitude response plots for Chebyshev Type I LPFs with $R_p = \{0.1, 1.0, 3.0\}$.

```
>>  omega_c = 2*pi*3500; f = linspace(0,7000,501);
>>  [B,A] = cheby1(6,.1,omega_c,'s');
>>  HLP1 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  [B,A] = cheby1(6,1,omega_c,'s');
>>  HLP2 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  [B,A] = cheby1(6,3,omega_c,'s');
```

```
>>  HLP3 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  plot(f,20*log10(abs(HLP1)),'k-',...
        f,20*log10(abs(HLP2)),'k--',...
        f,20*log10(abs(HLP3)),'k:');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
>>  legend('R_p = 0.1','R_p = 1.0','R_p = 3.0','Location','Best');
```
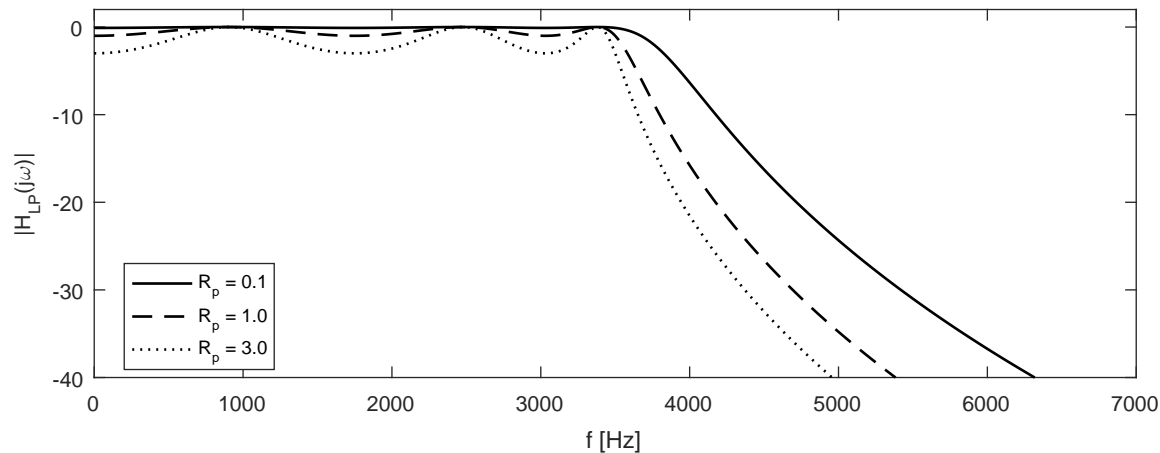

Figure S4.12-10e

Thus, reducing the allowable passband ripple R_p tends to broaden the transition bands of the filter.

Solution 4.12-11

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Chebyshev Type II LPF with $\omega_c = 2\pi 3500$.

```
>>  omega_c = 2*pi*3500;
>>  [z,p,k] = cheby2(6,20,omega_c,'s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = cheby2(6,20,omega_c,'s');
>>  HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HLP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

(b) Order-6 Chebyshev Type II HPF with $\omega_c = 2\pi 3500$.

```
>>  omega_c = 2*pi*3500;
>>  [z,p,k] = cheby2(6,20,omega_c,'high','s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = cheby2(6,20,omega_c,'high','s');
```
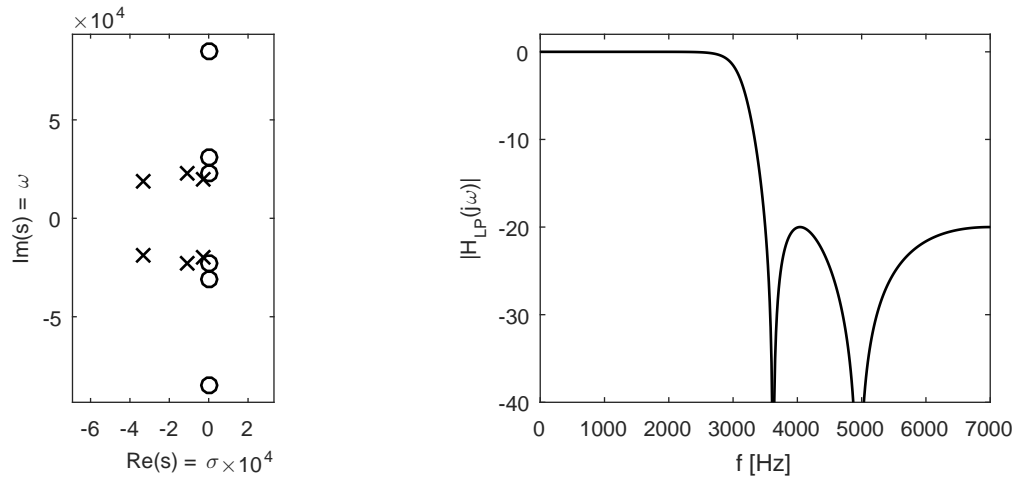
Figure S4.12-11a

```
>>  HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HHP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```
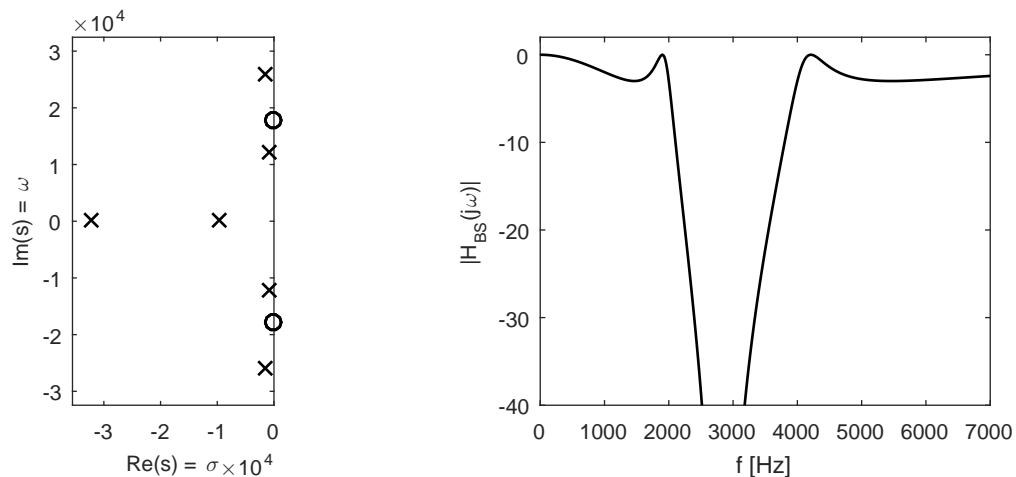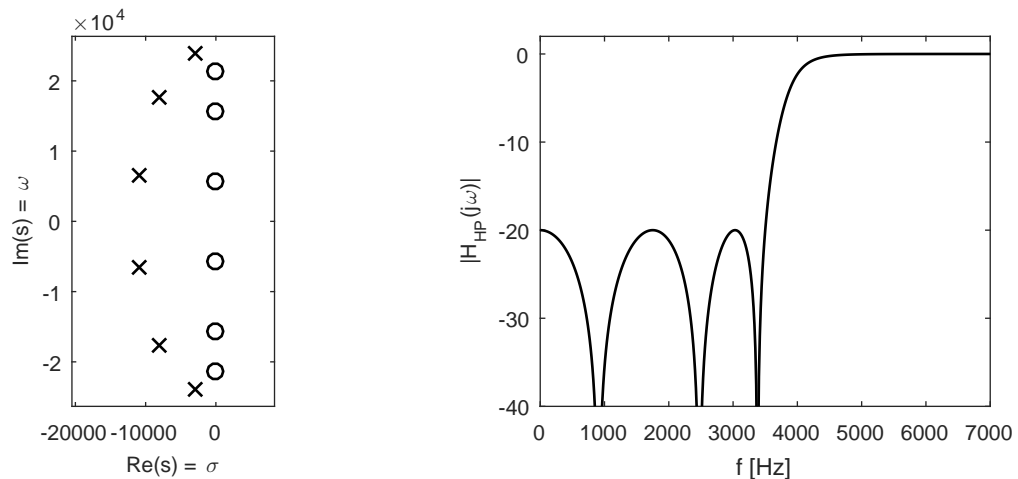

Figure S4.12-11b

(c) Order-6 Chebyshev Type II BPF with passband between 2kHz and 4kHz. Notice that the command `cheby2` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>>  omega_c = [2*pi*2000,2*pi*4000];
>>  [z,p,k] = cheby2(3,20,omega_c,'s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,501);
>>  [B,A] = cheby2(3,20,omega_c,'s');
>>  HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBP)),'k');
```

```
>>   axis([0 7000 -40 2])
>>   xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```
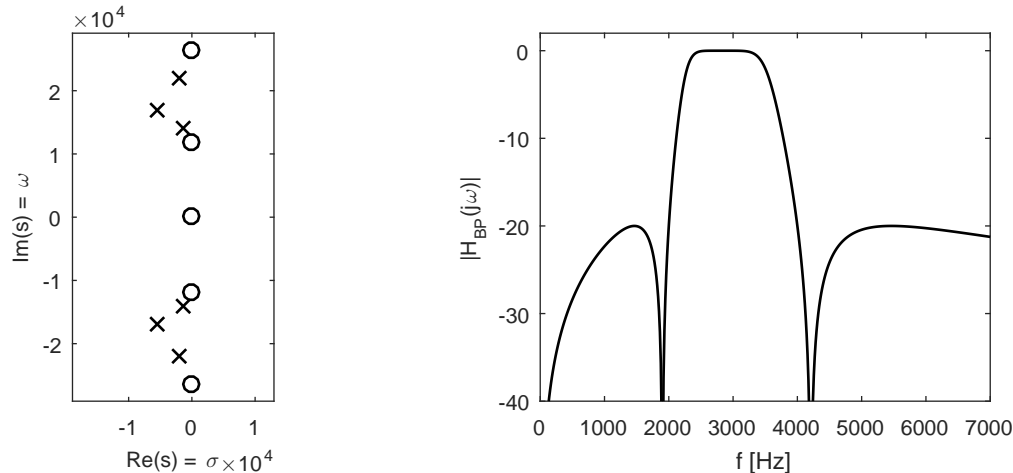

Figure S4.12-11c

(d) Order-6 Chebyshev Type II BSF with stopband between 2kHz and 4kHz. Notice that the command `cheby2` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>>   omega_c = [2*pi*2000,2*pi*4000];
>>   [z,p,k] = cheby2(3,20,omega_c,'stop','s');
>>   subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>   axis equal; axis(1.1*axis);
>>   xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>   f = linspace(0,7000,501);
>>   [B,A] = cheby2(3,20,omega_c,'stop','s');
>>   HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>   subplot(122),plot(f,20*log10(abs(HBS)),'k');
>>   axis([0 7000 -40 2])
>>   xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

(e) To demonstrate the effect of increasing R_s, consider magnitude response plots for Chebyshev Type II LPFs with $R_s = \{10, 20, 30\}$.

```
>>   omega_c = 2*pi*3500; f = linspace(0,7000,501);
>>   [B,A] = cheby2(6,10,omega_c,'s');
>>   HLP1 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>   [B,A] = cheby2(6,20,omega_c,'s');
>>   HLP2 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>   [B,A] = cheby2(6,30,omega_c,'s');
>>   HLP3 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>   plot(f,20*log10(abs(HLP1)),'k-',...
         f,20*log10(abs(HLP2)),'k--',...
         f,20*log10(abs(HLP3)),'k:');
>>   axis([0 7000 -40 2])
>>   xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
>>   legend('R_s = 10','R_s = 20','R_s = 30','Location','Best');
```

Thus, increasing R_s tends to broaden the transition bands of the filter.
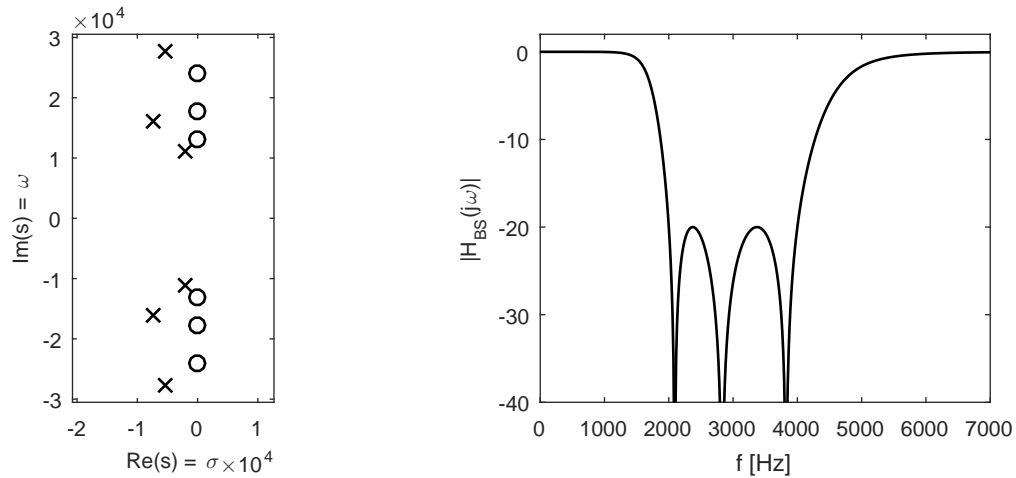
Figure S4.12-11d
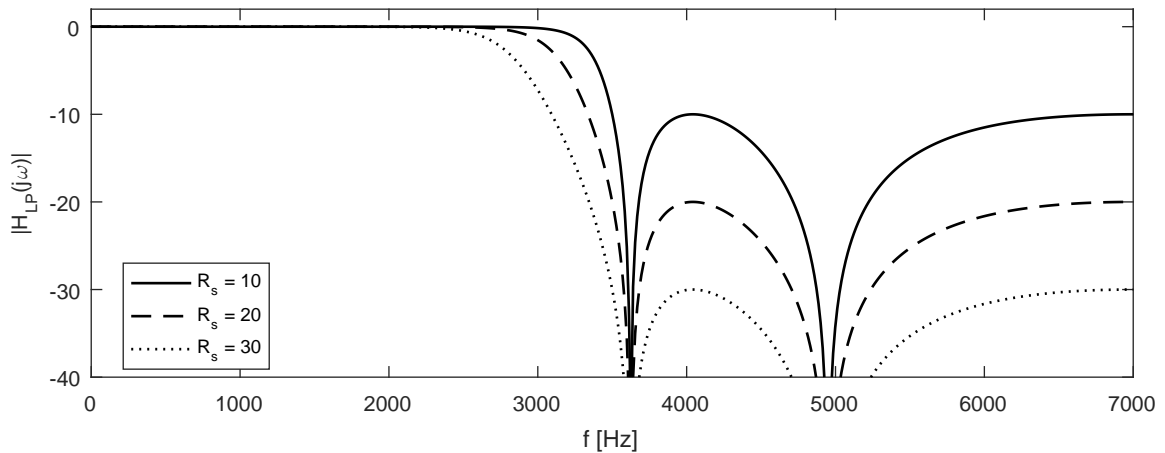
Figure S4.12-11e

Solution 4.12-12

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Elliptic LPF with $\omega_c = 2\pi 3500$.

```
>>  omega_c = 2*pi*3500;
>>  [z,p,k] = ellip(6,3,20,omega_c,'s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,2001);
>>  [B,A] = ellip(6,3,20,omega_c,'s');
>>  HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HLP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

(b) Order-6 Elliptic HPF with $\omega_c = 2\pi 3500$.
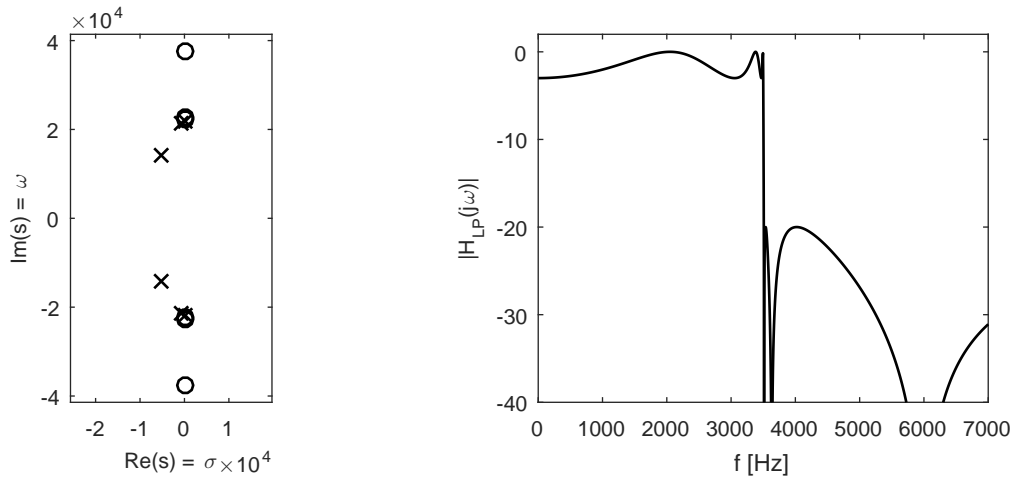
Figure S4.12-12a

```
>>   omega_c = 2*pi*3500;
>>   [z,p,k] = ellip(6,3,20,omega_c,'high','s');
>>   subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>   axis equal; axis(1.1*axis);
>>   xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>   f = linspace(0,7000,2001);
>>   [B,A] = ellip(6,3,20,omega_c,'high','s');
>>   HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>   subplot(122),plot(f,20*log10(abs(HHP)),'k');
>>   axis([0 7000 -40 2])
>>   xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```
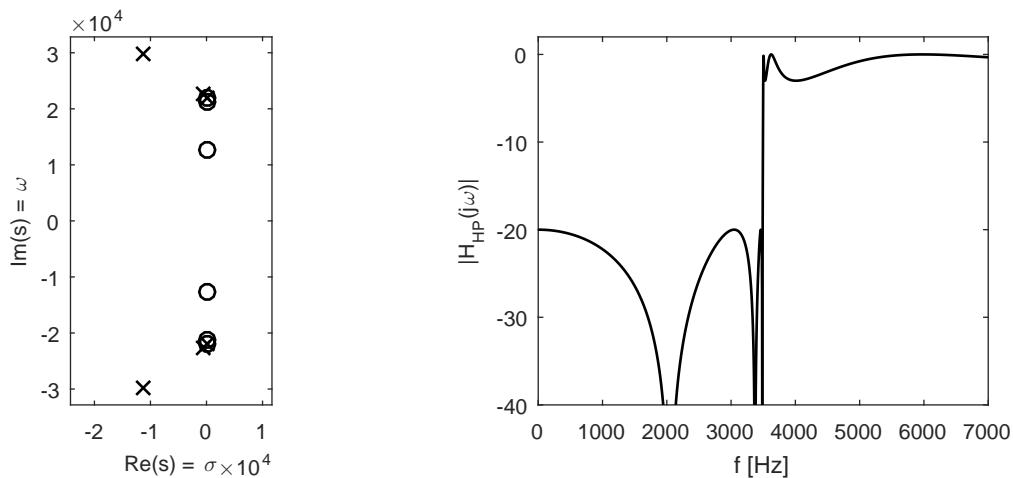

Figure S4.12-12b

(c) Order-6 Elliptic BPF with passband between 2kHz and 4kHz. Notice that the command `ellip` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>>   omega_c = [2*pi*2000,2*pi*4000];
>>   [z,p,k] = ellip(3,3,20,omega_c,'s');
>>   subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
```

```
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,2001);
>>  [B,A] = ellip(3,3,20,omega_c,'s');
>>  HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBP)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```
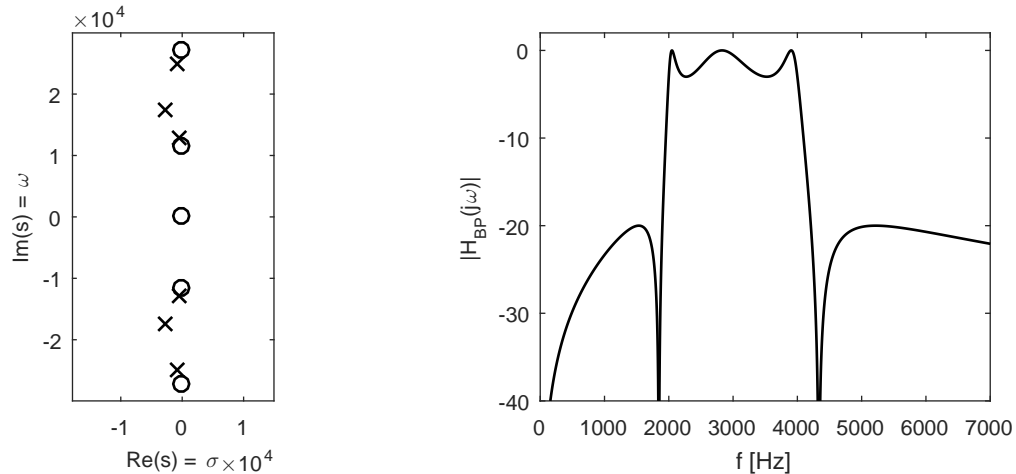

Figure S4.12-12c

(d) Order-6 Elliptic BSF with stopband between 2kHz and 4kHz. Notice that the command `ellip` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>>  omega_c = [2*pi*2000,2*pi*4000];
>>  [z,p,k] = ellip(3,3,20,omega_c,'stop','s');
>>  subplot(121),plot(real(p),imag(p),'kx',real(z),imag(z),'ko');
>>  axis equal; axis(1.1*axis);
>>  xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>>  f = linspace(0,7000,2001);
>>  [B,A] = ellip(3,3,20,omega_c,'stop','s');
>>  HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>>  subplot(122),plot(f,20*log10(abs(HBS)),'k');
>>  axis([0 7000 -40 2])
>>  xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```
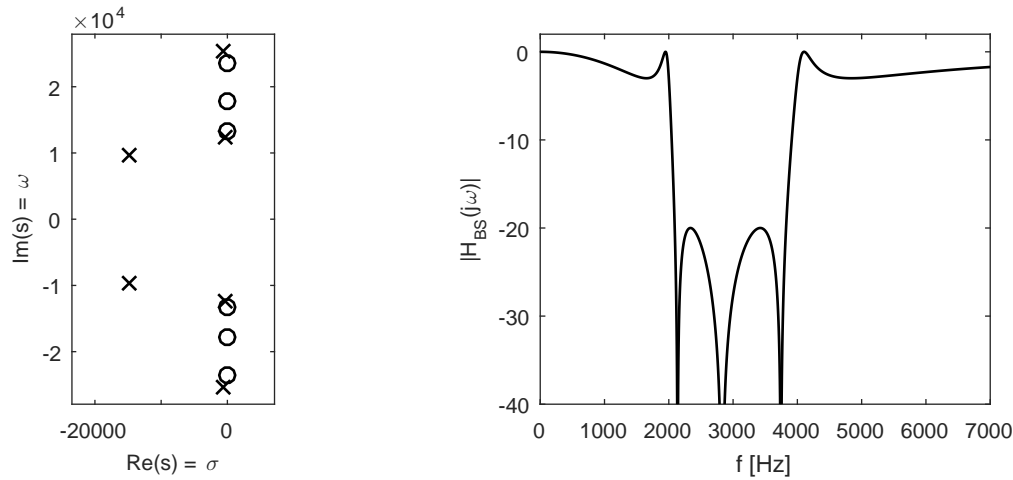
Figure S4.12-12d

Solution 4.12-13

First, the recursion relation $C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x)$ is rewritten as $C_{N+1}(x) = 2xC_N(x) - C_{N-1}(x)$ or $C_{N+1} + C_{N-1} = 2xC_N(x)$.

Letting $\gamma = \cosh^{-1}(x)$ and using Euler's formula, we know $C_N(x) = \cosh\left(N\cosh^{-1}(x)\right) = \cosh(N\gamma) = \frac{e^{N\gamma}+e^{-N\gamma}}{2}$. Thus, $C_{N+1} + C_{N-1} = \frac{e^{(N+1)\gamma}+e^{-(N+1)\gamma}}{2} + \frac{e^{(N-1)\gamma}+e^{-(N-1)\gamma}}{2} = \frac{e^{N\gamma}(e^{\gamma}+e^{-\gamma})+e^{-N\gamma}(e^{\gamma}+e^{-\gamma})}{2} = 2\frac{(e^{\gamma}+e^{-\gamma})}{2}\frac{(e^{N\gamma}+e^{-N\gamma})}{2} = 2\cosh(\gamma)\cosh(N\gamma)$. Replacing γ yields $C_{N+1} + C_{N-1} = 2\cosh(\cosh^{-1}(x))\cosh(N\cosh^{-1}(x)) = 2xC_N(x)$. Thus,

$$C_{N+1} + C_{N-1} = 2xC_N(x) \text{ or } C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x).$$

Solution 4.12-14

Note that $p_k = \sigma_k + j\omega_k = \omega_c\sinh(\xi)\sin(\phi_k) + j\omega_c\cosh(\xi)\cos(\phi_k)$. From the real portion, we know $\sigma_k = \omega_c\sinh(\xi)\sin(\phi_k)$ or $\sin(\phi_k) = \frac{\sigma_k}{\omega_c\sinh(\xi)}$. From the imaginary portion, we know $\omega_k = \omega_c\cosh(\xi)\cos(\phi_k)$ or $\cos(\phi_k) = \frac{\omega_k}{\omega_c\cosh(\xi)}$. From trigonometry, we know $1 = \cos^2(\phi_k)+\sin^2(\phi_k)$. Thus,

$$\left(\frac{\omega_k}{\omega_c\cosh(\xi)}\right)^2 + \left(\frac{\sigma_k}{\omega_c\sinh(\xi)}\right)^2 = 1.$$

This is the equation of an ellipse. Since the Chebyshev poles $p_k = \sigma_k + j\omega_k$ satisfy the equation, they must lie on the ellipse.

Chapter 5 Solutions

Solution 5.1-1

Given the fact that the time-domain signal is finite in duration, the region of convergence should include the entire z-plane, except possibly $z = 0$ or $z = \infty$. Now, $X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{7}(-1)^n z^{-n} = \sum_{n=0}^{7}(-1/z)^n = \frac{1-(-1/z)^8}{1-(-1/z)}$. Thus,

$$X[z] = \frac{1 - z^{-8}}{1 + z^{-1}}; \text{ ROC } |z| > 0.$$

In this form, $X[z]$ appears to have eight finite zeros and one finite pole. The eight zeros are the eight roots of unity, or $z = e^{2\pi k/8}$ for $k = (0, 1, \ldots, 7)$. The apparent pole is at $z = -1$. However, there is also a zero $z = -1$ $(k = 4)$ that cancels this pole. Thus, there are actually no finite poles and only seven finite zeros, $z = e^{2\pi k/8}$ for $k = (0, 1, 2, 3, 5, 6, 7)$. MATLAB is used to plot the zeros in the complex plane; the unit circle is also plotted for reference.

```
>>  k = [0:3,5:7]; zz = exp(j*2*pi*k/8); ang = linspace(0,2*pi,201);
>>  plot(real(zz),imag(zz),'ko',cos(ang),sin(ang),'k'); grid on;
>>  xlabel('Re(z)'); ylabel('Im(z)'); axis([-1.1 1.1 -1.1 1.1]); axis equal;
```
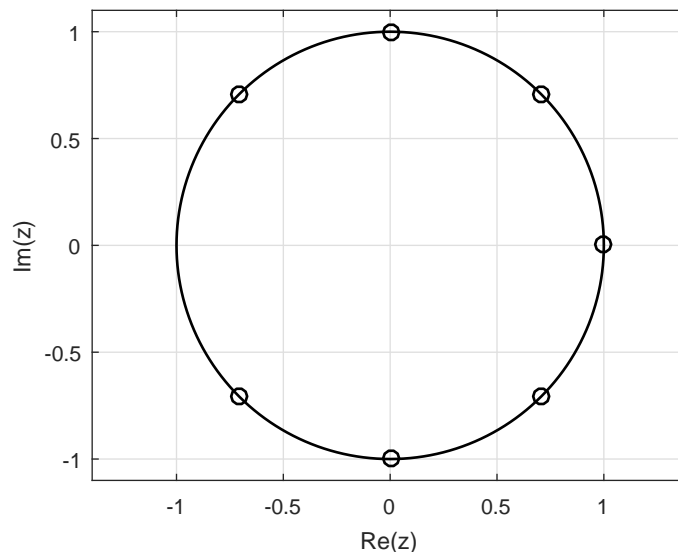

Figure S5.1-1

Solution 5.1-2

From the figure that shows $x[n]$, we see that

$$x[n] = 3\delta[n-3] + \delta[n-5] - \delta[n-7] - 3u[n-9].$$

Using Table 5.1, we see that

$$X[z] = \underbrace{3z^{-3} + z^{-5} - z^{-7}}_{|z|>0} \underbrace{-3z^{-9}\frac{z}{z-1}}_{|z|>1}.$$

Thus

$$X[z] = 3z^{-3} + z^{-5} - z^{-7} - \frac{3z^{-8}}{z-1}, \qquad \text{ROC: } |z| > 1.$$

Solution 5.1-3

(a) Since all three poles of $X[z]$ lie on the unit circle and $x[n]$ is known to be causal, we infer that

the ROC is $|z| > 1$.

Next, we obtain a series expansion of $X[z]$ in powers of z^{-1} by dividing the numerator by the denominator as follows:

$$
\begin{array}{r}
z^{-1} + z^{-4} + z^{-7} + \cdots \\
z^3 - 1 \overline{)\ z^2 } \\
\underline{z^2\ -\ z^{-1}} \\
z^{-1} \\
\underline{z^{-1}\ -\ z^{-4}} \\
z^{-4} \\
\underline{z^{-4}\ -\ z^{-7}} \\
z^{-7} \\
\vdots
\end{array}
$$

Thus,

$$X[z] = \sum_{i=0}^{\infty} z^{-(3i+1)}.$$

Inverting, we obtain

$$x[n] = \sum_{i=0}^{\infty} \delta[n - (3i+1)] = \delta[n-1] + \delta[n-4] + \delta[n-7] + \dots.$$

Figure S5.1-3 shows $x[n]$ over $-4 \le n \le 11$.
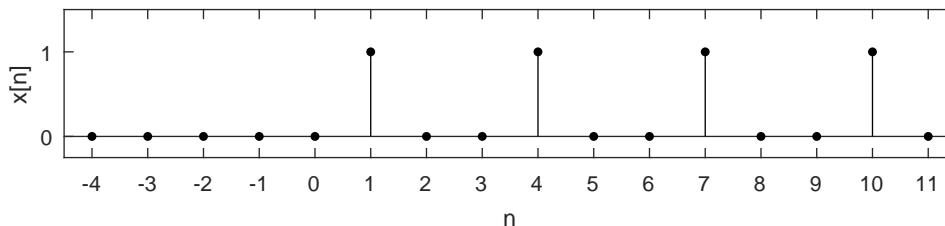
Figure S5.1-3

(b) Referring to $x[n]$ from part (a), we see that we can express $y[n]$ as

$$y[n] = x[n+1] + 2x[n] + 3x[n-1].$$

Taking the z-transform, we obtain

$$Y[z] = zX[z] + 2X[z] + 3z^{-1}X[z].$$

Substituting the expression $X[z] = \frac{z^2}{z^3-1}$, we obtain

$$Y[z] = \frac{z^3 + 2z^2 + 3z}{z^3 - 1}.$$

Solution 5.1-4

(a)

$$X[z] = \sum_{n=m}^{\infty} z^{-n} = z^{-m} + z^{-(m+1)} + z^{-(m+2)} + \cdots$$

$$= z^{-m}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= z^{-m}\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{z}{z^m(z-1)}, \qquad |z| > 1.$$

(b) Notice that $\gamma^n \sin \pi n u[n] = 0$ for all n. Hence,

$$X[z] = 0, \qquad \text{all } z.$$

(c) Since $\gamma^n \cos \pi n u[n] = (-\gamma)^n u[n]$, we see that

$$X[z] = \frac{z}{z+\gamma}, \qquad |z| > |\gamma|.$$

(d) Here, $\gamma^n \sin \frac{\pi n}{2} u[n]$ is a sequence

$$0, \gamma^1, 0, -\gamma^3, 0, \gamma^5, 0, -\gamma^7, \ldots.$$

Hence,

$$X[z] = \left(\frac{\gamma}{z} + \frac{\gamma^5}{z^5} + \frac{\gamma^9}{z^9} + \cdots\right) - \left(\frac{\gamma^3}{z^3} + \frac{\gamma^7}{z^7} + \frac{\gamma^{11}}{z^{11}} + \cdots\right)$$

$$= \frac{\gamma}{z}\left[1 + \left(\frac{\gamma}{z}\right)^4 + \left(\frac{\gamma}{z}\right)^8 + \cdots\right] - \left(\frac{\gamma}{z}\right)^3\left[1 + \left(\frac{\gamma}{z}\right)^4 + \left(\frac{\gamma}{z}\right)^8 + \cdots\right]$$

$$= \frac{\gamma}{z}\left[\frac{1}{1-\left(\frac{\gamma}{z}\right)^4}\right] - \left(\frac{\gamma}{z}\right)^3\left[\frac{1}{1-\left(\frac{\gamma}{z}\right)^4}\right] \qquad \left|\frac{\gamma}{z}\right| < 1$$

$$= \frac{\gamma}{z}\left[1 - \left(\frac{\gamma}{z}\right)^2\right]\left[\frac{1}{1-\left(\frac{\gamma}{z}\right)^4}\right]$$

$$= \frac{\gamma z}{z^2 + \gamma^2}, \qquad |z| > |\gamma|.$$

(e) In this case, $\gamma^n \cos \frac{\pi n}{2} u[n]$ is a sequence

$$1, 0, -\gamma^2, 0, \gamma^4, 0, -\gamma^6, 0, \gamma^8, \ldots.$$

Hence,

$$X[z] = 1 - \frac{\gamma^2}{z^2} + \frac{\gamma^4}{z^4} - \frac{\gamma^6}{z^6} + \frac{\gamma^8}{z^8} - \cdots$$

$$= \left[1 + \left(\frac{\gamma}{z}\right)^4 + \left(\frac{\gamma}{z}\right)^8 + \cdots \right] - \left(\frac{\gamma}{z}\right)^2 \left[1 + \left(\frac{\gamma}{z}\right)^4 + \left(\frac{\gamma}{z}\right)^8 + \cdots \right]$$

$$= \left(1 - \frac{\gamma^2}{z^2} \right) \left(\frac{1}{1 - \frac{\gamma^4}{z^4}} \right)$$

$$= \frac{z^2}{z^2 + \gamma^2} \qquad |z| > |\gamma|.$$

(f)

$$\sum_{k=0}^{\infty} 2^{2k}\delta[n - 2k] = \delta[n] + 4\delta[n - 2] + 16\delta[n - 4] + \cdots$$

$$X[z] = 1 + \frac{4}{z^2} + \frac{16}{z^4} + \frac{64}{z^6} + \cdots.$$

This is a geometric progression with common ratio $\frac{4}{z^2}$. Hence,

$$X[z] = \frac{1}{1 - \frac{4}{z^2}} = \frac{z^2}{z^2 - 4} \qquad |z| > 2.$$

(g)

$$X[z] = \sum_{n=1}^{\infty} \gamma^{n-1} z^{-n} = \frac{1}{\gamma} \sum_{n=1}^{\infty} \left(\frac{\gamma}{z}\right)^n$$

$$= \frac{1}{\gamma} \left[\frac{\gamma}{z} + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots \right]$$

$$= \frac{1}{\gamma} \left[-1 + \left(1 + \frac{\gamma}{z} + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots \right) \right]$$

$$= \frac{1}{\gamma} \left[-1 + \frac{1}{1 - \frac{\gamma}{z}} \right] = \frac{1}{z - \gamma}, \qquad |z| > |\gamma|.$$

(h) Here, $x[n] = n\gamma^n u[n]$ is a sequence

$$0, \ \gamma, \ 2\gamma^2, \ 3\gamma^3, \ \ldots.$$

Hence,

$$X[z] = \frac{\gamma}{z} + 2\left(\frac{\gamma}{z}\right)^2 + 3\left(\frac{\gamma}{z}\right)^3 + \cdots.$$

Using the results in Sec. B.8.3 with $n \to \infty$ and $|z| > |\gamma|$, we obtain

$$X[z] = \frac{\gamma/z}{[(\gamma/z) - 1]^2} = \frac{\gamma z}{(z - \gamma)^2} \qquad |z| > |\gamma|.$$

(i)

$$x[n] = nu[n] \qquad \text{and} \qquad X[z] = \sum_{n=0}^{\infty} nz^{-n}.$$

Using the results in Sec. B.8.3 with $|z| > 1$, we obtain

$$X[z] = \frac{(1/z)}{[(1/z) - 1]^2} = \frac{z}{(z - 1)^2} \qquad |z| > 1.$$

(j)

$$X[z] = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\gamma}{z}\right)^n.$$

Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Therefore,

$$X[z] = e^{\gamma/z}, \qquad |z| > 0.$$

(k)

$$x[n] = \left[2^{n-1} - (-2)^{n-1}\right] u[n]$$

$$X[z] = \frac{1}{2} \left[\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{-2}{z}\right)^n\right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - \frac{2}{z}} + \frac{1}{1 + \frac{2}{z}}\right]$$

$$= \frac{1}{2} \left[\frac{z}{z - 2} + \frac{z}{z + 2}\right]$$

$$= \frac{z^2}{z^2 - 4}, \qquad |z| > |2|.$$

(l)

$$X[z] = \sum_{n=0}^{\infty} \frac{1}{n!} (\ln \alpha)^n z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\ln \alpha}{z}\right)^n.$$

From the result in part (j) it follows that

$$X[z] = e^{\ln \alpha/z} = (e^{\ln \alpha})^{1/z} = \alpha^{1/z}, \qquad |z| > 0.$$

Solution 5.1-5

Note that the signal $x[n] = nu[n]$ has z-transform $X[z] = \sum_{n=0}^{\infty} nz^{-n} = \frac{z}{(z-1)^2}$. Thus, $\sum_{n=0}^{\infty} n(-3/2)^{-n}$ is easily found by evaluating the $X[z]|_{z=-3/2}$. That is, $\sum_{n=0}^{\infty} n(-3/2)^{-n} = \frac{z}{(z-1)^2}\Big|_{z=-3/2} = \frac{-3/2}{(-3/2-1)^2} = -\frac{3/2}{25/4} = -\frac{12}{50} = -\frac{6}{25}$. Thus,

$$\sum_{n=0}^{\infty} n(-3/2)^{-n} = -\frac{6}{25} = -0.24.$$

Solution 5.1-6

(a)

$$u[n] - u[n-2] = \delta[n] + \delta[n-1]$$

Hence,

$$u[n] - u[n-2] \Longleftrightarrow 1 + \frac{1}{z} = \frac{z+1}{z}.$$

(b)

$$\gamma^{n-2}u[n-2] = \frac{1}{\gamma^2}\left\{\gamma^n u[n] - \delta[n] - \gamma\delta[n-1]\right\}$$

Hence,

$$\gamma^{n-2}u[n-2] \Longleftrightarrow \frac{1}{\gamma^2}\left[\frac{z}{z-\gamma} - 1 - \frac{\gamma}{z}\right] = \frac{1}{z(z-\gamma)}.$$

(c)

$$x[n] = (2)^{n+1}u[n-1] + (e)^{n-1}u[n] = 4(2)^{n-1}u[n-1] + \frac{1}{e}(e)^n u[n]$$

Therefore,

$$X[z] = \frac{4}{z-2} + \frac{1}{e}\frac{z}{z-e}.$$

(d)

$$x[n] = \left[(2)^{-n}\cos\frac{\pi n}{3}\right]u[n-1] = (2)^{-n}\cos\frac{\pi n}{3}u[n] - \delta[n]$$

Therefore,

$$X[z] = \frac{z(z-0.25)}{z^2 - 0.5z + 0.25} - 1 = \frac{0.25(z-1)}{z^2 - 0.5z + 0.25}.$$

(e)

$$x[n] = n\gamma^n u[n-1] = n\gamma^n u[n] - 0 = n\gamma^n u[n]$$

Therefore,

$$X[z] = \frac{\gamma z}{(z-\gamma)^2}.$$

(f) Because $n(n-1)(n-2) = 0$ for $n = 0$, 1, and 2,

$$x[n] = n(n-1)(n-2)2^{n-3}u[n-m] = n(n-1)(n-2)(2)^{n-3}u[n].$$

Therefore,

$$x[n] = (2)^{-3}\{n(n-1)(n-2)2^n u[n]\}$$

and

$$X[z] = (2)^{-3}\left[\frac{3!(2)^3 z}{(z-2)^4}\right] = \frac{6z}{(z-2)^4}.$$

(g)

$$x[n] = (-1)^n n u[n]$$

$$\begin{aligned}
X[z] &= 0 - \frac{1}{z} + \frac{2}{z^2} - \frac{3}{z^3} + \frac{4}{z^4} - \frac{5}{z^5} + \cdots \\
&= \sum_{n=0}^{\infty} 2n\left(\frac{1}{z^2}\right)^n - \sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{z^2}\right)^{2n-1} \\
&= 2\sum_{n=0}^{\infty} n\left(\frac{1}{z^2}\right)^n - \frac{2}{z}\sum_{n=0}^{\infty} n\left(\frac{1}{z^2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n
\end{aligned}$$

Using the entries in Sec. B.8.3, we obtain

$$X[z] = \frac{2z^2}{(z^2 - 1)^2} - \frac{2}{z}\frac{z^2}{(z^2 - 1)^2} - \frac{z}{z^2 - 1}$$

$$= \frac{-z^3 + 2z^2 - z}{(z^2 - 1)^2}$$

$$= \frac{-z(z - 1)^2}{(z + 1)^2(z - 1)^2}$$

$$= \frac{-z}{(z + 1)^2}.$$

(h)

$$x[n] = \sum_{k=0}^{\infty} k\delta[n - 2k + 1]$$

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} k\delta[n - 2k + 1] \right) z^{-n}$$

Interchanging the order of summation and noting that

$$\delta[n - 2k + 1] = \begin{cases} 1 & n = 2k - 1 \\ 0 & n \neq 2k - 1 \end{cases}.$$

Thus,

$$X[z] = \sum_{k=0}^{\infty} k \left(\sum_{n=0}^{\infty} \delta[n - 2k + 1]z^{-n} \right)$$

$$= \sum_{k=0}^{\infty} kz^{-(2k-1)}$$

$$= z \sum_{k=0}^{\infty} k \left(\frac{1}{z^2} \right)^k$$

$$= z \frac{1/z^2}{[(1/z)^2 - 1]^2}$$

$$= \frac{z^3}{(z^2 - 1)^2}.$$

Solution 5.1-7

(a)

$$\frac{X[z]}{z} = \frac{z - 4}{(z - 2)(z - 3)} = \frac{2}{z - 2} - \frac{1}{z - 3}$$

$$X[z] = 2\frac{z}{z - 2} - \frac{z}{z - 3}$$

$$x[n] = [2(2)^n - (3)^n]u[n]$$

(b)

$$\frac{X[z]}{z} = \frac{z-4}{z(z-2)(z-3)} = \frac{-2/3}{z} + \frac{1}{z-2} - \frac{1/3}{z-3}$$

$$X[z] = -\frac{2}{3} + \frac{z}{z-2} - \frac{1}{3}\frac{z}{z-3}$$

$$x[n] = -\frac{2}{3}\delta[n] + \left[(2)^n - \frac{1}{3}(3)^n\right]u[n]$$

(c)

$$\frac{X[z]}{z} = \frac{e^{-2}-2}{(z-e^{-2})(z-2)} = \frac{1}{z-e^{-2}} - \frac{1}{z-2}$$

$$X[z] = \frac{z}{z-e^{-2}} - \frac{z}{z-2}$$

$$x[n] = \left[e^{-2n} - 2^n\right]u[n]$$

(d)

$$X[z] = \frac{(z-1)^2}{z^3} = \frac{z^2-2z+1}{z^3} = \frac{1}{z} - \frac{2}{z^2} + \frac{1}{z^3}$$

Hence,

$$x[n] = \delta[n-1] - 2\delta[n-2] + \delta[n-3].$$

(e)

$$\frac{X[z]}{z} = \frac{2z+3}{(z-1)(z-2)(z-3)} = \frac{5/2}{z-1} - \frac{7}{z-2} + \frac{9/2}{z-3}$$

$$X[z] = \frac{5}{2}\frac{z}{z-1} - 7\frac{z}{z-2} + \frac{9}{2}\frac{z}{z-3}$$

$$x[n] = \left[\frac{5}{2} - 7(2)^n + \frac{9}{2}(3)^n\right]u[n]$$

(f)

$$\frac{X[z]}{z} = \frac{-5z+22}{(z+1)(z-2)^2} = \frac{3}{z+1} + \frac{k}{z-2} + \frac{4}{(z-2)^2}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = 3 + k + 0 \implies k = -3.$$

Thus,

$$X[z] = 3\frac{z}{z+1} - 3\frac{z}{z-2} + 4\frac{z}{(z-2)^2}$$

$$x[n] = [3(-1)^n - 3(2)^n + 2n(2)^n]u[n].$$

(g)

$$\frac{X[z]}{z} = \frac{1.4z+0.08}{(z-0.2)(z-0.8)^2} = \frac{1}{z-0.2} + \frac{k}{z-0.8} + \frac{2}{(z-0.8)^2}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = 1 + k \implies k = -1.$$

Thus,

$$X[z] = \frac{z}{z - 0.2} - \frac{z}{z - 0.8} + 2\frac{z}{(z - 0.8)^2}$$

$$x[n] = \left[(0.2)^n - (0.8)^n + \frac{5}{2}n(0.8)^n\right]u[n].$$

(h) From Table 5.1, we use pair 12c with $A = 1$, $B = -2$, $a = -0.5$, $|\gamma| = 1$. Therefore,

$$r = \sqrt{4} = 2 \qquad \beta = \cos^{-1}(\frac{0.5}{1}) = \frac{\pi}{3} \qquad \theta = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{3}$$

$$x[n] = 2(1)^n \cos(\frac{\pi n}{3} + \frac{\pi}{3})u[n] = 2\cos(\frac{\pi n}{3} + \frac{\pi}{3})u[n].$$

(i)

$$\frac{X[z]}{z} = \frac{2z^2 - 0.3z + 0.25}{z(z^2 + 0.6z + 0.25)} = \frac{1}{z} + \frac{Az + B}{z^2 + 0.6z + 25}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$2 = 1 + A \implies A = 1.$$

Setting $z = 1$ on both sides yields

$$\frac{1.95}{1.85} = 1 + \frac{1 + B}{1.85} \implies B = -0.9$$

$$X[z] = 1 + \frac{z(z - 0.9)}{z^2 + 0.6z + 0.25}.$$

For the second fraction on right side, we use Table 5.1 pair 12c with $A = 1$, $B = -0.9$, $a = 0.3$, and $|\gamma| = 0.5$. This yields

$$r = \sqrt{10} \qquad \beta = \cos^{-1}(\frac{-0.3}{0.5}) = 2.214 \qquad \theta = \tan^{-1}(\frac{1.2}{0.4}) = 1.249$$

$$x[n] = \delta[n] + \sqrt{10}(0.5)^n \cos(2.214n + 1.249)u[n].$$

(j)

$$\frac{X[z]}{z} = \frac{2(3z - 23)}{(z - 1)(z^2 - 6z + 25)} = \frac{-2}{z - 1} + \frac{Az + B}{z^2 - 6z + 25}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = -2 + A \implies A = 2.$$

Set $z = 0$ on both sides to obtain

$$\frac{46}{25} = 2 + \frac{B}{25} \implies B = -4$$

$$X[z] = -2\frac{z}{z - 1} + \frac{z(2z - 4)}{z^2 - 6z + 25}.$$

For the second fraction on the right-hand side, we use Table 5.1 pair 12c with $A = 2$, $B = -4$, $a = -3$, and $|\gamma| = 5$. This yields

$$r = \frac{\sqrt{17}}{2} \qquad \beta = \cos^{-1}(\frac{3}{5}) = 0.927 \qquad \theta = \tan^{-1}(\frac{-1}{4}) = -0.25$$

$$x[n] = \left[-2 + \frac{\sqrt{17}}{2}(5)^n \cos(0.927n - 0.25)\right]u[n].$$

(k)

$$\frac{X[z]}{z} = \frac{3.83z + 11.34}{(z-2)(z^2 - 5z + 25)} = \frac{1}{z-2} + \frac{Az + B}{z^2 - 5z + 25}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = 1 + A \implies A = -1.$$

Setting $z = 0$ on both sides yields

$$\frac{11.34}{-50} = -\frac{1}{2} + \frac{B}{25} \implies B = 6.83$$

$$X[z] = \frac{z}{z-2} + \frac{z(-z + 6.83)}{z^2 - 5z + 25}.$$

For the second fraction on right-hand side, use Table 5.1 pair 12c with $A = -1$, $B = 6.83$, $a = -2.5$, and $|\gamma| = 5$.

$$r = \sqrt{2} \qquad \beta = \cos^{-1}(0.5) = \frac{\pi}{3} \qquad \theta = \tan^{-1}(\frac{-4.33}{-4.33}) = -\frac{3\pi}{4}$$

$$x[n] = \left[(2)^n + \sqrt{2}(5)^n \cos(\frac{\pi}{3}n - \frac{3\pi}{4}) \right] u[n]$$

(l)

$$\frac{X[z]}{z} = \frac{z(-2z^2 + 8z - 7)}{(z-1)(z-2)^3} = \frac{1}{z-1} + \frac{k_1}{z-2} + \frac{k_2}{(z-2)^2} + \frac{2}{(z-2)^3}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$-2 = 1 + k_1 \implies k_1 = -3.$$

Set $z = 0$ on both sides to obtain

$$0 = -1 + \frac{3}{2} + \frac{k_2}{4} - \frac{1}{4} \implies k_2 = -1.$$

Thus,

$$X[z] = \frac{z}{z-1} - 3\frac{z}{z-2} - \frac{z}{(z-2)^2} + 2\frac{z}{(z-2)^3}$$

$$x[n] = \left[1 - 3(2)^n - \frac{n}{2}(2)^n + \frac{1}{4}n(n-1)(2)^n \right] u[n]$$

Solution 5.1-8

(a) Long division of $2z^3 + 13z^2 + z$ by $z^3 + 7z^2 + 2z + 1$ yields

$$X[z] = 2 - \frac{1}{z} + \frac{4}{z^2} + \cdots.$$

Therefore,

$$x[0] = 2, \ x[1] = -1, \ x[2] = 4.$$

(b) Long division of $2z^4 + 16z^3 + 17z^2 + 3z$ by $z^3 + 7z^2 + 2z + 1$ yields

$$X[z] = 2z + 2 - \frac{1}{z} + \frac{4}{z^2} + \cdots.$$

Therefore,

$$x[-1] = 2, \ x[0] = 2, \ x[1] = -1, \ x[2] = 4.$$

Solution 5.1-9

We obtain a series expansion of $X[z]$ in powers of z^{-1} by dividing the numerator by the denominator as follows:

$$z^4 - 1 \overline{\smash{\big)}\ z^6 \ + \ 2z^5 \ + \ 3z^4 \ + \ 4z^3}$$

$$\begin{array}{r} z^2 + 2z + 3 + 4z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + \cdots \\ \hline z^6 \qquad\qquad\qquad\qquad - z^2 \\ \hline 2z^5 \ + \ 3z^4 \ + \ 4z^3 \ + \ z^2 \\ 2z^5 \qquad\qquad\qquad\qquad - 2z \\ \hline 3z^4 \ + \ 4z^3 \ + \ z^2 \ + \ 2z \\ 3z^4 \qquad\qquad\qquad\qquad - 3 \\ \hline 4z^3 \ + \ z^2 \ + \ 2z \ + \ 3 \\ 4z^3 \qquad\qquad\qquad\qquad - 4z^{-1} \\ \hline z^2 \ + \ 2z \ + \ 3 \ + \ 4z^{-1} \\ \vdots \end{array}$$

.

After obtaining the first four terms $(z^2 + 2z + 3 + 4z^{-1})$, the remainder has exactly the same form as the original denominator, just a 4-lower power of z. Thus, continued division will yield a 4-repeating sequence of coefficients and

$$X[z] = z^2 + 2z + 3 + 4z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + \cdots$$

$$= \sum_{k=0}^{\infty} z^{2-4k} + 2z^{1-4k} + 3z^{-4k} + 4z^{-1-4k}.$$

Inverting, we see that

$$x[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 4\delta[n-1] + \delta[n-2] + 2\delta[n-3] + 3\delta[n-4] + 4\delta[n-5] + \cdots$$

$$= \sum_{k=0}^{\infty} \delta[n+2-4k] + 2\delta[n+1-4k] + 3\delta[n-4k] + 4\delta[n-1-4k].$$

Over $-5 \le n \le 5$, we see that

$$\{x[n]\}_{n=-5}^{5} = \{0,\ 0,\ 0,\ 1,\ 2,\ \overset{\overset{n=0}{\downarrow}}{3},\ 4,\ 1,\ 2,\ 3,\ 4\}.$$

Solution 5.1-10

Here,

$$X[z] = \frac{\gamma z}{z^2 - 2\gamma z + \gamma^2}.$$

Long division yields

$$\frac{\gamma z}{z^2 - 2\gamma z + \gamma^2} = \frac{\gamma}{z} + 2\left(\frac{\gamma}{z}\right)^2 + 3\left(\frac{\gamma}{z}\right)^3 + \cdots.$$

Therefore, $x[0] = 0$, $x[1] = \gamma$, $x[2] = 2\gamma^2$, $x[3] = 3\gamma^3, \cdots$, and

$$x[n] = n\gamma^n u[n].$$

Solution 5.1-11

(a) We can express

$$X[z] = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \cdots$$

Let $X_n[z]$ and $X_d[z]$ be the numerator and the denominator polynomials of $X[z]$ with powers M and N, respectively. If $M = N$, then the long division of X_n with X_d in power series of z^{-1} yields $x[0]$ as a nonzero constant. If $N - M = 1$, the term $x[0] = 0$, but $x[1]$, $x[2]$, $\cdots$ are generally nonzero. In general, if $N - M = m$, then the long division shows that all $x[0]$, $x[1]$, $\cdots$, $x[m-1]$ are zero. Only the terms from $x[m]$ on are generally nonzero. The difference $N - M$ indicates that the first $N - M$ samples of $x[n]$ are zero.

(b) In this case the first four samples of $x[n]$ are zero. Thus, $N - M = 4$.

Solution 5.2-1

(a) By definition of the z-transform, we know that

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{m} z^{-n}.$$

This sum can be found using the result in Sec. B.8.3, as

$$X[z] = \frac{(1/z)^{m-1} - 1}{(1/z) - 1} = \frac{1 - z^{-m}}{1 - z^{-1}}$$

(b) Since $x[n] = u[n] - u[n - m]$, Table 5.1 tells us that

$$X[z] = \frac{z}{z - 1} - z^{-m}\frac{z}{z - 1} = \frac{1 - z^{-m}}{1 - z^{-1}}.$$

Solution 5.2-2

Using properties, we establish that

$$\cos(\pi n/2)u[n] \Longleftrightarrow \frac{z^2}{z^2+1} \qquad \text{(Table 5.1, pair 11a)}$$

$$-n\cos(\pi n/2)u[n] \Longleftrightarrow z\frac{d}{dz}\left(\frac{z^2}{z^2+1}\right) \qquad \text{(z-domain differentiation)}$$

$$= z\left(\frac{2z}{z^2+1} - \frac{z^2}{(z^2+1)^2}(2z)\right) = \frac{2z^2}{(z^2+1)^2}$$

$$x[n] = -(n-1)\cos(\pi(n-1)/2)u[n-1] \Longleftrightarrow z^{-1}\frac{2z^2}{(z^2+1)^2} \qquad \text{(time shift)}$$

Thus,

$$X[z] = \frac{2z}{(z^2+1)^2} = \frac{2z}{z^4 + 2z^2 + 1}.$$

Solution 5.2-3

$$X[z] \Longleftrightarrow x[n] = 2(u[n-10] - u[n-6]) \qquad \overset{n=6}{\underset{\downarrow}{[-2, -2, -2, -2]}}$$

$$X[2z] \Longleftrightarrow (\tfrac{1}{2})^n x[n] \qquad \overset{n=6}{\underset{\downarrow}{[-\tfrac{1}{32}, -\tfrac{1}{64}, -\tfrac{1}{128}, -\tfrac{1}{256}]}}$$

$$-z\tfrac{d}{dz}X[2z] \Longleftrightarrow n(\tfrac{1}{2})^n x[n] \qquad \overset{n=6}{\underset{\downarrow}{[-\tfrac{3}{16}, -\tfrac{7}{64}, -\tfrac{1}{16}, -\tfrac{9}{256}]}}$$

$$\underbrace{-\frac{1}{2}z^2\left(-z\frac{d}{dz}X[2z]\right)}_{Y[z]} \Longleftrightarrow \underbrace{-\frac{1}{2}(n+2)(\tfrac{1}{2})^{(n+2)}x[n+2]}_{y[n]} \qquad \overset{n=4}{\underset{\downarrow}{[\tfrac{3}{32}, \tfrac{7}{128}, \tfrac{1}{32}, \tfrac{9}{512}]}}$$

More generally in equation form, we have

$$y[n] = -\tfrac{1}{2}(n+2)(\tfrac{1}{2})^{n+2}2(u[n-8]-u[n-4]) = (n+2)(\tfrac{1}{2})^{n+2}(u[n-4]-u[n-8]).$$

Solution 5.2-4

$$X[z] \Longleftrightarrow x[n] = 3(u[n]-u[n-5]) \qquad [\overset{n=0}{\underset{\downarrow}{3}}, 3, 3, 3, 3]$$

$$X[\tfrac{z}{2}] \Longleftrightarrow (2)^n x[n] \qquad [\overset{n=0}{\underset{\downarrow}{3}}, 6, 12, 24, 48]$$

$$-z\tfrac{d}{dz}X[\tfrac{z}{2}] \Longleftrightarrow n(2)^n x[n] \qquad [\overset{n=0}{\underset{\downarrow}{0}}, 6, 24, 72, 192]$$

$$\underbrace{-2z^{-5}\left(-z\tfrac{d}{dz}X[\tfrac{z}{2}]\right)}_{Y[z]} \Longleftrightarrow \underbrace{-2(n-5)(2)^{n-5}x[n-5]}_{y[n]} \quad [\overset{n=5}{\underset{\downarrow}{0}}, -12, -48, -144, -384]$$

More generally in equation form, we have

$$y[n] = -6(n-5)(2)^{n-5}(u[n-5]-u[n-10]).$$

Solution 5.2-5

$$x[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 4\delta[n-4] + 3\delta[n-5] + 2\delta[n-6] + \delta[n-7]$$

Therefore,

$$X[z] = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \frac{3}{z^5} + \frac{2}{z^6} + \frac{1}{z^7}$$

$$= \frac{z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1}{z^7}.$$

Alternate Method:

$$x[n] = n\{u[n] - u[n-5]\} + (-n+8)\{u[n-5] - u[n-9]\}$$
$$= nu[n] - 2nu[n-5] + nu[n-9] + 8u[n-5] - 8u[n-9]$$
$$= nu[n] - 2\{(n-5)u[n-5] + 5u[n-5]\}$$
$$+ (n-9)u[n-9] + 9u[n-9] + 8u[n-5] - 8u[n-9]$$
$$= nu[n] - 2(n-5)u[n-5] + (n-9)u[n-9] - 2u[n-5] + u[n-9]$$

Therefore,

$$X[z] = \frac{z}{(z-1)^2} - \frac{2z}{z^5(z-1)^2} + \frac{z}{z^9(z-1)^2} - \frac{2z}{z^5(z-1)} + \frac{z}{z^9(z-1)}$$

$$= \frac{z}{z^9(z-1)^2}\left[z^9 - 2z^4 + 1 - 2z^4(z-1) + (z-1)\right]$$

$$= \frac{1}{z^7(z-1)^2}\left[z^8 - 2z^4 + 1\right].$$

To verify that this answer matches the first, we use MATLAB. First, we root out the numerator polynomial.

```
>>  numroots = roots([1 0 0 0 -2 0 0 0 1]).'
    numroots =  -1    -1    1i    -1i    1i    -1i    1    1
```

Thus,

$$X[z] = \frac{(z+1)^2(z-j)^2(z+j)^2(z-1)^2}{z^7(z-1)^2} = \frac{(z+1)^2(z-j)^2(z+j)^2}{z^7}.$$

Next, we expand the remaining numerator terms.

```
>>  poly([-1 -1 -1j -1j 1j 1j])
    ans =   1     2     3     4     3     2     1
```

Thus,

$$X[z] = \frac{(z+1)^2(z-j)^2(z+j)^2}{z^7} = \frac{z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1}{z^7}.$$

This result verifies the answer from the second method exactly matches the result from the first method.

Solution 5.2-6

To begin, we note that

$$\left(\frac{1}{2}\right)^n u[n-3] = \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{n-3} u[n-3] \iff \left(\frac{1}{2}\right)^3 z^{-3}\frac{z}{z-\frac{1}{2}}$$

and

$$\left(\frac{1}{3}\right)^{n-6} u[n-4] = \left(\frac{1}{3}\right)^{-2} \left(\frac{1}{3}\right)^{n-4} u[n-4] \iff 9z^{-4}\frac{z}{z-\frac{1}{3}}.$$

We are intersted in $y[n] = (\frac{1}{2})^n u[n-3] * (\frac{1}{3})^{n-6}u[n-4]$. From Table 5.2 we know that convolution in the time-domain yields multiplication in the z-domain. Thus,

$$Y[z] = \left(\frac{1}{2}\right)^3 z^{-3}\frac{z}{z-\frac{1}{2}}9z^{-4}\frac{z}{z-\frac{1}{3}} = \frac{9}{8}z^{-7}\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}.$$

Using modified partial fractions, we see that

$$\frac{Y[z]}{z} = \frac{9}{8}z^{-6}\left(\frac{6}{z-\frac{1}{2}} + \frac{-6}{z-\frac{1}{3}}\right).$$

Thus,

$$Y[z] = \frac{54}{8}z^{-6}\left(\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}}\right).$$

Inverting, we obtain

$$y[n] = \frac{27}{4}\left[\left(\frac{1}{2}\right)^{n-6} - \left(\frac{1}{3}\right)^{n-6}\right]u[n-6].$$

Comparing to the form $y[n] = c_1\gamma_1^{n-N_1}u[n-N_1] + c_2\gamma_2^{n-N_2}u[n-N_2]$, we see that

$$c_1 = \frac{27}{4}, \quad c_2 = -\frac{27}{4}, \quad \gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{1}{3}, \quad N_1 = 6, \quad N_2 = 6.$$

Solution 5.2-7

To begin, we express $X[z]$ as

$$X[z] = (\underbrace{z^{-1}}_{\substack{\text{delay} \\ \text{by 1}}} \underbrace{z\frac{d}{dz}}_{\substack{\text{mult} \\ \text{by } -n}})^7 \left[z^{-5}\left(\frac{\frac{2}{7}z}{z-\frac{1}{2}} + \frac{-\frac{2}{7}z}{z+3}\right)\right].$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{apply } 7\times}$$

Inverting the part in square brackets yields

$$z^{-5}\left(\frac{\frac{2}{7}z}{z-\frac{1}{2}} + \frac{-\frac{2}{7}z}{z+3}\right) \Longleftrightarrow \frac{2}{7}\left[(\frac{1}{2})^{n-5} - (-3)^{n-5}\right]u[n-5].$$

Applying the first $z^{-1}z\frac{d}{dz}$ term yields

$$z^{-1}z\frac{d}{dz}\left[z^{-5}\left(\frac{\frac{2}{7}z}{z-\frac{1}{2}} + \frac{-\frac{2}{7}z}{z+3}\right)\right] \Longleftrightarrow (1-n)\frac{2}{7}\left[(\frac{1}{2})^{n-6} - (-3)^{n-6}\right]u[n-6].$$

Applying the second $z^{-1}z\frac{d}{dz}$ term yields

$$\left(z^{-1}z\frac{d}{dz}\right)^2\left[z^{-5}\left(\frac{\frac{2}{7}z}{z-\frac{1}{2}} + \frac{-\frac{2}{7}z}{z+3}\right)\right] \Longleftrightarrow (1-n)(2-n)\frac{2}{7}\left[(\frac{1}{2})^{n-7} - (-3)^{n-7}\right]u[n-7].$$

Continuing to the seventh $z^{-1}z\frac{d}{dz}$ term yields the final result of

$$x[n] = \frac{2}{7}\left(\prod_{i=1}^{7}(i-n)\right)\left[(\frac{1}{2})^{n-12} - (-3)^{n-12}\right]u[n-12].$$

Solution 5.2-8

(a) Consider

$$\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma}.$$

Application of the multiplication property Eq. (5.18) to this pair yields

$$n\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-\gamma}\right) = \frac{\gamma z}{(z-\gamma)^2}.$$

One more application of the multiplication property yields

$$n^2\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{\gamma z}{(z-\gamma)^2}\right) = \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}.$$

Letting $\gamma = 1$, we obtain

$$n^2 u[n] \Longleftrightarrow \frac{z(z+1)}{(z-1)^3}.$$

(b) Consider

$$\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma}.$$

Application of the multiplication property Eq. (5.18) to this pair yields

$$n\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-\gamma}\right) = \frac{\gamma z}{(z-\gamma)^2}.$$

One more application of the multiplication property yields

$$n^2\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{\gamma z}{(z-\gamma)^2}\right) = \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}.$$

(c) Application of Eq. (5.18) to $n^2\gamma^n u\,[n] \iff \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$ (found in part a) yields

$$n^3\gamma^n u\,[n] = -z\frac{d}{dz}\left(\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}\right) = \frac{\gamma z(z^2 + 4\gamma z + \gamma^2)}{(z-\gamma)^4}.$$

Setting $\gamma = 1$ in this result yields

$$n^3 u\,[n] = \frac{z(z^2 + 4z + 1)}{(z-1)^4}$$

(d)

$$x\,[n] = a^n\{u\,[n] - u\,[n-m]\}$$
$$= a^n u\,[n] - a^m a^{(n-m)} u\,[n-m]$$
$$X[z] = \frac{z}{z-a} - \frac{a^m z}{z-a}z^{-m} = \frac{z}{z-a}\left[1 - \left(\frac{a}{z}\right)^m\right]$$

(e)

$$x\,[n] = ne^{-2n}u\,[n-m] = (n-m+m)e^{-2(n-m+m)}u\,[n-m]$$
$$= e^{-2m}(n-m)e^{-2(n-m)}u\,[n-m] + me^{-2m}e^{-2(n-m)}u\,[n-m]$$

Using $\gamma^n u[n] \iff z/(z-\gamma)$ and the intermediate result of part (a), we obtain

$$X[z] = e^{-2m}\frac{e^{-2}z}{(z-e^{-2})^2}z^{-m} + me^{-2m}\left(\frac{z}{z-e^{-2}}\right)z^{-m}$$
$$= \frac{z^{-m+1}e^{-2m}}{(z-e^{-2})^2}\left[\frac{1}{e^2}(1-m) + mz\right].$$

(f)

$$x[n] = (n-2)(0.5)^{n-3}u[n-4]$$
$$= \frac{1}{2}(n-4+2)(0.5)^{n-4}u[n-4]$$
$$= \frac{1}{2}(n-4)(0.5)^{n-4}u[n-4] + (0.5)^{n-4}u[n-4]$$

Application of shift property yields

$$x[n] \iff \frac{1}{2}\left[\frac{0.5z}{z^4(z-0.5)^2}\right] + \frac{z}{z^4(z-0.5)}$$

or

$$X[z] = \frac{0.25}{z^3(z-0.5)^2} + \frac{1}{z^3(z-0.5)} = \frac{z-0.25}{z^3(z-0.5)^2}.$$

Solution 5.2-9

Using only pair 1 in Table 5.1 and appropriate properties of the z-transform, we here iteratively derive pairs 2 through 9. From pair 1, we know that

$$\delta[n] \iff 1$$

To derive pair 2, we first express $u[n]$ as

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots = \sum_{k=0}^{\infty} \delta[n-k].$$

Using the time-shifting and linearity properties, we therefore see that

$$u[n] \Longleftrightarrow \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

To derive pair 3, we use the differentiation-in-z property as

$$nu[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}.$$

Applying the differentiation-in-z property to this result yields pair 4 as

$$n^2u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{z(z+1)}{(z-1)^3}.$$

Again applying the differentiation-in-z property to this result yields pair 5 as

$$n^3u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z(z+1)}{(z-1)^3}\right) = \frac{z(z^2+4z+1)}{(z-1)^4}.$$

To derive pair 6, we use the z-domain scaling property, which states that

$$\gamma^n x[n] \Longleftrightarrow X\left(\frac{z}{\gamma}\right).$$

Applying this property to pair 2, we obtain pair 6 as

$$\gamma^n u[n] = \frac{\frac{z}{\gamma}}{\frac{z}{\gamma}-1} = \frac{z}{z-\gamma}.$$

Applying the time-shifting property to this result, we obtain pair 7 as

$$\gamma^{n-1}u[n-1] \Longleftrightarrow z^{-1}\frac{z}{z-\gamma} = \frac{1}{z-\gamma}.$$

We next apply the z-domain differentiation property to pair 6 to obtain pair 8 as

$$n\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-\gamma}\right) = \frac{-z}{z-\gamma} + \frac{z^2}{(z-\gamma)^2} = \frac{\gamma z}{(z-\gamma)^2}.$$

Applying the z-domain differentiation property to this result, we obtain pair 9 as

$$n^2\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{\gamma z}{(z-\gamma)^2}\right) = \frac{-\gamma z}{(z-\gamma)^2} + 2\frac{\gamma z^2}{(z-\gamma)^3} = \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}, \qquad |z| > |\gamma|.$$

Solution 5.2-10

Letting $|\gamma| = 1$, pair 11b in Table 5.1 tells us that

$$\sin(\beta n)u[n] \stackrel{\mathcal{Z}}{\Longleftrightarrow} \frac{z\sin(\beta)}{z^2 - 2\cos(\beta)z + 1}.$$

To make use of this pair, we next express $x[n]$ as

$$
\begin{aligned}
x[n] = \cos\left(\frac{\pi}{4}n\right)u[n] &= -\sin\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)u[n] = -\sin\left[\frac{\pi}{4}(n-2)\right]u[n] \\
&= -\sin\left[\frac{\pi}{4}(n-2)\right]u[n-2] - \sin\left(-\frac{\pi}{2}\right)\delta[n] - \sin\left(-\frac{\pi}{4}\right)\delta[n-1] \\
&= -\sin\left[\frac{\pi}{4}(n-2)\right]u[n-2] + \delta[n] + \frac{1}{\sqrt{2}}\delta[n-1].
\end{aligned}
$$

Now, applying pairs 1 and 11b in Table 5.1 and the time-shifting property, we obtain

$$
\begin{aligned}
X[z] = \mathcal{Z}\left\{\cos\left(\frac{\pi}{4}n\right)u[n]\right\} &= z^{-2}\frac{\frac{-z}{\sqrt{2}}}{z^2 - \sqrt{2}z + 1} + 1 + \frac{\frac{1}{\sqrt{2}}}{z} \\
&= \frac{-\frac{1}{\sqrt{2}} + z^3 - \sqrt{2}z^2 + z + \frac{1}{\sqrt{2}}z^2 - z + \frac{1}{\sqrt{2}}}{z(z^2 - \sqrt{2}z + 1)} \\
&= \frac{z\left(z - \frac{1}{\sqrt{2}}\right)}{z^2 - \sqrt{2}z + 1}.
\end{aligned}
$$

As expected, this result matches pair 11a of Table 5.1.

Solution 5.2-11

Application of the time-reversal property to pair 6 yields

$$
\beta^{-n}u[-n] \iff \frac{1/z}{1/z - \beta} = \frac{1}{1 - \beta z}, \qquad |z| < 1/\beta.
$$

Moreover,

$$
\beta^{-n}u[-n-1] = \beta^{-n}u[-n] - \delta[n].
$$

Hence,

$$
\beta^{-n}u[-n-1] \iff \frac{1}{1 - \beta z} - 1 = \frac{\beta z}{1 - \beta z}, \qquad |z| < 1/\beta.
$$

Setting $\beta = 1/\gamma$, we obtain the desired result of

$$
\gamma^n u[-n-1] = \frac{-z}{z - \gamma} \qquad |z| < |\gamma|.
$$

Solution 5.2-12

(a)

$$
(-1)^n x[n] \iff \sum_{n=0}^{\infty}\frac{(-1)^n x[n]}{z^n} = \sum_{n=0}^{\infty}\frac{x[n]}{(-z)^n} = X[-z]
$$

(b) Application of (a) to pair 6 of Table 5.1 yields

$$
(-1)^n \gamma^n u[n] = (-\gamma)^n u[n] \iff \frac{-z}{-z - \gamma} = \frac{z}{z + \gamma}.
$$

(c) (i)

$$
2^{n-1}u[n] = \frac{1}{2}2^n u[n] \iff \frac{1}{2}\frac{z}{z - 2}
$$

Application of (b) to this result yields

$$(-2)^{n-1}u[n] = -\frac{1}{2}(-2)^n u[n] \iff -\frac{1}{2}\frac{z}{z+2}.$$

Hence,

$$x_i[n] = \left[2^{n-1} - (-2)^{n-1}\right] u[n] \iff X_i[z] = \frac{1}{2}\left[\frac{z}{z-2} + \frac{z}{z+2}\right] = \frac{z^2}{z^2 - 4}.$$

(ii) To begin, we note that
$$\gamma^n \cos \pi n u[n] = (-\gamma)^n u[n].$$

Hence,

$$x_{ii}[n] = \gamma^n \cos \pi n u[n] \iff X_{ii}[z] = \frac{z}{z+\gamma}$$

Solution 5.2-13

(a) To begin, notice that we can represent the accumulation of $x[n]$ as

$$\sum_{k=0}^{n} x[k] = \sum_{k=0}^{\infty} x[k]u[n-k] = x[n] * u[n] = u[n] * x[n].$$

Using the time-convolution property and pair 2 of Table 5.1 yields

$$\sum_{k=0}^{n} x[k] = u[n] * x[n] \iff \frac{z}{z-1}X[z].$$

This is the desired result.

(b) From pair 1 of Table 5.1, we know that $x[n] = \delta[n]$ has z-transform $X[z] = 1$. To derive pair 2, we first note that

$$u[n] = \sum_{k=0}^{n} \delta[n] = \sum_{k=0}^{n} x[n].$$

Using the results of part (a), we see that

$$u[n] = \sum_{k=0}^{n} x[n] \iff \frac{z}{z-1}(1) = \frac{z}{z-1}.$$

This matches pair 2 of Table 5.1, as desired.

Solution 5.2-14

(a) In the time-domain, $\frac{z^2}{(z-0.75)^2}$ is a convolution of two causal, decaying exponentials. Thus, either plot 1 or plot 13 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z^2}{(z-0.75)^2} = 1$. Thus,

$$\text{Plot 13 corresponds to } \frac{z^2}{(z-0.75)^2}.$$

(b) In the time-domain, $\frac{z^2-0.9z/\sqrt{2}}{z^2-0.9\sqrt{2}z+0.81}$ is a decaying sinusoid. Thus, either plot 8 or plot 9 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z^2-0.9z/\sqrt{2}}{z^2-0.9\sqrt{2}z+0.81} = 1$. Thus,

$$\text{Plot 9 corresponds to } \frac{z^2 - 0.9z/\sqrt{2}}{z^2 - 0.9\sqrt{2}z + 0.81}.$$

(c) Note that $\sum_{k=0}^{4} z^{-2k} = 1 + z^{-2} + z^{-4} + z^{-6} + z^{-8}$. Thus, the time-domain signal is $\delta[n] + \delta[n-2] + \delta[n-4] + \delta[n-6] + \delta[n-8]$ and

$$\text{Plot 18 corresponds to } \sum_{k=0}^{4} z^{-2k}.$$

(d) By inspection, $\frac{z^{-5}}{1-z^{-1}}$ corresponds to a unit step that is shifted to the right by five. Thus,

$$\text{Plot 10 corresponds to } \frac{z^{-5}}{1 - z^{-1}}.$$

(e) Using synthetic division on $\frac{z^2}{z^4-1}$ yields $(z^{-2} + z^{-6} + z^{-10} + \dots)$. In the time-domain, the first non-zero term therefore occurs at $n = 2$. Thus,

$$\text{Plot 15 corresponds to } \frac{z^2}{z^4 - 1}.$$

(f) In the time-domain, $\frac{0.75z}{(z-0.75)^2}$ is a convolution of two causal, decaying exponentials. Thus, either plot 1 or plot 13 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{0.75z}{(z-0.75)^2} = 0$. Thus,

$$\text{Plot 1 corresponds to } \frac{0.75z}{(z - 0.75)^2}.$$

(g) In the time-domain, $\frac{z^2-z/\sqrt{2}}{z^2-\sqrt{2}z+1}$ has the form of a sinusoid. Thus, either plot 3 or plot 4 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z^2-z/\sqrt{2}}{z^2-\sqrt{2}z+1} = 1$. Thus,

$$\text{Plot 4 corresponds to } \frac{z^2 - z/\sqrt{2}}{z^2 - \sqrt{2}z + 1}.$$

(h) The apparent repeated root at $z = 1$ of $\frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2}$ suggests a signal that grows linearly in the time-domain. Thus, plot 6 or plot 12 are possible. To distinguish between the two, first determine whether or not a root at $z = 1$ really exists. First, $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = \lim_{z\to 1} \frac{z^5-5z+4}{5(z^3-z^2)^2} = \lim_{z\to 1} \frac{z^5-5z+4}{5(z^6-2z^5+z^4)} = 0/0$. This is indeterminant so use L'Hospital's rule: $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = \lim_{z\to 1} \frac{\frac{d}{dz}(z^5-5z+4)}{\frac{d}{dz}(5z^6-10z^5+5z^4)} = \lim_{z\to 1} \frac{5z^4-5}{30z^5-50z^4+20z^3} = 0/0$. This too is indeterminant so use L'Hospital's rule again: $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = \lim_{z\to 1} \frac{\frac{d^2}{dz^2}(z^5-5z+4)}{\frac{d^2}{dz^2}(5z^6-10z^5+5z^4)} = \lim_{z\to 1} \frac{20z^3}{150z^4-200z^3+60z^2} = 20/10 = 2$. Since $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = 2 \neq \infty$, no root exists at $z = 1$ and

$$\text{Plot 6 corresponds to } \frac{z^{-1} - 5z^{-5} + 4z^{-6}}{5(1 - z^{-1})^2}.$$

(i) In the time-domain, $\frac{z}{z-1.1}$ is a growing exponential due to the root outside the unit circle. Thus, plot 16 or plot 17 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z}{z-1.1} = 1$. Thus,

$$\text{Plot 16 corresponds to } \frac{z}{z - 1.1}.$$

(j) In the time-domain, $\frac{0.25z^{-1}}{(1-z^{-1})(1-0.75z^{-1})}$ is the convolution of a decaying sinusoid and a unit step. Thus, plot 7 or plot 11 are possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{0.25z^{-1}}{(1-z^{-1})(1-0.75z^{-1})} = 0$. Thus,

$$\text{Plot 7 corresponds to } \frac{0.25z^{-1}}{(1-z^{-1})(1-0.75z^{-1})}.$$

Solution 5.2-15

To begin, we express $x[n]$ using the sifting property and show the corresponding z-transform as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \iff X[z] = \sum_{k=-\infty}^{\infty} x[k]z^{-k}.$$

This notation is particularly convenient for upsampling, which replaces $\delta[n-k]$ with $\delta[n-Nk]$. Thus, upsampling $x[n]$ by N produces $y[n]$ as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-Nk] \iff Y[z] = \sum_{k=-\infty}^{\infty} x[k]z^{-Nk} = X[z^N].$$

Clearly,

$$Y[z] = X[z^N], \text{ where ROC } R_y \text{ is } R_x \text{ with } z \text{ replaced by } z^N.$$

Solution 5.3-1

Here, we use z-transform techniques to find the output $y[n]$ of an LTID system specified by the equation $y[n] - \frac{1}{3}y[n-1] = x[n-1]$ when the initial condition is $y[-1] = 2$ and the input is $x[n] = -u[n]$. First, we take the unilateral z-transform of the difference equation to obtain

$$Y[z] - \frac{1}{3}(z^{-1}Y[z] + y[-1]) = z^{-1}X[z] = -\frac{1}{z-1}.$$

Rearranging, we obtain

$$Y[z](1 - \frac{1}{3}z^{-1}) = -\frac{1}{z-1} + \frac{2}{3}.$$

Solving for $Y[z]/z$, we obtain

$$\frac{Y[z]}{z} = \frac{-1}{(z-\frac{1}{3})(z-1)} + \frac{\frac{2}{3}}{z-\frac{1}{3}} = \frac{\frac{3}{2}}{z-\frac{1}{3}} + \frac{-\frac{3}{2}}{z-1} + \frac{\frac{2}{3}}{z-\frac{1}{3}} = \frac{\frac{13}{6}}{z-\frac{1}{3}} + \frac{-\frac{3}{2}}{z-1}.$$

Inverting, the solution is therefore

$$y[n] = \left(\frac{13}{6} \left(\frac{1}{3} \right)^n - \frac{3}{2} \right) u[n].$$

Solution 5.3-2

This problem considers an LTID system $y[n] - y[n-2] = x[n]$ with $y[-1] = 0$, $y[-2] = 1$, and $x[n] = u[n]$.

(a) Taking the unilateral z-transform of the difference equation, we obtain

$$Y[z](1 - z^{-2}) - z^{-1}y[-1] - y[-2] = X[z] = \frac{z}{z-1}.$$

Rearranging yields

$$Y[z](1 - z^{-2}) = \frac{z}{z-1} + 1 = \frac{2z-1}{z-1}.$$

Solving for $Y[z]$ we obtain

$$Y[z] = \left(\frac{2z-1}{z-1} \right) \frac{1}{1-z^{-2}} = \frac{z^2(2z-1)}{(z-1)(z^2-1)}.$$

Written in factored form, we obtain

$$Y[z] = \frac{2(z)(z)(z - \frac{1}{2})}{(z - 1)(z - 1)(z + 1)}.$$

(b) Using the result from part (a), we see that

$$\frac{Y[z]}{z} = \frac{2z(z - \frac{1}{2})}{(z - 1)(z - 1)(z + 1)} = \frac{a_0}{(z - 1)^2} + \frac{a_1}{z - 1} + \frac{k_1}{z + 1}.$$

Using the Heaviside cover-up method, we see that

$$a_0 = \frac{2(1)(\frac{1}{2})}{2} = \frac{1}{2} \quad \text{and} \quad k_1 = \frac{2(-1)(-\frac{3}{2})}{(-2)(-2)} = \frac{3}{4}.$$

To find a_1, multiply both sides by $(z - 1)$ and let $z \to \infty$,

$$2 = a_1 + 0 + k_1 \quad \Rightarrow \quad a_1 = \frac{5}{4}.$$

Thus,

$$Y[z] = \frac{\frac{1}{2}z}{(z - 1)^2} + \frac{\frac{5}{4}z}{z - 1} + \frac{\frac{3}{4}z}{z + 1}.$$

Using Table 5.1 to invert, the result is

$$y[n] = \left(\tfrac{1}{2}n + \tfrac{5}{4} + \tfrac{3}{4}(-1)^n\right) u[n].$$

Solution 5.3-3

We model the system as

$$y[n + 1] - \gamma y[n] = x[n + 1]$$

with $y[0] = -M$, $x[n] = Pu[n - 1]$. Thus,

$$X[z] = \frac{P}{z - 1}$$

$$y[n] \Longleftrightarrow Y[z] \qquad y[n + 1] \Longleftrightarrow zY[z] + Mz.$$

The z-transform of the system equation is

$$zY[z] + Mz - \gamma Y[z] = \frac{Pz}{z - 1}$$

$$(z - \gamma)Y[z] = -M + \frac{Pz}{z - 1}$$

and

$$Y[z] = \frac{-Mz}{z - \gamma} + \frac{Pz}{(z - \gamma)(z - 1)}$$

$$\frac{Y[z]}{z} = \frac{-M}{z - \gamma} + \frac{P}{(z - \gamma)(z - 1)} = \frac{-M}{z - \gamma} + \frac{P}{\gamma - 1}\left[\frac{1}{z - \gamma} - \frac{1}{z - 1}\right]$$

$$Y[z] = -M\frac{z}{z - \gamma} + \frac{P}{\gamma - 1}\left[\frac{z}{z - \gamma} - \frac{z}{z - 1}\right]$$

$$y[n] = \left[-M\gamma^n + \frac{P(\gamma^n - 1)}{r}\right] u[n] \qquad r = \gamma - 1$$

The loan balance is zero for $n = N$, that is, $y[N] = 0$. Setting $n = N$ in the above equation we obtain

$$y[N] = \left[-M\gamma^N + \frac{P(\gamma^N - 1)}{r} \right] = 0.$$

This yields

$$P = \frac{r\gamma^N}{\gamma^N - 1} M = \frac{rM}{1 - (1 + r)^{-N}}.$$

Solution 5.3-4

Here,

$$H[z] = \frac{Y[z]}{X[z]} = \frac{2z - 2}{z - \frac{1}{2}} \quad \Rightarrow \quad (z - \tfrac{1}{2})Y[z] = (2z - 2)X[z].$$

Inverting, the system difference equation is

$$y[n] - \tfrac{1}{2}y[n-1] = 2x[n] - 2x[n-1].$$

The zero-input response requires we set $x[n] = 0$. Thus,

$$y_{\text{zir}}[n] - \tfrac{1}{2}y_{\text{zir}}[n-1] = 0.$$

Taking the unilateral z-transform yields

$$Y_{\text{zir}}[z] - \tfrac{1}{2}\left(z^{-1}Y_{\text{zir}}[z] + y[-1] \right) - 0.$$

Using $y[-1] = 1$, we see that

$$Y_{\text{zir}}[z]\left(1 - \tfrac{1}{2}z^{-1} \right) = \tfrac{1}{2}y[-1] = \tfrac{1}{2}.$$

Thus,

$$Y_{\text{zir}}[z] = \frac{\frac{1}{2}z}{z - \frac{1}{2}}$$

and

$$y_{\text{zir}}[n] = \tfrac{1}{2}(\tfrac{1}{2})^n u[n].$$

Solution 5.3-5

(a) Because part (b) requires us to separate the response into zero-input and zero-state components, we shall start with the delay operator form of the equations, as

$$y[n] + 2y[n-1] = x[n].$$

To determine the initial condition $y[-1]$, we set $n = 0$ in this equation and substitute $y[0] = 1$ to obtain

$$1 + 2y[-1] = x[0] = e \quad \Longrightarrow y[-1] = (e - 1)/2.$$

The z-transform of the delay form of equation yields

$$Y[z] + 2\left[\frac{1}{z}Y[z] + \frac{e-1}{2} \right] = \frac{ez}{z - e^{-1}}.$$

Rearranging the terms yields

$$\frac{Y[z]}{z} = \frac{1}{z + 2}\left[(1 - e) + \frac{ez}{z - e^{-1}} \right].$$

The term $(1-e)$ on the right-hand side is due to the initial condition, and hence represents the zero-input component. The second term on the right-hand side represents the zero-state component of the response. Thus,

$$
\begin{aligned}
\frac{Y[z]}{z} &= \frac{1-e}{z+2} + \left[\frac{ez}{(z-e^{-1})(z+2)}\right] \\
&= \frac{1-e}{z+2} + \frac{2e^2}{(2e+1)(z+2)} + \frac{e}{(2e+1)(z-e^{-1})}
\end{aligned}
$$

and

$$
Y[z] = (1-e)\frac{z}{z+2} + \frac{2e^2}{2e+1}\frac{z}{z+2} + \frac{e}{2e+1}\frac{z}{z-e^{-1}}.
$$

The first term on the right-hand side is the zero-input component and the remaining two terms represent the zero-state component. Thus,

$$
y[n] = \underbrace{(1-e)(-2)^n u[n]}_{\text{zir}} + \underbrace{\frac{2e^2}{2e+1}(-2)^n u[n] + \frac{e}{2e+1}e^{-n}u[n]}_{\text{zsr}}.
$$

The total response is

$$
y[n] = \frac{1}{2e+1}\left[(e+1)(-2)^n + e^{-(n-1)}\right]u[n].
$$

(b) Referring to part (a), we see that

$$
y_{\text{zir}}[n] = (1-e)(-2)^n u[n]
$$

and

$$
y_{\text{zsr}}[n] = \frac{2e^2}{2e+1}(-2)^n u[n] + \frac{e}{2e+1}e^{-n}u[n].
$$

Solution 5.3-6

(a) Taking the z-transform of the difference equation (0 ICs) yields

$$
Y_{\text{zsr}}[z] - \tfrac{1}{4}z^{-2}Y_{\text{zsr}}[z] = z^{-1}X[z].
$$

Since $x[n] = 3u[n-5]$, we see that

$$
Y_{\text{zsr}}[z]\left(1 - \tfrac{1}{4}z^{-2}\right) = z^{-1}3z^{-5}\frac{z}{z-1}.
$$

Solving for $Y_{\text{zsr}}[z]$ yields

$$
Y_{\text{zsr}}[z] = \frac{3z^{-5}}{(z-1)(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})} = \frac{3z^{-3}}{(z-1)(z-\frac{1}{2})(z+\frac{1}{2})}.
$$

The modified partial fraction expansion is thus

$$
\frac{Y_{\text{zsr}}[z]}{z} = 3z^{-4}\left(\frac{\frac{4}{3}}{z-1} + \frac{-2}{z-\frac{1}{2}} + \frac{\frac{2}{3}}{z+\frac{1}{2}}\right).
$$

Inverting, we obtain

$$
y_{\text{zsr}}[n] = \left(4 - 6(\tfrac{1}{2})^{n-4} + 2(-\tfrac{1}{2})^{n-4}\right)u[n-4].
$$

(b) Taking the unilateral z-transform of the difference equation with $x[n] = 0$ yields

$$Y_{\text{zir}}[z] - \tfrac{1}{4}\left[z^{-2}Y_{\text{zir}}[z] + z^{-2}(y[-1]z^1 + y[-2]z^2)\right] = 0.$$

Substituting $y_{\text{zir}}[-2] = y_{\text{zir}}[-1] = 1$ and then rearranging yield

$$Y_{\text{zir}}[z]\left(1 - \tfrac{1}{4}z^{-2}\right) = \tfrac{1}{4} + \tfrac{1}{4}z^{-1}.$$

Thus,

$$\frac{Y_{\text{zir}}[z]}{z} = \frac{\tfrac{1}{4}(z+1)}{z^2 - \tfrac{1}{4}} = \frac{\tfrac{3}{8}}{z - \tfrac{1}{2}} + \frac{-\tfrac{1}{8}}{z + \tfrac{1}{2}}.$$

Inverting yields

$$y_{\text{zir}}[n] = \left(\tfrac{3}{8}(\tfrac{1}{2})^n - \tfrac{1}{8}(-\tfrac{1}{2})^n\right)u[n].$$

Solution 5.3-7

(a) The system equation in delay form is

$$2y[n] - 3y[n-1] + y[n-2] = 4x[n] - 3x[n-1].$$

Also

$$y[n] \Longleftrightarrow Y[z], \quad y[n-1] \Longleftrightarrow \frac{1}{z}Y[z], \quad y[n-2] \Longleftrightarrow \frac{1}{z^2}Y[z] + 1,$$

$$x[n] \Longleftrightarrow X[z] = \frac{z}{z - 0.25}, \quad \text{and} \quad x[n-1] \Longleftrightarrow \frac{1}{z - 0.25}.$$

The z-transform of the equation is

$$2Y[z] - \frac{3}{z}Y[z] + \frac{1}{z^2}Y[z] + 1 = \frac{4z}{z - 0.25} - \frac{3}{z - 0.25} = \frac{4z - 3}{z - 0.25}$$

or

$$\left(2 - \frac{3}{z} + \frac{1}{z^2}\right)Y[z] = -1 + \frac{4z - 3}{z - 0.25} = \frac{3z - 2.75}{z - 0.25}.$$

Thus,

$$\frac{Y[z]}{z} = \frac{z(3z - 2.75)}{(2z^2 - 3z + 1)(z - 0.25)}$$

$$= \frac{z(3z - 2.75)}{2(z - 0.5)(z - 1)(z - 0.25)}$$

$$= \frac{5/2}{z - 1/2} + \frac{1/3}{z - 1} - \frac{4/3}{z - 0.25}$$

$$y[n] = \left[\frac{1}{3} + \frac{5}{2}(0.5)^n - \frac{4}{3}(0.25)^n\right]u[n]$$

$$= \left[\frac{1}{3} + \frac{5}{2}(2)^{-n} - \frac{4}{3}(4)^{-n}\right]u[n].$$

(b) From part (a), we have

$$\left(2 - \frac{3}{z} + \frac{1}{z^2}\right)Y[z] = \underbrace{-1}_{\text{zero-input}} + \underbrace{\frac{4z - 3}{z - 0.25}}_{\text{zero-state}}$$

$$\frac{2z^2 - 3z + 1}{z^2}Y[z] = -1 + \frac{4z - 3}{z - 0.25}.$$

Thus,

$$\frac{Y[z]}{z} = \underbrace{\frac{-z}{2(z - 0.5)(z - 1)}}_{\text{zero-input}} + \underbrace{\frac{z(4z - 3)}{2(z - 0.5)(z - 1)(z - 0.25)}}_{\text{zero-state}}$$

$$= \frac{0.5}{z - 0.5} - \frac{1}{z - 1} + \frac{2}{z - 0.5} + \frac{4}{3}\frac{1}{z - 1} - \frac{4}{3}\frac{1}{z - 0.25}$$

and

$$Y[z] = 0.5\frac{z}{z - 0.5} - \frac{z}{z - 1} + 2\frac{z}{z - 0.5} + \frac{4}{3}\frac{z}{z - 1} - \frac{4}{3}\frac{z}{z - 0.25}.$$

Inverting yields

$$y[n] = \underbrace{\left[\frac{1}{2}(0.5)^n - 1\right]u[n]}_{y_{\text{zir}}[n]} + \underbrace{\left[2(0.5)^n + \frac{4}{3} - \frac{4}{3}(0.25)^n\right]u[n]}_{y_{\text{zsr}}[n]}.$$

(c) Given the total response, it is easy to separate the transient and steady-state components as

$$y[n] = \underbrace{\left[2.5(0.5)^n - \frac{4}{3}(0.5)^n\right]u[n]}_{y_{\text{transient}}[n]} + \underbrace{\frac{1}{3}u[n]}_{y_{\text{steady-state}}[n]}$$

Solution 5.3-8

(a) For initial conditions $y[0]$, $y[1]$, we require the difference equation to be in advance form:

$$2y[n + 2] - 3y[n + 1] + y[n] = 4x[n + 2] - 3x[n + 1].$$

Also,

$$y[n] \Longleftrightarrow Y[z], \quad y[n + 1] \Longleftrightarrow zY[z] - \frac{3}{2}z, \quad y[n + 2] \Longleftrightarrow z^2Y[z] - \frac{3}{2}z^2 - \frac{35}{4}z,$$

$$x[n] \Longleftrightarrow X[z] = \frac{z}{z - 0.25}, \quad x[n + 1] \Longleftrightarrow zX[z] - z = \frac{0.25z}{z - 0.25},$$

and

$$x[n + 2] \Longleftrightarrow z^2X[z] - z^2 - \frac{1}{4}z = \frac{z}{16(z - 0.25)}.$$

The z-transform of the equation is

$$2\left[z^2Y[z] - \frac{3}{2}z^2 - \frac{35}{4}z\right] - 3\left[zY[z] - \frac{3}{2}z\right] + Y[z] = \frac{-z/2}{z - 0.25}$$

or

$$(2z^2 - 3z + 1)Y[z] = \frac{z(3z^2 + 12.25z - 3.75)}{(z - 0.25)}.$$

Thus,

$$\frac{Y[z]}{z} = \frac{3z^2 + 12.25z - 3.75}{2(z - 0.25)(z - 1)(z - 0.5)} = \frac{46/3}{z - 1} - \frac{4/3}{z - 0.25} - \frac{25/2}{z - 0.5}$$

and

$$Y[z] = \frac{46}{3}\frac{z}{z-1} - \frac{4}{3}\frac{z}{z-0.25} - \frac{25}{2}\frac{z}{z-0.5}.$$

Inverting, we obtain

$$y[n] = \left[\frac{46}{3} - \frac{4}{3}(0.25)^n - \frac{25}{2}(0.5)^n\right]u[n].$$

(b) The solution to Prob. 5.3-7 determined a system output of

$$\underbrace{\left[\frac{1}{2}(0.5)^n - 1\right]u[n]}_{y_{\text{zir}}[n]} + \underbrace{\left[2(0.5)^n + \frac{4}{3} - \frac{4}{3}(0.25)^n\right]u[n]}_{y_{\text{zsr}}[n]}.$$

Since the input and system are the same in this problem as they were in Prob. 5.3-7, the zero-state response remains unchanged as

$$y_{\text{zsr}}[n] = \left[2(0.5)^n + \tfrac{4}{3} - \tfrac{4}{3}(0.25)^n\right]u[n].$$

The zero-input response is just $y_{\text{zsr}}[n]$ subtracted from the total response found in part (a),

$$y_{\text{zir}}[n] = \left[\tfrac{42}{3} - \tfrac{29}{2}(0.5)^n\right]u[n].$$

(c) Using the result in part (a), it is easy to separate the transient and steady-state components as

$$y[n] = \underbrace{\left[-\frac{4}{3}(0.25)^n - \frac{25}{2}(0.5)^n\right]u[n]}_{y_{\text{transient}}[n]} + \underbrace{\frac{46}{3}u[n]}_{y_{\text{steady-state}}[n]} \quad .$$

Solution 5.3-9

(a) System equation in delay form is

$$4y[n] + 4y[n-1] + y[n-2] = x[n-1].$$

Also,

$$y[n] \Longleftrightarrow Y[z], \quad y[n-1] \Longleftrightarrow \frac{1}{z}Y[z], \quad y[n-2] \Longleftrightarrow \frac{1}{z^2}Y[z] + 1$$

$$x[n] \Longleftrightarrow \frac{z}{z-1}, \quad \text{and} \quad x[n-1] \Longleftrightarrow \frac{1}{z-1} \quad (x[-1]=0).$$

The z-transform of the system equation is

$$4Y[z] + \frac{4}{z}Y[z] + \frac{1}{z^2}Y[z] + 1 = \frac{1}{z-1} \tag{5.3-9a}$$

$$\frac{4z^2 + 4z + 1}{z^2}Y[z] = \frac{2-z}{z-1}. \tag{5.3-9b}$$

Thus,

$$\frac{Y[z]}{z} = \frac{z(2-z)}{4(z-1)(z^2+z+0.25)} = \frac{z(2-z)}{4(z-1)(z+0.5)^2}$$

$$= \frac{1}{4}\left[\frac{4/9}{z-1} - \frac{13/9}{z+0.5} + \frac{5/6}{(z+0.5)^2}\right]$$

and

$$Y[z] = \frac{1}{4}\left[\frac{4}{9}\frac{z}{z-1} - \frac{13}{9}\frac{z}{z+0.5} + \frac{5}{6}\frac{z}{(z+0.5)^2}\right].$$

Inverting yields

$$y[n] = \left[\frac{1}{9} - \frac{13}{36}(-0.5)^n - \frac{5}{12}n(-0.5)^n\right]u[n].$$

(b) To find the zero-input and the zero-state components, we observe that the only term arising because of the initial conditions is 1 on the left-hand side of Eq. (5.3-9a). Hence, we can rewrite Eq. (5.3-9b) with explicit zero-input and zero-state components as

$$\frac{4z^2 + 4z + 1}{z^2}Y[z] = -1 + \frac{1}{z-1}$$

Here, -1 on the right-hand side represents the zero-input term and the second term on the right-hand side represents the zero-state component. Rearranging the equation, we obtain

$$\frac{Y[z]}{z} = \frac{z}{4(z+0.5)^2}\left[-1 + \frac{1}{z-1}\right]$$

$$= \underbrace{\frac{-z}{4(z+0.5)^2}}_{\text{zero-input}} + \underbrace{\frac{z}{4(z-1)(z+0.5)^2}}_{\text{zero-state}}$$

$$= \underbrace{\frac{-1/4}{z+0.5} + \frac{1/8}{(z+0.5)^2}}_{\text{zero-input}} + \underbrace{\frac{1/9}{z-1} - \frac{1/9}{z+0.5} + \frac{1/12}{(z+0.5)^2}}_{\text{zero-state}}$$

Therefore

$$Y[z] = \underbrace{\frac{(-1/4)z}{z+0.5} + \frac{(1/8)z}{(z+0.5)^2}}_{\text{zero-input}} + \underbrace{\frac{(1/9)z}{z-1} - \frac{(1/9)z}{z+0.5} + \frac{(1/12)z}{(z+0.5)^2}}_{\text{zero-state}}$$

Inverting yields

$$y[n] = \underbrace{\left[\frac{-1}{4}(-0.5)^n - \frac{n(-0.5)^n}{4}\right]u[n]}_{y_{\text{zir}}[n]} + \underbrace{\left[\frac{1}{9} - \frac{(-0.5)^n}{9} - \frac{n(-0.5)^n}{6}\right]u[n]}_{y_{\text{zsr}}[n]}.$$

(c) The terms which vanish as $n \to \infty$ correspond to the transient component and the terms which do not vanish correspond to the steady-state component. Hence

$$y[n] = \underbrace{\left[-\frac{13}{36}(-0.5)^n - \frac{5}{12}n(-0.5)^n\right]u[n]}_{y_{\text{transient}}[n]} + \underbrace{\frac{1}{9}u[n]}_{y_{\text{steady−state}}[n]}.$$

Solution 5.3-10

The system in delay form is

$$y[n] - 3y[n-1] + 2y[n-2] = x[n-1].$$

Also,

$$y[n] \Longleftrightarrow Y[z], \quad y[n-1] \Longleftrightarrow \frac{1}{z}Y[z] + 2, \quad y[n-2] \Longleftrightarrow \frac{1}{z^2}Y[z] + \frac{2}{z} + 3,$$

$$x[n] \Longleftrightarrow X[z], \quad x[n-1] \Longleftrightarrow \frac{1}{z}X[z], \quad \text{and} \quad X[z] = \frac{z}{z-3}.$$

The z-transform of the system equation is

$$Y[z] - 3\left[\frac{1}{z}Y[z] + 2\right] + 2\left[\frac{1}{z^2}Y[z] + \frac{2}{z} + 3\right] = \frac{1}{z-3}$$

$$\left(1 - \frac{3}{z} + \frac{2}{z^2}\right)Y[z] = -\frac{4}{z} + \frac{1}{z-3} = \frac{-3z+12}{z(z-3)}.$$

Thus,

$$\frac{Y[z]}{z} = \frac{-3z+12}{(z^2-3z+2)(z-3)} = \frac{-3z+12}{(z-1)(z-2)(z-3)} = \frac{9/2}{z-1} - \frac{6}{z-2} + \frac{3/2}{z-3}$$

and

$$Y[z] = \frac{9}{2}\frac{z}{z-1} - 6\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}.$$

Inverting yields

$$y[n] = \left[\frac{9}{2} - 6(2)^n + \frac{3}{2}(3)^n\right]u[n].$$

Solution 5.3-11

The system equation in delay form is

$$y[n] - 2y[n-1] + 2y[n-2] = x[n-2].$$

Also,

$$y[n] \Longleftrightarrow Y[z], \quad y[n-1] \Longleftrightarrow \frac{1}{z}Y[z] + 1, \quad y[n-2] \Longleftrightarrow \frac{1}{z^2}Y[z] + \frac{1}{z},$$

$$x[n-2] \Longleftrightarrow \frac{1}{z^2}X[z], \quad \text{and} \quad X[z] = \frac{z}{z-1}$$

The z-transform of the difference equation is

$$Y[z] - 2\left[\frac{1}{z}Y[z] + 1\right] + 2\left[\frac{1}{z^2}Y[z] + \frac{1}{z}\right] = \frac{1}{z(z-1)}.$$

Thus,

$$\frac{(z^2-2z+2)}{z^2}Y[z] = \frac{2z^2-4z+3}{z(z-1)},$$

$$\frac{Y[z]}{z} = \frac{2z^2-4z+3}{(z-1)(z^2-2z+2)} = \frac{1}{z-1} + \frac{z-1}{z^2-2z+2},$$

and

$$Y[z] = \frac{z}{z-1} + \frac{z(z-1)}{z^2-2z+2}.$$

For the second fraction on the right-hand side, we use pair 12c with $A = 1$, $B = -1$, $a = -1$, $|\gamma|^2 = 2$. This yields $r = 1$, $\beta = \frac{\pi}{4}$, and $\theta = 0$. Therefore

$$y[n] = \left[1 + (\sqrt{2})^n \cos(\frac{\pi}{4}n)\right]u[n].$$

Solution 5.3-12

(a) Here,

$$H[z] = \frac{21(z^2+1)}{16(z^2 + \frac{1}{4}z - \frac{3}{8})} = \frac{\frac{21}{16} + \frac{21}{16}z^{-2})}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{Y[z]}{X[z]}.$$

Cross multiplying and inverting yields the desired difference equation of

$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = \frac{21}{16}x[n] + \frac{21}{16}x[n-2].$$

(b) Since $H[z]$ and $h[n]$ form a transform pair, we can obtain the impulse response $h[n]$ directly from $H[z]$. Using modified fractions we obtain

$$\frac{H[z]}{z} = \frac{\frac{21}{16}(z^2+1)}{z(z^2+\frac{1}{4}z-\frac{3}{8})} = \frac{\frac{21}{16}(-\frac{1}{3})}{z} + \frac{\frac{21}{16}(\frac{\frac{5}{4}}{\frac{1}{2}(\frac{5}{4})})}{z-\frac{1}{2}} + \frac{\frac{21}{16}(\frac{\frac{25}{16}}{-\frac{3}{4}(-\frac{5}{4})})}{z+\frac{3}{4}}.$$

Thus,

$$H[z] = -\frac{7}{2} + \frac{21}{8}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{35}{16}\left(\frac{z}{z+\frac{3}{4}}\right).$$

Inverting, we obtain the impulse response as

$$h[n] = \tfrac{7}{2}\delta[n] + \tfrac{21}{8}(\tfrac{1}{2})^n u[n] + \tfrac{35}{16}(-\tfrac{3}{4})^n u[n].$$

(c) Using the result from part (a) with $x[n] = 0$ yields

$$y_{\text{zir}}[n] + \tfrac{1}{4}y_{\text{zir}}[n-1] - \tfrac{3}{8}y_{\text{zir}}[n-2] = 0.$$

Taking the unilateral z-transform yields

$$Y_{\text{zir}}[z] + \tfrac{1}{4}\left(z^{-1}Y_{\text{zir}}[z] + y_{\text{zir}}[-1]\right) - \tfrac{3}{8}\left(z^{-2}Y_{\text{zir}}[z] + z^{-1}y_{\text{zir}}[-1] + y_{\text{zir}}[-2]\right) = 0.$$

Substituting $y_{\text{zir}}[-1] = 16$ and $y_{\text{zir}}[-2] = 8$ yields

$$\left(1 + \tfrac{1}{4}z^{-1} - \tfrac{3}{8}z^{-2}\right)Y_{\text{zir}}[z] = -1 + 6z^{-1}.$$

Using modified fractions we obtain

$$\frac{Y_{\text{zir}}[z]}{z} = \frac{-z+6}{(z-\frac{1}{2})(z+\frac{3}{4})} = \frac{\frac{\frac{11}{2}}{\frac{5}{4}}}{z-\frac{1}{2}} + \frac{\frac{\frac{27}{4}}{-\frac{5}{4}}}{z+\frac{3}{4}}.$$

Thus,

$$Y_{\text{zir}}[z] = \frac{22}{5}\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{27}{5}\left(\frac{z}{z+34}\right).$$

Inverting, we obtain $y_{\text{zir}}[n]$ as

$$y_{\text{zir}}[n] = \tfrac{22}{5}(\tfrac{1}{2})^n u[n] - \tfrac{27}{5}(-\tfrac{3}{4})^n u[n].$$

Solution 5.3-13

(a) By inspection of $y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = \frac{3}{2}x[n-1] + \frac{3}{2}x[n-2]$, the transfer function is

$$H[z] = \frac{\frac{3}{2}z + \frac{3}{2}}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{\frac{3}{2}(z+1)}{(z-\frac{1}{2})(z-\frac{1}{3})}.$$

Figure S5.3-13 shows the corresponding pole-zero plot. Note that one zero is at ∞.

(b) Setting $x[n] = 0$, we obtain $y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 0$. Taking the unilaterial z-transform yields

$$Y[z] - \tfrac{5}{6}(z^{-1}Y[z] + y[-1]) + \tfrac{1}{6}(z^{-2}Y[z] + z^{-1}y[-1] + y[-2]) = 0.$$

Thus,

$$\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right)Y[z] = \frac{5}{3} + \frac{1}{3} - \frac{1}{3}z^{-1}, \qquad Y[z] = \frac{2z^2 - \frac{1}{3}z}{(z-\frac{1}{2})(z-\frac{1}{3})},$$

$$\frac{Y[z]}{z} = \frac{2z - \frac{1}{3}}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{\frac{\frac{2}{3}}{\frac{1}{6}}}{z-\frac{1}{2}} + \frac{\frac{\frac{1}{3}}{-\frac{1}{6}}}{z-\frac{1}{3}}, \qquad \text{and} \qquad Y[z] = \frac{4z}{z-\frac{1}{2}} - \frac{2z}{z-\frac{1}{3}}.$$

Inverting, we obtain

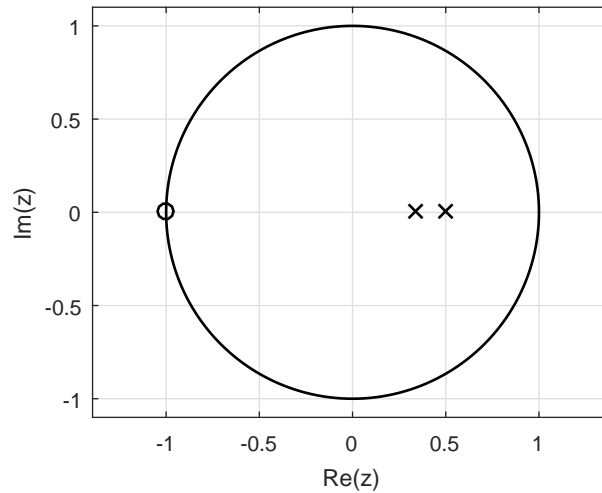$$z_{\text{zir}}[n] = \left(4(\tfrac{1}{2})^n - 2(\tfrac{1}{3})^n\right)u[n].$$

Figure S5.3-13

Solution 5.3-14

The equation in advance form is

$$y[n+2] + 2y[n+1] + 2y[n] = x[n+1] + 2x[n].$$

Further, we know that

$$y[n] \Longleftrightarrow Y[z], \quad y[n+1] \Longleftrightarrow zY[z], \quad y[n+2] \Longleftrightarrow z^2Y[z] - z,$$

$$x[n] \Longleftrightarrow X[z], \quad x[n+1] \Longleftrightarrow zX[z] - z, \quad \text{and} \quad X[z] = \frac{z}{z-e}.$$

The z-transform of the difference equation is

$$z^2Y[z] - z + 2zY[z] + 2Y[z] = \frac{z^2}{z-e} - z + \frac{2z}{z-e} = \frac{z(e+2)}{z-e}$$

$$(z^2 + 2z + 2)Y[z] = z + \frac{z(e+2)}{z-e} = \frac{z(z+2)}{z-e}.$$

Therefore,

$$\frac{Y[z]}{z} = \frac{z+2}{(z-e)(z^2+2z+2)} = \frac{0.318}{z-e} + \frac{-0.318z - 0.502}{z^2 + 2z + 2}$$

and

$$Y[z] = 0.318\frac{z}{z-e} - \frac{z(0.318z + 0.502)}{z^2 + 2z + 2}.$$

For the second fraction on the right-hand side, we use pair 12c with $A = 0.318$, $B = 0.502$, $a = 1$, $|\gamma|^2 = 2$ and

$$r = 0.367, \quad \beta = \cos^{-1}(\frac{-1}{\sqrt{2}}) = \frac{3\pi}{4}, \quad \text{and} \quad \theta = \tan^{-1}(\frac{-0.184}{0.318}) = -0.525.$$

Thus,

$$y[n] = \left[0.318(e)^n - 0.367(\sqrt{2})^n \cos(\frac{3\pi}{4}n - 0.525)\right] u[n].$$

Solution 5.3-15

In transform domain, $H[z] = z^{-1}\frac{2z/3}{z-1/3}$ and $Y[z] = z^{-1}\frac{-2z}{z+2}$. Since $Y[z] = H[z]X[z]$, we know $X[z] = Y[z]/H[z] = \frac{z^{-1}\frac{-2z}{z+2}}{z^{-1}\frac{2z/3}{z-1/3}}$. Thus, $X[z] = -3\frac{z-1/3}{z+2}$. Using tables, $x[n] = -3\left((-2)^n u[n] - \frac{1}{3}(-2)^{n-1}u[n-1]\right) = -3\left(-2(-2)^{n-1}u[n] - \frac{1}{3}(-2)^{n-1}u[n-1]\right)$ or

$$x[n] = -3\delta[n] + 7(-2)^{n-1}u[n-1].$$

Solution 5.3-16

A professor invests \$10 into a savings account that earns 0.5% interest compounded monthly (6.17% APY) and furthermore decides to supplement this initial investment with an additional \$5 deposit made every month, beginning the month immediately following the initial \$10 investment.

(a) Designating $y[n]$ as the account balance at month n, where $n = 0$ corresponds to the first month that interest is awarded (and that her \$5 deposits begin), the savings account can be modeled using a difference equation as

$$y[n] - 1.005y[n-1] = x[n], \quad \text{with } y[-1] = 10 \text{ and } x[n] = 5u[n].$$

(b) Taking the unilateral z-transform of the difference equation from part (a), we obtain

$$Y[z] - 1.005\left(z^{-1}Y[z] + y[-1]\right) = X[z] = \frac{5z}{z-1} = \frac{5}{1-z^{-1}}.$$

Thus,

$$Y[z]\left(1 - 1.005z^{-1}\right) = \frac{5}{z - z^{-1}} + 1.005y[-1]$$

and

$$Y[z] = \frac{5}{(1 - z^{-1})(z - 1.005z^{-1})} + \frac{1.005y[-1]}{1 - 1.005z^{-1}}.$$

Using a partial fraction expansion, we see that

$$Y[z] = \frac{-1000}{1 - z^{-1}} + \frac{1000}{1 - 1.005z^{-1}} + \frac{10.05}{1 - 1.005z^{-1}}.$$

Inverting, we obtain

$$y[n] = \left(1010.05(1.005)^n - 1000\right)u[n].$$

(c) Directly from the difference equation, we see that the system transfer function is

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1 - 1.005z^{-1}}.$$

Inverting $H[z]$, we the system impulse response is obtained as

$$h[n] = (1.005)^n u[n].$$

(d) For most useful engineering systems, the response of a system $H[z]$ to the everlasting exponential $x[n] = 1^n = 1$ is computed using the concept of frequency response as $y[n] = 1^n H[1] = H[1]$ (dc response). In this case, however, the output of the bank account to input $x[n] = 1^n = 1$ is not $y[n] = 1^n H[1] = H[1]$. To understand why, we note that the system $H[z]$ is not stable since its only root is outside the unit circle. Consequently, frequency response (or sinusoidal steady-state) concepts are meaningless, since the unit circle (where frequency response is evaluated) is not in the region of convergence of $H[z]$; frequency response only applies to stable systems.

Solution 5.3-17

Taking the z-transform of $b[m] = (1.01)b[m-1] + p[m]$ and solving for $B[z]$ yields $B[z] = P[z]\frac{1}{1-1.01z^{-1}}$. Thus, $P[z]$ is required to solve for $b[m]$. One way to represent Sally's deposit schedule is $p[m] = 100\left(u[m] - \sum_{k=0}^{\infty}\delta[m - (12k + 11)]\right)$. Defined this way, Sally deposits one hundred dollars on the first day of every month m except for Decembers, ($m = 12k + 11$ for $k = \{0, 1, 2, \dots\}$).

Taking the z-transform yields

$$P[z] = 100\left(\frac{1}{1 - z^{-1}} - \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\delta[m - (12k+11)]z^{-m}\right)$$

$$= 100\left(\frac{1}{1 - z^{-1}} - \sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}\delta[m - (12k+11)]z^{-m}\right)$$

$$= 100\left(\frac{1}{1 - z^{-1}} - \sum_{k=0}^{\infty}z^{-(12k+11)}\right).$$

Substituting $P[z]$ into the expression for $B[z]$ yields

$$B[z] = 100\left(\frac{1}{1 - z^{-1}} - \sum_{k=0}^{\infty}z^{-(12k+11)}\right)\frac{1}{1 - 1.01z^{-1}}$$

$$= 100\left(\frac{1}{(1 - 1.01z^{-1})(1 - z^{-1})} + \sum_{k=0}^{\infty}\frac{z^{-(12k+11)}}{1 - 1.01z^{-1}}\right)$$

$$= 100\left(\frac{101}{1 - 1.01z^{-1}} + \frac{-100}{1 - z^{-1}} + \sum_{k=0}^{\infty}\frac{z^{-(12k+11)}}{1 - 1.01z^{-1}}\right).$$

The first two terms are easily inverted using a table of z-transform pairs, while the last sum is inverted using tables and the shifting property.

$$b[m] = 100\left(101(1.01)^{m}u[m] - 100u[m] - \sum_{k=0}^{\infty}(1.01)^{m-(12k+11)}u[m - (12k + 11)]\right).$$

Solution 5.3-18

(a) Note, $h_1[n] = \left(-1 + (0.5)^n\right)u[n] = -(1)^n u[n] + (1/2)^n u[n]$. Thus, two real poles are evident at $z = 1$ and $z = 1/2$. Since $h[n]$ is not absolutely summable, the system is not BIBO stable. Thought of another way, the pole on the unit-circle makes the system marginally stable, at best. Marginally stable systems are not BIBO stable.

(b) Notice, $h_2[n] = (j)^n \left(u[n] - u[n - 10]\right)$ is a finite duration, causal signal. Thus, $h_2[n]$ has no poles (other than at zero). Since $h_2[n]$ is absolutely summable, the system is BIBO stable.

Solution 5.3-19

(a) If we let $y[n] = \sum_{k=0}^{n}k$, then

$$y[n] - y[n - 1] = n \qquad \text{with} \qquad y[0] = 0.$$

Setting $n = 0$ in this equation and $y[0] = 0$, yields

$$y[0] - y[-1] = 0 \Longrightarrow y[-1] = 0.$$

The z-transform of the difference equation is

$$\left(1 - \frac{1}{z}\right) Y[z] = \frac{z}{(z-1)^2}$$

and

$$\frac{Y[z]}{z} = \frac{z}{(z-1)^3} = \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}.$$

Hence

$$Y[z] = \frac{z}{(z-1)^2} + \frac{z}{(z-1)^3}$$

and

$$y[n] = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \qquad (n \geq 0).$$

(b) If we let $y[n] = \sum_{k=0}^{n} k^2$, then

$$y[n] - y[n-1] = n^2 \qquad \text{with} \qquad y[0] = 0.$$

Setting $n = 0$, we get

$$0 - y[-1] = 0 \Longrightarrow y[-1] = 0.$$

The z-transform of the difference equation is

$$\frac{z-1}{z} Y[z] = \frac{z(z+1)}{(z-1)^3}.$$

Hence

$$\frac{Y[z]}{z} = \frac{z(z+1)}{(z-1)^4} = \frac{1}{(z-1)^2} + \frac{3}{(z-1)^3} + \frac{2}{(z-1)^4}$$

$$Y[z] = \frac{z}{(z-1)^2} + \frac{3z}{(z-1)^3} + \frac{2z}{(z-1)^4}$$

$$y[n] = \left[n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3}\right]$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} \qquad (n \geq 0).$$

Solution 5.3-20

If we let $y[n] = \sum_{k=0}^{n} k^3$, then

$$y[n] - y[n-1] = n^3 \qquad \text{with} \qquad y[0] = 0.$$

Setting $n = 0$ in this equation and using $y[0] = 0$, we get

$$0 - y[-1] = 0 \Longrightarrow y[-1] = 0.$$

The z-transform of the difference equation is

$$\frac{z-1}{z} Y[z] = \frac{z(z^2 + 4z + 1)}{(z-1)^4}.$$

Thus,

$$\frac{Y[z]}{z} = \frac{z(z^2 + 4z + 1)}{(z-1)^5} = \frac{1}{(z-1)^2} + \frac{7}{(z-1)^3} + \frac{12}{(z-1)^4} + \frac{6}{(z-1)^5},$$

$$Y[z] = \frac{z}{(z-1)^2} + 7\frac{z}{(z-1)^3} + 12\frac{z}{(z-1)^4} + 6\frac{z}{(z-1)^5}, \quad \text{and}$$

$$y[n] = n + \frac{7}{2}n(n-1) + 2n(n-1)(n-2) + \frac{1}{4}n(n-1)(n-2)(n-3)$$

$$= \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4} \quad (n \geq 0).$$

Solution 5.3-21

If we let $y[n] = \sum_{k=0}^{n} ka^k \quad a \neq 1$, then

$$y[n] - y[n-1] = na^k \quad \text{with} \quad y[0] = 0.$$

Setting $n = 0$ and $y[0] = 0$ in this equation yields

$$0 - y[-1] = 0 \Longrightarrow y[-1] = 0.$$

The z-transform of the difference equation is

$$\frac{z-1}{z}Y[z] = \frac{az}{(z-a)^2}$$

or

$$\frac{Y[z]}{z} = \frac{az}{(z-1)(z-a)^2} = \frac{\frac{a}{(a-1)^2}}{z-1} - \frac{\frac{a}{(a-1)^2}}{z-a} + \frac{\frac{a^2}{a-1}}{(z-a)^2}.$$

Thus,

$$Y[z] = \frac{a}{(a-1)^2}\left[\frac{z}{z-1} - \frac{z}{z-a} + a(a-1)\frac{z}{(z-a)^2}\right]$$

and

$$y[n] = \frac{a}{(a-1)^2}\left[1 - a^n + (a-1)na^n\right]$$

$$= \frac{a + a^{n+1}[n(a-1) - 1]}{(a-1)^2} \quad a \neq 1 \text{ and } (n \geq 0).$$

Solution 5.3-22

(a) Let $x[n] = nu[n]$. Then,

$$X[z] = \frac{z}{(z-1)^2}.$$

Use of the result in Prob. 5.2-13a yields

$$\sum_{k=0}^{n} k \Longleftrightarrow \frac{z^2}{(z-1)^3} = \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^3}.$$

Hence

$$\sum_{k=0}^{n} k = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \quad (n \geq 0).$$

(b) Let $x[n] = n^2 u[n]$. Then,

$$X[z] = \frac{z(z+1)}{(z-1)^3}.$$

Use of the result in Prob. 5.2-13a yields

$$\sum_{k=0}^{n} k^2 \iff \frac{z^2(z+1)}{(z-1)^4} = \frac{z(z^2+4z+1)}{(z-1)^4} - \frac{3z(z-1)}{(z-1)^4} - \frac{4z}{(z-1)^4}.$$

Hence

$$\sum_{k=0}^{n} k^2 = n^3 - \frac{3n(n-1)}{2} - \frac{2n(n-1)(n-2)}{3} = \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} \qquad (n \geq 0).$$

Solution 5.3-23

Let $x[n] = n^3 u[n]$. Then,

$$X[z] = \frac{z(z^2 + 4z + 1)}{(z-1)^4}.$$

Use of the result in Prob. 5.2-13a yields

$$\sum_{k=0}^{n} k^3 \iff \frac{z^2(z^2+4z+1)}{(z-1)^5} = \frac{z}{(z-1)^2} + \frac{7z}{(z-1)^3} + \frac{12z}{(z-1)^4} + \frac{6z}{(z-1)^5}.$$

Hence

$$\sum_{k=0}^{n} k^3 = n + \frac{7}{2}n(n-1) + 2n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \frac{n^2(n+1)^2}{4} \qquad (n \geq 0).$$

Solution 5.3-24

Let $x[n] = na^n u[n]$. Then,

$$X[z] = \frac{az}{(z-a)^2}.$$

Use of the result in Prob. 5.2-13a yields

$$\sum_{k=0}^{n} ka^k \iff \frac{az^2}{(z-1)(z-a)^2} = \frac{a}{(a-1)^2}\left[\frac{z}{z-1} - \frac{z}{z-a} + a(a-1)\frac{z}{(z-a)^2}\right].$$

Hence

$$\sum_{k=0}^{n} ka^k = \frac{a}{(a-1)^2}[1 - a^n + (a-1)na^n]$$

$$= \frac{a + a^{n+1}[n(a-1) - 1]}{(a-1)^2} \qquad a \neq 1 \text{ and } (n \geq 0).$$

Solution 5.3-25

(a) Here,

$$x[n] = ee^n u[n], \qquad X[z] = \frac{ez}{z-e}, \quad \text{and}$$

$$Y[z] = X[z]H[z] = \frac{ez^2}{(z-e)(z+0.2)(z-0.8)}.$$

Therefore,

$$\frac{Y[z]}{z} = \frac{ez}{(z-e)(z+0.2)(z-0.8)} = \frac{1.32}{z-e} - \frac{0.186}{z+0.2} - \frac{1.13}{z-0.8},$$

$$Y[z] = 1.32\frac{z}{z-e} - 0.186\frac{z}{z+0.2} - 1.13\frac{z}{z-0.8}, \quad \text{and}$$

$$y[n] = [1.32(e)^n - 0.186(-0.2)^n - 1.13(0.8)^n]\, u[n].$$

(b) From the given $H[z]$, we can write

$$(z^2 - 0.6z - 0.16)Y[z] = zX[z].$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = x[n+1]$$

or

$$y[n] - 0.6y[n-1] - 0.16y[n-2] = x[n-1].$$

Solution 5.3-26

(a) Here,

$$Y[z] = X[z]H[z] = \frac{z(2z+3)}{(z-1)(z-2)(z-3)}.$$

Therefore,

$$\frac{Y[z]}{z} = \frac{2z+3}{(z-1)(z-2)(z-3)} = \frac{5/2}{z-1} - \frac{7}{z-2} + \frac{9/2}{z-3},$$

$$Y[z] = \frac{5}{2}\frac{z}{z-1} - 7\frac{z}{z-2} + \frac{9}{2}\frac{z}{z-3}, \quad \text{and}$$

$$y[n] = \left[\frac{5}{2} - 7(2)^n + \frac{9}{2}(3)^n\right] u[n].$$

(b) From the given $H[z]$, we can write

$$(z^2 - 5z + 6)Y[z] = (2z+3)X[z].$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - 5y[n+1] + 6y[n] = 2x[n+1] + 3x[n]$$

or

$$y[n] - 5y[n-1] + 6y[n-2] = 2x[n-1] + 3x[n-2].$$

Solution 5.3-27

All cases use the same transfer function. From the given $H[z]$ (after dividing the numerator and the denominator by 6), we can write

$$\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right) Y[z] = (5z - 1)X[z].$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - \frac{5}{6}y[n+1] + \frac{1}{6}y[n] = 5x[n+1] - x[n]$$

or

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 5x[n-1] - x[n-2].$$

(a) Here, $x[n] = 4^{-n}u[n] = (\frac{1}{4})^n u[n]$, so that $X[z] = \frac{z}{z-\frac{1}{4}}$, and

$$Y[z] = X[z]H[z] = \frac{6z(5z-1)}{(z-\frac{1}{4})(6z^2 - 5z + 1)} = \frac{z(5z-1)}{(z-\frac{1}{4})(z-\frac{1}{3})(z-\frac{1}{2})}.$$

Therefore,

$$\frac{Y[z]}{z} = \frac{5z-1}{(z-\frac{1}{4})(z-\frac{1}{3})(z-\frac{1}{2})} = \frac{12}{z-\frac{1}{4}} - \frac{48}{z-\frac{1}{3}} + \frac{36}{z-\frac{1}{2}},$$

$$Y[z] = 12\frac{z}{z-\frac{1}{4}} - 48\frac{z}{z-\frac{1}{3}} + 36\frac{z}{z-\frac{1}{2}}, \quad \text{and}$$

$$y[n] = \left[12(\frac{1}{4})^n - 48(\frac{1}{3})^n + 36(\frac{1}{2})^n\right] u[n]$$

$$= 12\left[4^{-n} - 4(3)^{-n} + 3(2)^{-n}\right] u[n].$$

(b) The input here is $4^{-(n-2)}u[n-2]$, which is identical to the input in part (a) delayed by 2 units. Therefore, the response will be the output in part (a) delayed by 2 units (time-invariance property). That is,

$$y[n] = 12\left[4^{-(n-2)} - 4(3)^{-(n-2)} + 3(2)^{-(n-2)}\right] u[n-2].$$

(c) Here the input can be expressed as

$$x[n] = 4^{-(n-2)}u[n] = 16(4)^{-n}u[n].$$

This input is 16 times the input in part (a). Therefore, the response will be 16 times the output in part (a) (linearity property). Thus,

$$y[n] = 192\left[4^{-n} - 4(3)^{-n} + 3(2)^{-n}\right] u[n].$$

(d) Here the input can be expressed as

$$x[n] = 4^{-n}u[n-2] = \frac{1}{16}(4)^{-(n-2)}u[n-2].$$

This input is $\frac{1}{16}$ times the input in part (b). Therefore the response will be $\frac{1}{16}$ times the output in part (b). Therefore,

$$y[n] = \frac{3}{4}\left[4^{-(n-2)} - 4(3)^{-(n-2)} + 3(2)^{-(n-2)}\right] u[n-2].$$

Solution 5.3-28

(a) Here,

$$Y[z] = X[z]H[z] = \frac{z(2z-1)}{(z-1)(z^2-1.6z+0.8)},$$

$$\frac{Y[z]}{z} = \frac{2z-1}{(z-1)(z^2-1.6z+0.8)} = \frac{5}{z-1} - \frac{5(z-1)}{z^2-1.6z+0.8}, \quad \text{and}$$

$$Y[z] = 5\frac{z}{z-1} - 5\frac{z(z-1)}{z^2-1.6z+0.8}.$$

For the second fraction on the right-hand side, we use pair 12c with $A = 1$, $B = -1$, $a = -0.8$, $\gamma = \frac{2}{\sqrt{5}}$, $|\gamma|^2 = 0.8$. Therefore,

$$r = 1.118, \qquad \beta = \cos^{-1}(\frac{0.8\sqrt{5}}{2}) = 0.464, \qquad \theta = \tan^{-1}(\frac{0.2}{0.4}) = 0.464,$$

and

$$y[n] = \left[5 - 5(1.118)\left(\frac{2}{\sqrt{5}}\right)^n \cos(0.464n + 0.464)\right] u[n]$$

$$= \left[5 - 5.59\left(\frac{2}{\sqrt{5}}\right)^n \cos(0.464n + 0.464)\right] u[n].$$

(b) From the given $H[z]$, we can write

$$\left(z^2 - 1.6z + 0.8\right) Y[z] = (2z - 1)X[z].$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - 1.6y[n+1] + 0.8y[n] = 2x[n+1] - x[n]$$

or

$$y[n] - 1.6y[n-1] + 0.8y[n-2] = 2x[n-1] - x[n-2].$$

Solution 5.3-29

(a) For Prob. 5.3-5, the transfer function is

$$H[z] = \frac{z}{z+2}.$$

(b) For Prob. 5.3-7, the transfer function is

$$H[z] = \frac{4z^2 - 3z}{2z^2 - 3z + 1}.$$

(c) For Prob. 5.3-9, the transfer function is

$$H[z] = \frac{z}{4z^2 + 4z + 1}.$$

(d) For Prob. 5.3-14, we convert the equation to advance operator form. This yields $(E^2 + 2E + 2)y[n] = (E + 2)x[n]$. Hence, the transfer function is

$$H[z] = \frac{z+2}{z^2 + 2z + 2}.$$

Solution 5.3-30

(a) In this case,

$$H[z] = \frac{z^2 + 3z + 3}{z^2 + 3z + 2} = \frac{z^2 + 3z + 3}{(z+1)(z+2)}.$$

Therefore,

$$\frac{H[z]}{z} = \frac{z^2 + 3z + 3}{z(z+1)(z+2)} = \frac{3/2}{z} - \frac{1}{z+1} + \frac{1/2}{z+2},$$

$$H[z] = \frac{3}{2} - \frac{z}{z+1} + \frac{1}{2}\frac{z}{z+2}, \quad \text{and}$$

$$h[n] = \left[\frac{3}{2}\delta[n] - (-1)^n + \frac{1}{2}(-2)^n\right] u[n]$$

(b) Here,

$$H[z] = \frac{2z^2 - z}{z^2 + 2z + 1} = \frac{z(2z-1)}{(z+1)^2}.$$

Therefore,

$$\frac{H[z]}{z} = \frac{2z - 1}{(z+1)^2} = \frac{2}{z+1} - \frac{3}{(z+1)^2},$$

$$H[z] = 2(\frac{z}{z+1}) - 3\frac{z}{(z+1)^2}, \quad \text{and}$$

$$h[n] = [2(-1)^n + 3n(-1)^n] u[n] = (2 + 3n)(-1)^n u[n].$$

(c) In this part,

$$H[z] = \frac{z^2 + 2z}{z^2 - z + 0.5} = \frac{z(z+2)}{z^2 - z + 0.5}.$$

Therefore,

$$\frac{H[z]}{z} = \frac{z + 2}{z^2 - z + 0.5}.$$

We use pair 12c with $A = 1$, $B = 2$, $a = -0.5$, $|\gamma|^2 = 0.5$, and $|\gamma| = \frac{1}{\sqrt{2}}$. Thus,

$$r = 5.099, \qquad \beta = \cos^{-1}(0.5\sqrt{5}) = \frac{\pi}{4}, \qquad \theta = \tan^{-1}(\frac{-2.5}{0.5}) = -1.373,$$

and

$$h[n] = 5.099 \left(\frac{1}{\sqrt{2}}\right)^n \cos(\frac{\pi}{4}n - 1.373)u[n].$$

Solution 5.3-31

(a) For Prob. 5.3-25,

$$\frac{H[z]}{z} = \frac{1}{(z+0.2)(z-0.8)} = \frac{-1}{z+0.2} + \frac{1}{z-0.8},$$

$$H[z] = -\frac{z}{z+0.2} + \frac{z}{z-0.8}, \quad \text{and}$$

$$h[n] = [-(-0.2)^n + (0.8)^n] u[n].$$

(b) For Prob. 5.3-26,

$$\frac{H[z]}{z} = \frac{2z+3}{z(z-2)(z-3)} = \frac{1/2}{z} - \frac{7/2}{z-2} + \frac{3}{z-3},$$

$$H[z] = \frac{1}{2} - \frac{7}{2}\frac{z}{z-2} + 3\frac{z}{z-3}, \quad \text{and}$$

$$h[n] = \left[\frac{1}{2}\delta[n] - \frac{7}{2}(2)^n + 3(3)^n\right]u[n].$$

(c) For Prob. 5.3-28,

$$\frac{H[z]}{z} = \frac{2z-1}{z(z^2 - 1.6z + 0.8)} = \frac{-1.25}{z} + \frac{1.25z}{z^2 - 1.6z + 0.8}.$$

For the second fraction on the right-hand side, we use pair 12c with $A = 1.25$, $B = 0$, $a = -0.8$, $|\gamma|^2 = 0.8$, and $|\gamma| = \frac{2}{\sqrt{5}}$. Thus,

$$r = 2.795, \qquad \beta = \cos^{-1}\left(\frac{0.8\sqrt{5}}{2}\right) = 0.464, \qquad \theta = \tan^{-1}(-2) = -1.107,$$

and

$$h[n] = -1.25\delta[n] + 2.795\left(\frac{2}{\sqrt{5}}\right)^n \cos(0.464n - 1.107)u[n].$$

Solution 5.3-32

(a) Noting that $H[z] = z^{-3}\frac{z}{z-1}$, we see that $H^{-1}[z] = \frac{1}{H[z]} = \frac{z-1}{z-2} = z^3 - z^2$. Thus,

$$h^{-1}[n] = \delta[n+3] - \delta[n+2].$$

(b) Since $h^{-1}[n]$ is absolutely summable, the system inverse is stable. However, $h^{-1}[n] \neq 0$ for $n < 0$ so the system is not causal.

(c) For systems that have time as the independent variable, it is only possible to realize causal systems. Shifting $h^{-1}[n]$ by three makes it causal and therefore realizable. That is, implement $h^{-1}_{\text{causal}}[n] = h^{-1}[n-3] = \delta[n] - \delta[n-1]$, as shown in Fig. S5.3-32c. Within a delay factor, this implementation functions as the system inverse.
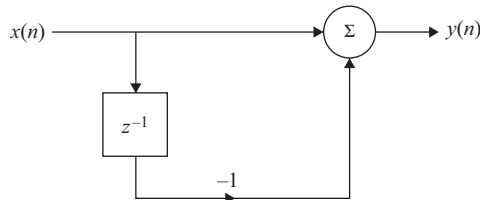
Figure S5.3-32c

Solution 5.4-1

For convenience, let $\gamma = \frac{1+j}{\sqrt{8}}$ and rewrite $h[n] = (\gamma^n + (\gamma^*)^n)u[n]$.

(a) By inspection, the structure is a parallel implementation of two modes, which are easily identified in $h[n]$. The transfer function of the structure is $H[z] = \frac{1}{1+A_1 z^{-1}} + \frac{1}{1+A_2 z^{-1}}$. Taking the transform of $h[n]$, $H[z] = \frac{1}{1-\gamma z^{-1}} + \frac{1}{1-\gamma^* z^{-1}}$. Thus,

$$A_1 = -\gamma = -\frac{1+j}{\sqrt{8}} \quad \text{and} \quad A_2 = -\gamma^* = -\frac{1-j}{\sqrt{8}}.$$

(b) Here, $y_0[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \left\{ \sum_{k=0}^{n+3} \gamma^n + (\gamma^*)^n \right\} u[n+3]$. Thus,

$$y_0[n] = \left\{ \frac{1-\gamma^{n+4}}{1-\gamma} + \frac{1-(\gamma^*)^{n+4}}{1-\gamma^*} \right\} u[n+3], \quad \text{where } \gamma = \frac{1+j}{\sqrt{8}}.$$

Written another way, $y_0[n] = 2\text{Re}\left\{ \frac{1-\gamma^{n+4}}{1-\gamma} \right\} u[n+3]$.

Solution 5.4-2

(a) We want to realize the system

$$H[z] = \frac{z(3z-1.8)}{z^2 - z + 0.16} = \frac{3 - 1.8z^{-1}}{1 - z^{-1} + 0.16z^{-2}}.$$

Figure S5.4-2a shows the canonical direct form (DFII) of the system.
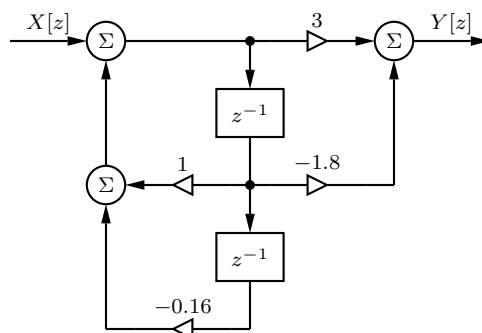
Figure S5.4-2a

To construct a parallel realization, we use partial fractions. To begin, notice that

$$\frac{H[z]}{z} = \frac{3z - 1.8}{z^2 - z + 0.16}.$$

We use MATLAB to compute the necessary partial fraction expansion.

```
>>  [r,p,k] = residue([3 -1.8],[1 -1 0.16])
    r =  1.0000  2.0000
    p =  0.8000  0.2000
    k = []
```

Thus,

$$H[z] = \frac{2z}{z - 0.2} + \frac{z}{z - 0.8}.$$

Figure S5.4-2b shows a parallel realization based on this expression for $H[z]$.

To construct a series realization, we simply factor $H[z]$ as

$$H[z] = \left(\frac{3z}{z - 0.2} \right) \left(\frac{z - 0.6}{z - 0.8} \right).$$

Figure S5.4-2c shows a series realization based on this expression for $H[z]$. Notice that the parallel and series representations of the system are not unique.

(b) We can obtain the transpose of a block diagram by the following operations:
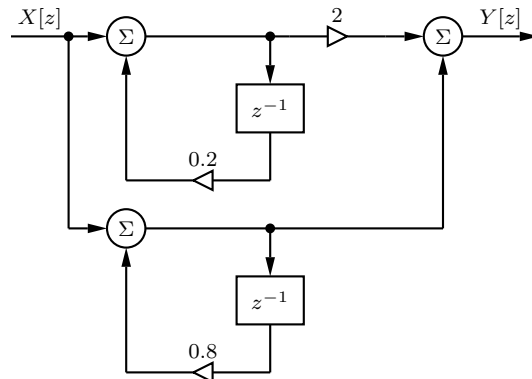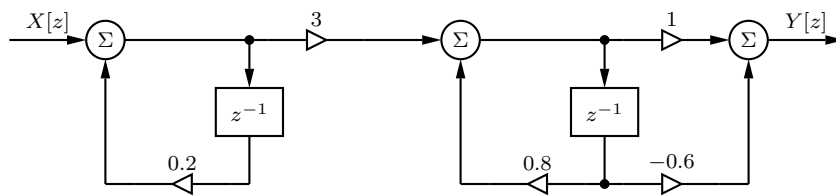
Figure S5.4-2b

Figure S5.4-2c

1. Reverse the directions of all paths.
2. Replace summing nodes with pick-off nodes and pick-off nodes with summing nodes.
3. Interchange the input $x[n]$ and the output $y[n]$.

Figures S5.4-2d, S5.4-2e, and S5.4-2f are the transpose realizations of Figs. S5.4-2a, S5.4-2b, and S5.4-2c, respectively.
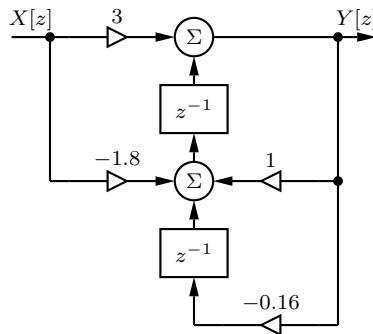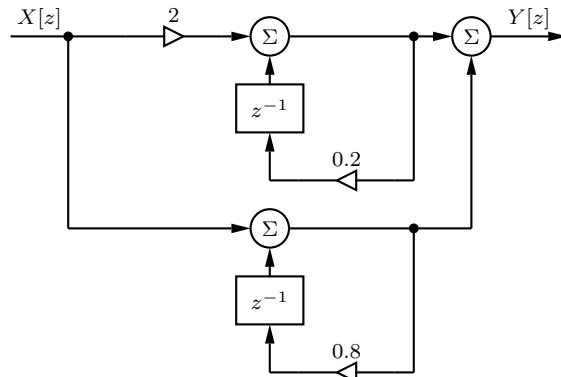
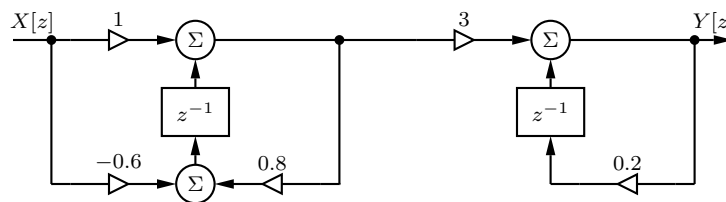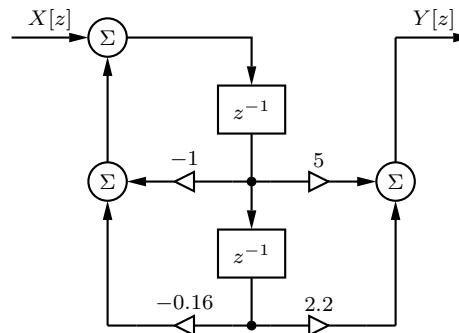

Figure S5.4-2d

Figure S5.4-2e

Figure S5.4-2f

Solution 5.4-3

(a) We want to realize the system

$$H[z] = \frac{5z + 2.2}{z^2 + z + 0.16} = \frac{5z^{-1} + 2.2z^{-2}}{1 + z^{-1} + 0.16z^{-2}}.$$

Figure S5.4-3a shows the canonical direct form (DFII) of the system.

Figure S5.4-3a

To construct a parallel realization, we use MATLAB to expand $H[z]$ using partial fractions.

```
>>  [r,p,k] = residue([5 2.2],[1 1 0.16])
    r =   3.0000   2.0000
    p = -0.8000 -0.2000
    k = []
```

Thus,

$$H[z] = \frac{2}{z + 0.2} + \frac{3}{z + 0.8}.$$
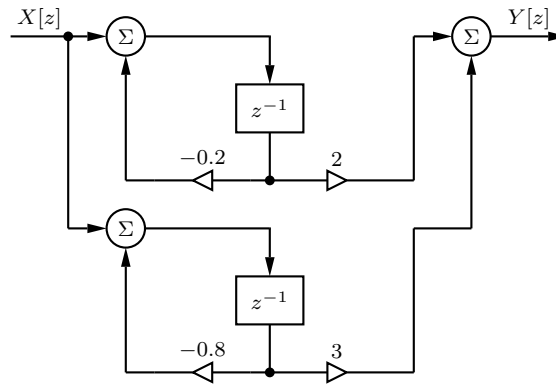
Figure S5.4-3b

Figure S5.4-3b shows a parallel realization based on this expression for $H[z]$.

To construct a series realization, we simply factor $H[z]$ as

$$H[z] = \left(\frac{1}{z + 0.2}\right)\left(\frac{5z + 2.2}{z + 0.8}\right).$$

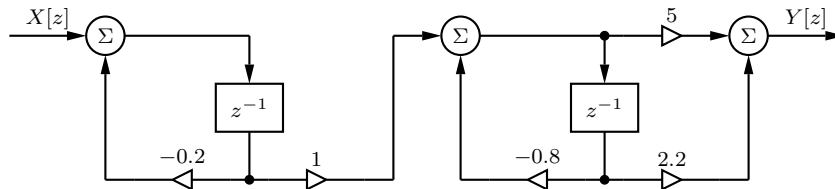Figure S5.4-3c shows a series realization based on this expression for $H[z]$. Notice that the

Figure S5.4-3c

parallel and series representations of the system are not unique.

(b) We can obtain the transpose of a block diagram by the following operations:

1. Reverse the directions of all paths.
2. Replace summing nodes with pick-off nodes and pick-off nodes with summing nodes.
3. Interchange the input $x[n]$ and the output $y[n]$.

Figures S5.4-3d, S5.4-3e, and S5.4-3f are the transpose realizations of Figs. S5.4-3a, S5.4-3b, and S5.4-3c, respectively.
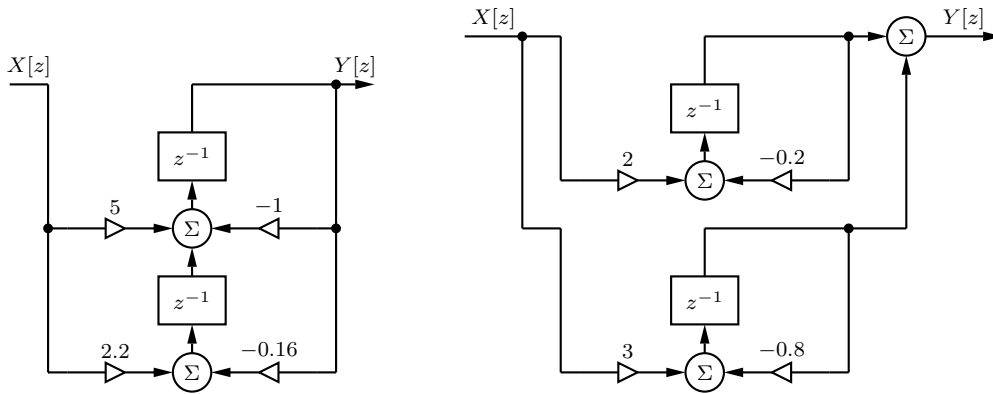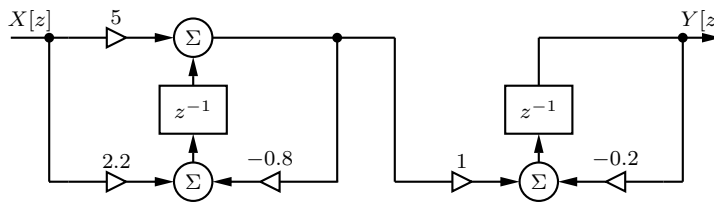
Figures S5.4-3d and S5.4-3e

Figure S5.4-3f

Solution 5.4-4

(a) We want to realize the system

$$H[z] = \frac{3.8z - 1.1}{(z - 0.2)(z^2 - 0.6z + 0.25)} = \frac{3.8z^{-2} - 1.1z^{-3}}{1 - 0.8z^{-1} + 0.37z^{-2} - 0.05z^{-3}}.$$

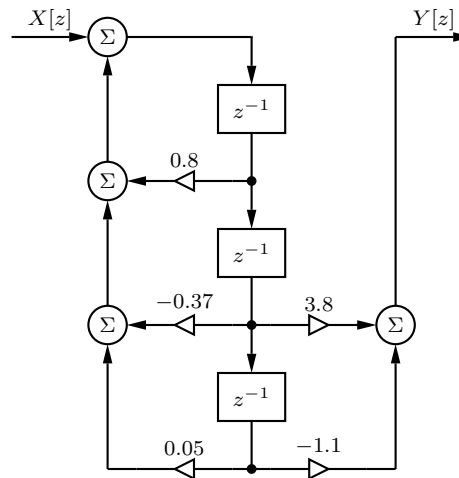Figure S5.4-4a shows the canonical direct form (DFII) of the system.

Figure S5.4-4a

To construct a parallel realization, we use MATLAB to expand $H[z]$ using partial fractions.

```
>>  [r,p,k] = residue([3.8 -1.1],[1 -0.8 0.37 -0.05])
    r =  1.0000-4.5000i  1.0000+4.5000i  -2.0000
    p =  0.3000+0.4000i  0.3000-0.4000i   0.2000
    k =  []
```

Thus,

$$H[z] = \frac{1 - 4.5j}{z - 0.3 - 0.4j} + \frac{1 + 4.5j}{z - 0.3 + 0.4j} + \frac{-2}{z - 0.2} = \frac{-2}{z - 0.2} + \frac{2z + 3}{z^2 - 0.6z + 0.25}.$$

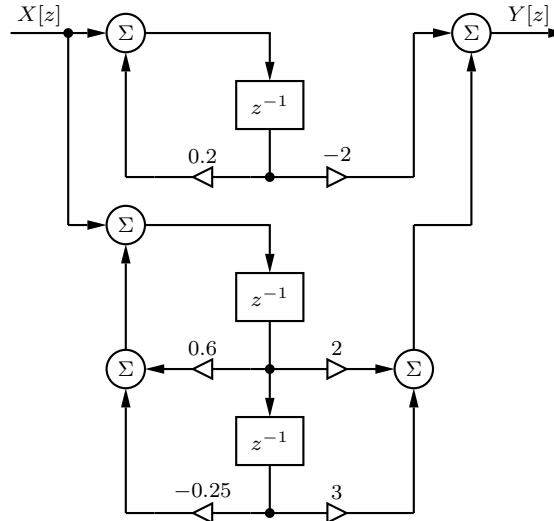Figure S5.4-4b shows a parallel realization based on this expression for $H[z]$.

Figure S5.4-4b

To construct a series realization, we simply factor $H[z]$ as

$$H[z] = \left(\frac{1}{z - 0.2} \right) \left(\frac{3.8z - 1.1}{z^2 - 0.6z + 0.25} \right).$$

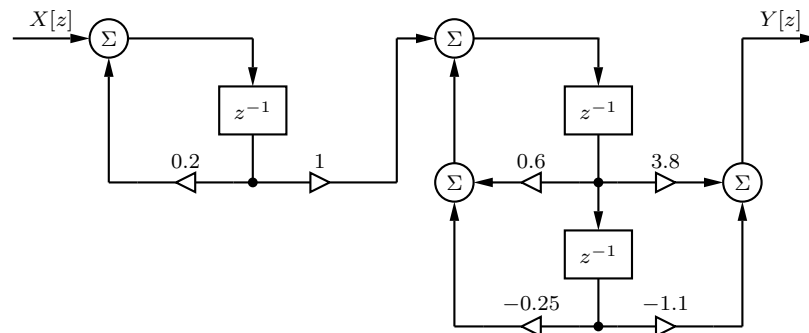Figure S5.4-4c shows a series realization based on this expression for $H[z]$. Notice that the

Figure S5.4-4c

parallel and series representations of the system are not unique.

(b) We can obtain the transpose of a block diagram by the following operations:

1. Reverse the directions of all paths.
2. Replace summing nodes with pick-off nodes and pick-off nodes with summing nodes.
3. Interchange the input $x[n]$ and the output $y[n]$.

Figures S5.4-4d, S5.4-4e, and S5.4-4f are the transpose realizations of Figs. S5.4-4a, S5.4-4b, and S5.4-4c, respectively.
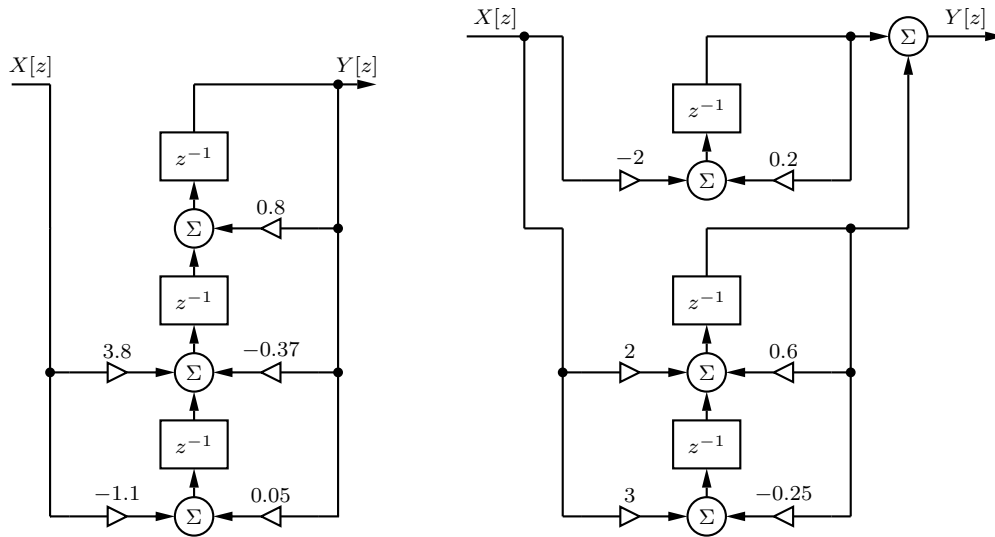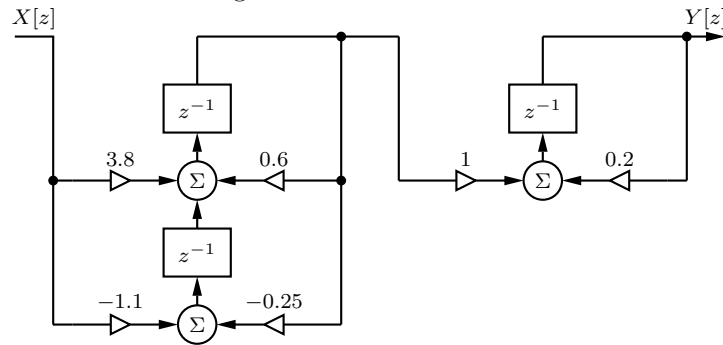
Figures S5.4-4d and S5.4-4e

Figure S5.4-4f

Solution 5.4-5

(a) We want to realize the system

$$H[z] = \frac{z(1.6z - 1.8)}{(z - 0.2)(z^2 + z + 0.5)} = \frac{1.6z^{-1} - 1.8z^{-2}}{1 + 0.8z^{-1} + 0.3z^{-2} - 0.1z^{-3}}.$$

Figure S5.4-5a shows the canonical direct form (DFII) of the system.

To construct a parallel realization, we use MATLAB to expand $\frac{H[z]}{z}$ using partial fractions.

```
>>  [r,p,k] = residue([1.6 -1.8],[1 0.8 0.3 -0.1])
    r =  1.0000-3.0000i  1.0000+3.0000i -2.0000
    p = -0.5000+0.5000i -0.5000-0.5000i  0.2000
    k =  []
```

Thus,

$$\frac{H[z]}{z} = \frac{1 - 3j}{z + 0.5 - 0.5j} + \frac{1 + 3j}{z + 0.5 + 0.5j} + \frac{-2}{z - 0.2} = \frac{-2}{z - 0.2} + \frac{2z + 4}{z^2 + z + 0.5}.$$

or

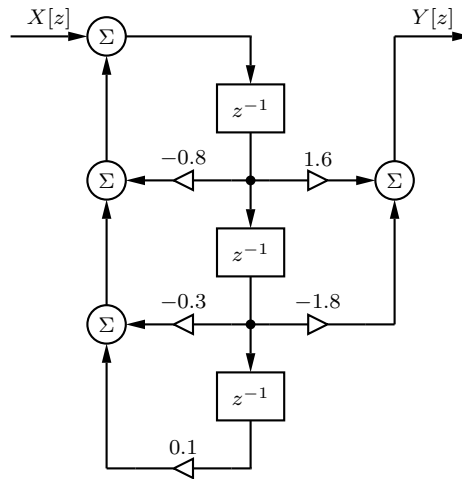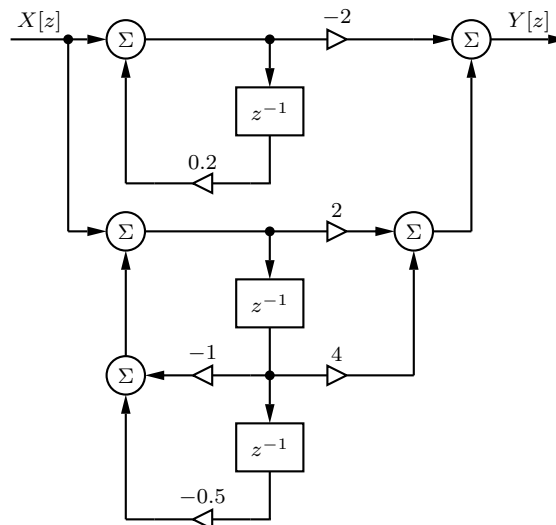$$H[z] = \frac{-2z}{z - 0.2} + \frac{2z^2 + 4z}{z^2 + z + 0.5}.$$

Figure S5.4-5a

Figure S5.4-5b

Figure S5.4-5b shows a parallel realization based on this expression for $H[z]$.

To construct a series realization, we simply factor $H[z]$ as

$$H[z] = \left(\frac{z}{z - 0.2}\right)\left(\frac{1.6z - 1.8}{z^2 + z + 0.5}\right).$$

Figure S5.4-5c shows a series realization based on this expression for $H[z]$. Notice that the parallel and series representations of the system are not unique.

(b) We can obtain the transpose of a block diagram by the following operations:

    1. Reverse the directions of all paths.
    2. Replace summing nodes with pick-off nodes and pick-off nodes with summing nodes.
    3. Interchange the input $x[n]$ and the output $y[n]$.

Figures S5.4-5d, S5.4-5e, and S5.4-5f are the transpose realizations of Figs. S5.4-5a, S5.4-5b, and S5.4-5c, respectively.
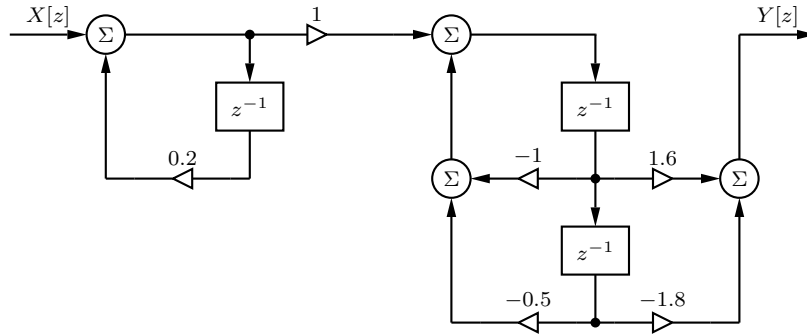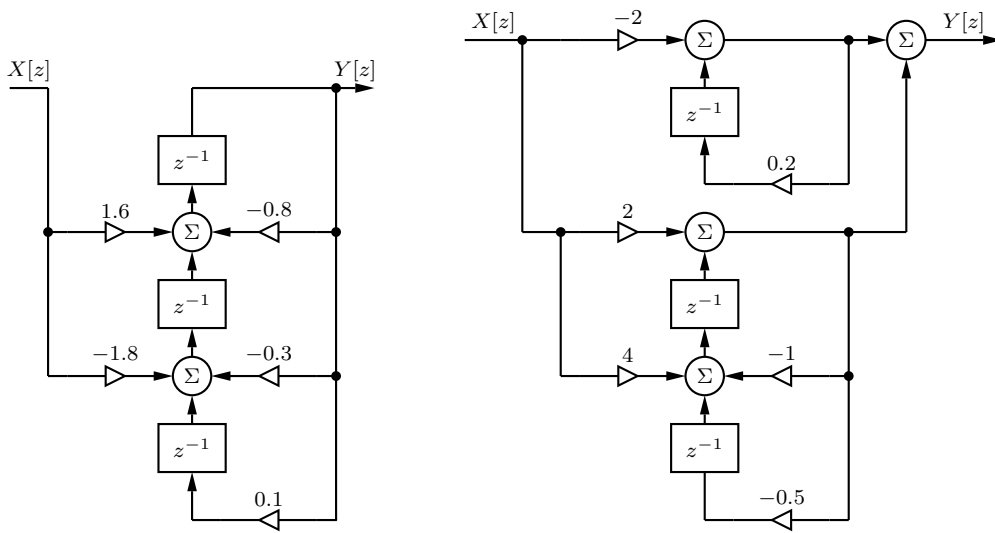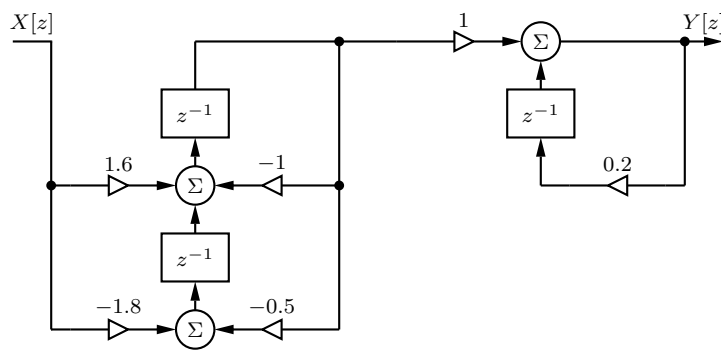
Figure S5.4-5c

Figures S5.4-5d and S5.4-5e

Figure S5.4-5f

Solution 5.4-6

(a) we want to realize the system

$$H[z] = \frac{z(2z^2 + 1.3z + 0.96)}{(z + 0.5)(z - 0.4)^2} = \frac{2 + 1.3z^{-1} + 0.96z^{-2}}{1 - 0.3z^{-1} - 0.24z^{-2} + 0.08z^{-3}}.$$

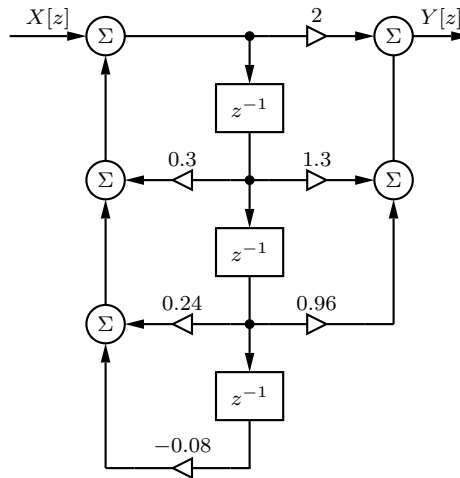Figure S5.4-6a shows the canonical direct form (DFII) of the system.

Figure S5.4-6a

To construct a parallel realization, we use MATLAB to expand $\frac{H[z]}{z}$ using partial fractions.

```
>>  [r,p,k] = residue([2 1.3 0.96],[1 -0.3 -0.24 0.08])
    r =   1.0000   1.0000   2.0000
    p = -0.5000   0.4000   0.4000
    k =   []
```

Thus,

$$\frac{H[z]}{z} = \frac{1}{z + 0.5} + \frac{1}{z - 0.4} + \frac{2}{(z - 0.4)^2}.$$

or

$$H[z] = \frac{z}{z + 0.5} + \frac{z}{z - 0.4} + \frac{2z}{(z - 0.4)^2}.$$

Figure S5.4-6b shows a parallel realization based on this expression for $H[z]$.

To construct a series realization, we simply factor $H[z]$ as

$$H[z] = \left(\frac{z}{z + 0.5}\right)\left(\frac{2z^2 + 1.3z + 0.96}{z^2 - 0.8z + 0.16}\right).$$

Figure S5.4-6c shows a series realization based on this expression for $H[z]$. Notice that the parallel and series representations of the system are not unique.

(b) We can obtain the transpose of a block diagram by the following operations:

1. Reverse the directions of all paths.
2. Replace summing nodes with pick-off nodes and pick-off nodes with summing nodes.
3. Interchange the input $x[n]$ and the output $y[n]$.

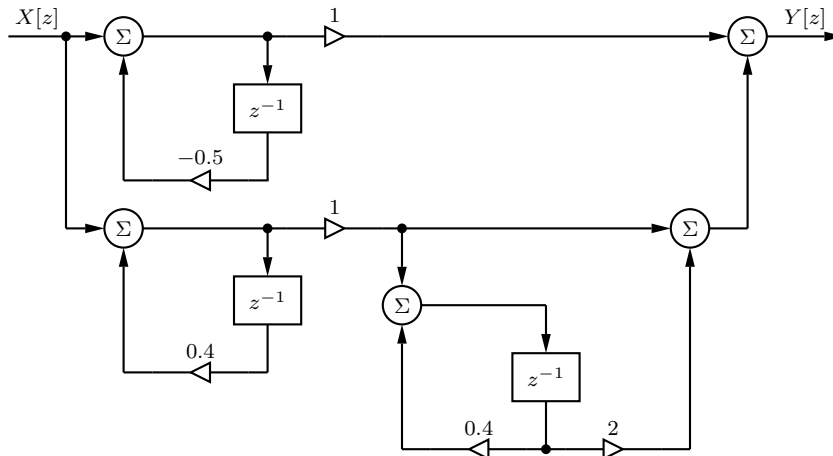Figures S5.4-6d, S5.4-6e, and S5.4-6f are the transpose realizations of Figs. S5.4-6a, S5.4-6b, and S5.4-6c, respectively.
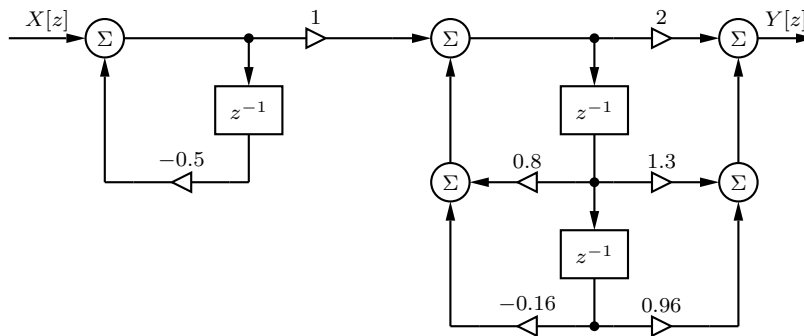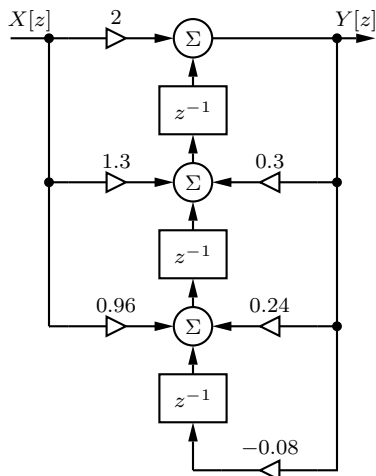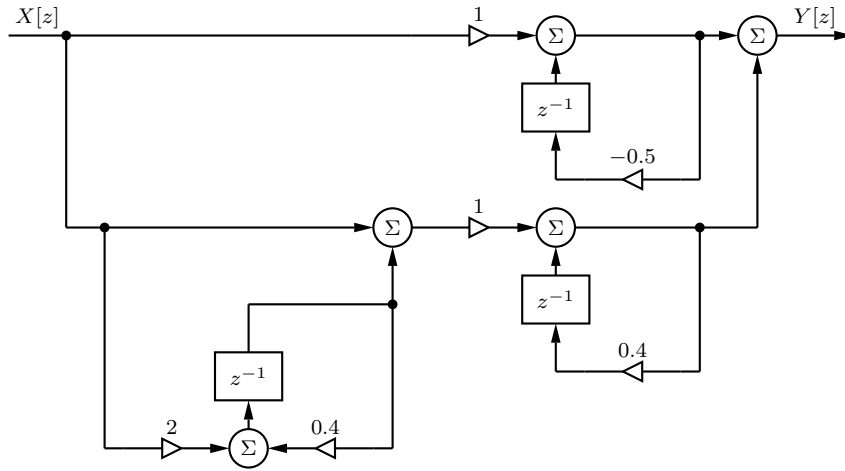
Figure S5.4-6b

Figure S5.4-6c

Figure S5.4-6d

Figure S5.4-6e
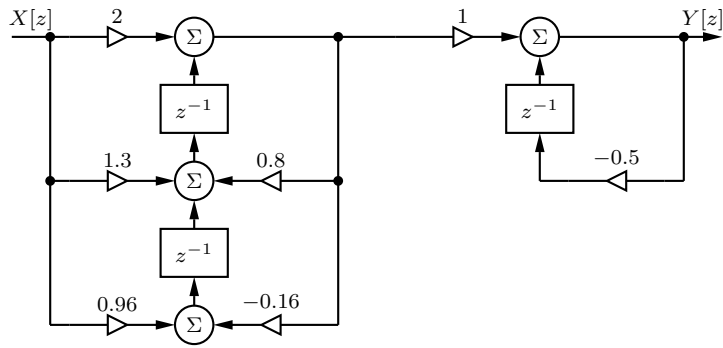
Figure S5.4-6f

Solution 5.4-7

(a) The equation governing the summer output is

$$\frac{1}{2}y[n+1] = y[n] + 3x[n] + 3x[n-1].$$

Scale by 2, delay by 1, and rearrange to get standard form as

$$y[n] - 2y[n-1] = 6x[n-1] + 6x[n-2].$$

(b) Taking the z-transform of the result of part (a) yields

$$Y[z](1 - 2z^{-1}) = X[z](6z^{-1} + 6z^{-2}).$$

Thus, the transfer function is

$$H[z] = \frac{Y[z]}{X[z]} = \frac{6(z+1)}{z(z-2)} = \frac{-3}{z} + \frac{9}{z-2} = -3z^{-1} + 9z^{-1}\frac{z}{z-2}.$$

Inverting, the system impulse response is

$$h[n] = -3\delta[n-1] + 9(2)^{n-1}u[n-1].$$

(c) From (a), the system difference equation is $y[n] - 2y[n-1] = 6x[n-1] + 6x[n-2]$. Since the number of delay blocks in the system realization exactly matches the largest delay in the difference equation, the system is canonical.

<div align="center">Yes. The system realization is canonical.</div>

(d) From $H[z] = \frac{6(z+1)}{z(z-2)}$, we see that the system has a pole at $z = 0$ and $z = 2$. Since a system pole is outside the unit circle, the system is not stable.

<div align="center">No. The system is not stable.</div>

(e) Any system that can be physically realized with summers, scale multipliers, and delay blocks must be causal. This conclusion is also verified by noticing that the impulse response found in part (b) is zero for all n less than zero ($h[n] = 0$ for $n < 0$).

<div align="center">Yes. The system is causal.</div>

Solution 5.4-8

We want to realize the system

$$H[z] = \frac{2z^4 + z^3 + 0.8z^2 + 2z + 8}{z^4} = 2 + z^{-1} + 0.8z^{-2} + 2z^{-3} + 8z^{-4}.$$

Since there is no feedback, we see that $H[z]$ represents a finite impulse response (FIR) system. We can simply realize the system using the direct form structure shown in Fig. S5.4-8.
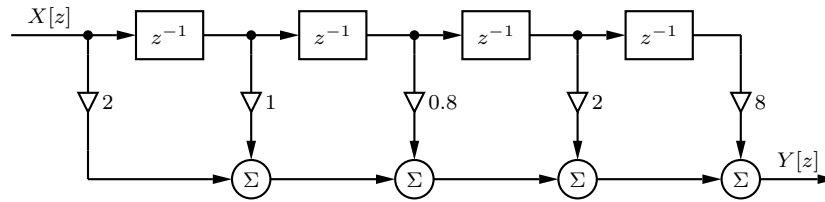
Figure S5.4-8

Solution 5.4-9

We now want to realize the system

$$H[z] = \sum_{n=0}^{6} nz^{-n} = z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 5z^{-5} + 6z^{-6}.$$

Since there is again no feedback, we see that $H[z]$ represents a finite impulse response (FIR) system. We can simply realize the system using the direct form structure shown in Fig. S5.4-9.
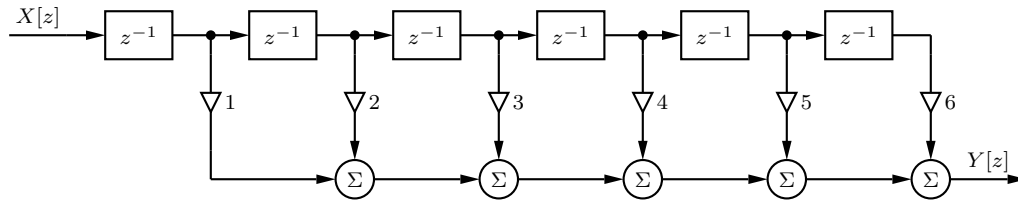
Figure S5.4-9

Solution 5.4-10

(a) We begin by labeling some nodes on the system block diagram, as shown in Fig. S5.4-10a.
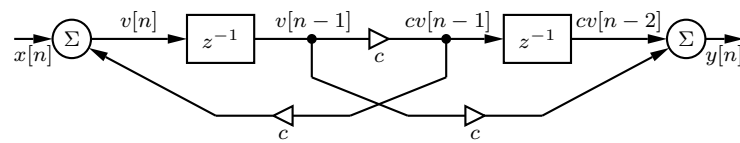
Figure S5.4-10a

Using Fig. S5.4-10a and the z-transform, the output of the second summer is

$$Y[z] = cz^{-2}V[z] + cz^{-1}V[z] = (cz^{-2} + cz^{-1})V[z].$$

Similarly, the output of the first summer is

$$V[z] = c^2 z^{-1}V[z] + X[z] \quad \Rightarrow \quad X[z] = (1 - c^2 z^{-1})V[z].$$

Combining, we obtain

$$H[z] = \frac{Y[z]}{X[z]} = \frac{(cz^{-2} + cz^{-1})V[z]}{(1 - c^2 z^{-1})V[z]} = \frac{c(z+1)}{z(z - c^2)}.$$

(b) By inspection of $H[z]$ from part (a), we see that

the system has two poles ($z = 0$ and $z = c^2$) and two zeros ($z = -1$ and $z = \infty$).

(c) Since the system has two poles and two zeros, the system order is two. Further, the number of delay blocks in the system realization exactly equals the system order. Thus, the system is canonical.

$$\text{Yes. The system is canonical } (\# \text{ of delays blocks} = \text{system order}).$$

(d) To be stable, the pole at c^2 must be inside the unit circle. Furthermore we know c is real. Thus,

$$\text{the system is stable if } -1 < c < 1.$$

Solution 5.4-11

To begin, we represent a general second-order transfer function with two real zeros and two real poles as

$$H[z] = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + z_2} = b_0 \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = b_0 + \frac{k_1}{z - p_1} + \frac{k_2}{z - p_2}.$$

Next, we investigate realizing this system with canonic direct, cascade, parallel, and corresponding transposed forms. To simplify this process, let us represent a canonic direct (DFII) realization as $[\cdot]$ and a transposed canonic direct realization (TDFII) as $[\cdot]^T$. Thus, we represent the second-order canonic direct (DFII) realization of $H[z]$ as

$$\left[\frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + z_2} \right]$$

and we represent the second-order transposed canonic direct (TDFII) realization as

$$\left[\frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + z_2} \right]^T.$$

Graphical depiction of these, and later, realizations are easily generated following the discussion and figures in Sec. 5.4 and are not given here.

We can also realize the system with at least 16 different realizations that cascade two first-order systems, represented as

$$b_0 \left[\tfrac{z-z_1}{z-p_1} \right] \left[\tfrac{z-z_2}{z-p_2} \right], \quad b_0 \left[\tfrac{z-z_1}{z-p_1} \right]^T \left[\tfrac{z-z_2}{z-p_2} \right], \quad b_0 \left[\tfrac{z-z_1}{z-p_1} \right] \left[\tfrac{z-z_2}{z-p_2} \right]^T, \quad b_0 \left[\tfrac{z-z_1}{z-p_1} \right]^T \left[\tfrac{z-z_2}{z-p_2} \right]^T,$$

$$b_0 \left[\tfrac{z-z_1}{z-p_2} \right] \left[\tfrac{z-z_2}{z-p_1} \right], \quad b_0 \left[\tfrac{z-z_1}{z-p_2} \right]^T \left[\tfrac{z-z_2}{z-p_1} \right], \quad b_0 \left[\tfrac{z-z_1}{z-p_2} \right] \left[\tfrac{z-z_2}{z-p_1} \right]^T, \quad b_0 \left[\tfrac{z-z_1}{z-p_2} \right]^T \left[\tfrac{z-z_2}{z-p_1} \right]^T,$$

$$b_0 \left[\tfrac{z-z_2}{z-p_1} \right] \left[\tfrac{z-z_1}{z-p_2} \right], \quad b_0 \left[\tfrac{z-z_2}{z-p_1} \right]^T \left[\tfrac{z-z_1}{z-p_2} \right], \quad b_0 \left[\tfrac{z-z_2}{z-p_1} \right] \left[\tfrac{z-z_1}{z-p_2} \right]^T, \quad b_0 \left[\tfrac{z-z_2}{z-p_1} \right]^T \left[\tfrac{z-z_1}{z-p_2} \right]^T,$$

$$b_0 \left[\tfrac{z-z_2}{z-p_2} \right] \left[\tfrac{z-z_1}{z-p_1} \right], \quad b_0 \left[\tfrac{z-z_2}{z-p_2} \right]^T \left[\tfrac{z-z_1}{z-p_1} \right], \quad b_0 \left[\tfrac{z-z_2}{z-p_2} \right] \left[\tfrac{z-z_1}{z-p_1} \right]^T, \quad b_0 \left[\tfrac{z-z_2}{z-p_2} \right]^T \left[\tfrac{z-z_1}{z-p_1} \right]^T.$$

Additionally, there are at least 4 different parallel forms, represented as

$$b_0 + \left[\tfrac{k_1}{z-p_1} \right] + \left[\tfrac{k_2}{z-p_2} \right], \quad b_0 + \left[\tfrac{k_1}{z-p_1} \right]^T + \left[\tfrac{k_2}{z-p_2} \right],$$

$$b_0 + \left[\tfrac{k_1}{z-p_1} \right] + \left[\tfrac{k_2}{z-p_2} \right]^T, \quad b_0 + \left[\tfrac{k_1}{z-p_1} \right]^T + \left[\tfrac{k_2}{z-p_2} \right]^T.$$

Together, we have shown a total of 22 realizations. We have not considered realizations using the DFI or TDFI forms, combination of DFI and TDFI forms with the DFII and TDFII forms, distribution or placement of the constant b_0, or any of the various nonstandard realizations. There are, in fact, a limitless number of realizations, even for this relatively low-order system.

Solution 5.4-12

(a) Following the code given in the problem, Figure S5.4-12a presents the system block diagram.

Figure S5.4-12a

As shwon in Fig. S5.4-12a, the code implements at DFII.

(b) From part (a), we see that

$$H[z] = \frac{\frac{7}{16}(z^2 - 1)}{z^2 + \frac{9}{16}} = \frac{\frac{7}{16}(z - 1)(z + 1)}{(z - \frac{j3}{4})(z + \frac{j3}{4})}.$$

(c) We use MATLAB to plot the system magnitude response.

```
>> Omega = linspace(-pi,pi,1001); H = @(z) 7/16*(z.^2-1)./(z.^2+9/16);
>> plot(Omega,abs(H(exp(1j*Omega))),'k'); axis tight;
>> xlabel('\Omega'); ylabel('|H[e^{j \Omega}]|');
```
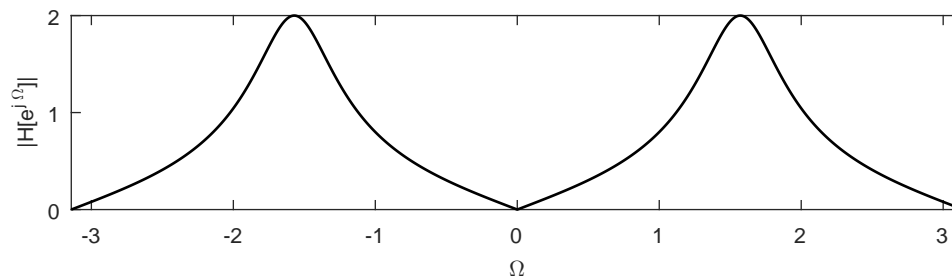

Figure S5.4-12c

As Fig. S5.4-12b makes clear, this digital system is a bandpass filter (BPF).

(d) From part (b), we know that $H[z] = \frac{\frac{7}{16}(z^2-1)}{z^2+\frac{9}{16}}$. Thus, the inverse system has transfer function

$$H^{-1}[z] = \frac{\frac{16}{7}(z^2 + \frac{9}{16})}{z^2 - 1} = \frac{\frac{16}{7}z^2 + \frac{9}{7}}{z^2 - 1}.$$

Figure S5.4-12d shows the DFI block implementation of $H^{-1}[z]$. Since the inverse system has poles on the unit circle ($z = \pm 1$), the inverse system is marginally stable (and BIBO unstable).

The inverse system is BIBO unstable and will not operate well.
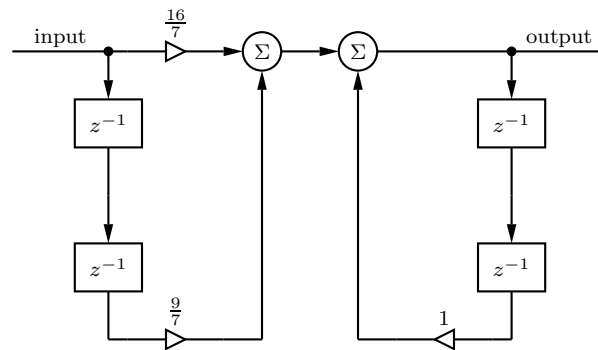
Figure S5.4-12d

Solution 5.4-13

(a) Following the code given in the problem, Figure S5.4-13a presents the system block diagram.
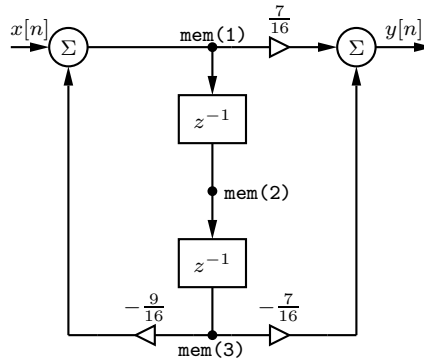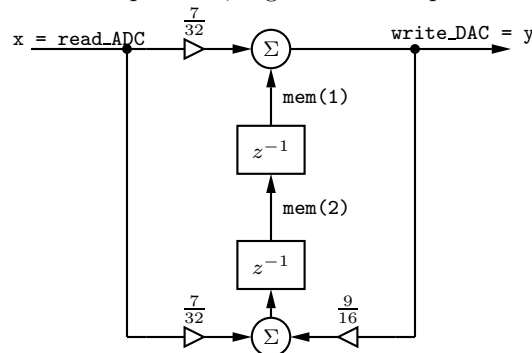

Figure S5.4-13a

As shwon in Fig. S5.4-13a, the code implements at TDFII.

(b) From part (a), we see that

$$H[z] = \frac{\frac{7}{32}(z^2 + 1)}{z^2 - \frac{9}{16}} = \frac{\frac{7}{32}(z - j)(z + j)}{(z - \frac{3}{4})(z + \frac{3}{4})}.$$

(c) We use MATLAB to plot the system magnitude response.

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) 7/32*(z.^2+1)./(z.^2-9/16);
>>  plot(Omega,abs(H(exp(1j*Omega))),'k'); axis tight;
>>  xlabel('\Omega'); ylabel('|H[e^{j \Omega}]|');
```

As Fig. S5.4-13b makes clear, this digital system is a bandstop filter (BSF).

(d) From part (b), we know that $H[z] = \frac{\frac{7}{32}(z^2+1)}{z^2 - \frac{9}{16}}$. Thus, the inverse system has transfer function

$$H^{-1}[z] = \frac{\frac{32}{7}(z^2 - \frac{9}{16})}{z^2 + 1} = \frac{\frac{32}{7}z^2 - \frac{18}{7}}{z^2 + 1}.$$

Figure S5.4-13d shows the DFI block implementation of $H^{-1}[z]$. Since the inverse system has poles on the unit circle ($z = \pm j$), the inverse system is marginally stable (and BIBO unstable).

The inverse system is BIBO unstable and will not operate well.
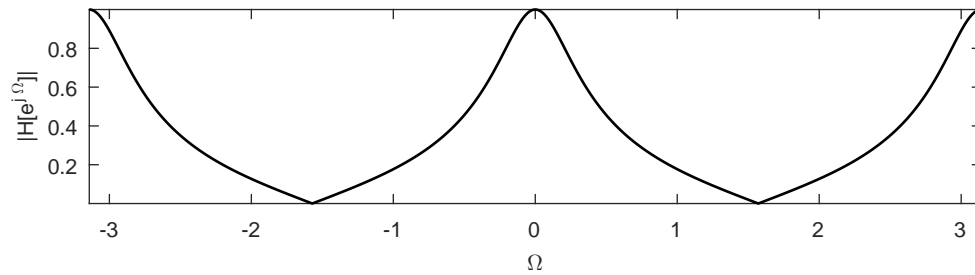
Figure S5.4-13c
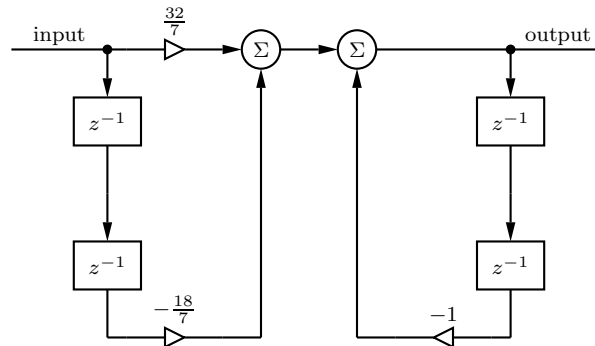
Figure S5.4-13d

Solution 5.5-1

Sampling $\cos(\omega t)$ at uniform instants $t = nT$ yields a discrete-time sinusoid $\cos(\omega T n) = \cos(\Omega n)$. Thus, we see that $\Omega = \omega T$ or $\omega = \Omega/T$. Using $F_\mathrm{s} = 1/T$, this becomes

$$\omega = \Omega F_\mathrm{s}.$$

(a) For $\Omega = \frac{\pi}{4}$, $\omega = \frac{\pi}{4}(1000) = 250\pi$ rad/s.

(b) For $\Omega = \frac{2\pi}{3}$, $\omega = \frac{2\pi}{3}(1000) = \frac{2000\pi}{3}$ rad/s.

(c) For $\Omega = \frac{7}{8}$, $\omega = \frac{7}{8}(1000) = 875$ rad/s.

Solution 5.5-2

(a) By inspection, the transfer function of Fig. P5.5-2a is

$$H_\mathrm{a}[z] = \frac{1}{z - 0.4}.$$

Thus,

$$H_\mathrm{a}\left[e^{j\Omega}\right] = \frac{1}{e^{j\Omega} - 0.4} = \frac{1}{\cos(\Omega) - 0.4 + j\sin(\Omega)}.$$

To determine the magnitude response, we see that

$$\left|H_\mathrm{a}\left[e^{j\Omega}\right]\right|^2 = H_\mathrm{a}\left[e^{j\Omega}\right] H_\mathrm{a}^*\left[e^{j\Omega}\right] = \frac{1}{(e^{j\Omega} - 0.4)(e^{-j\Omega} - 0.4)} = \frac{1}{1.16 - 0.8\cos(\Omega)}.$$

Thus,

$$\left|H_\mathrm{a}\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{1.16 - 0.8\cos(\Omega)}}.$$

Furthermore,

$$\angle H_{\mathrm{a}}\left[e^{j\Omega}\right] = -\tan^{-1}\left(\frac{\sin(\Omega)}{\cos(\Omega) - 0.4}\right).$$

We use MATLAB to compute and plot the system magnitude and phase responses, which are shown in Fig. S5.5-2a. Based on $|H_{\mathrm{a}}[e^{j\Omega}]|$, we see that this system is lowpass (or low-enhance) in nature.

```
>>  Ha = @(z) 1./(z-0.4); Omega = linspace(-pi,pi,1001);
>>  subplot(121); plot(Omega,abs(Ha(exp(1j*Omega))));
>>  ylabel('|H_a[e^{j\Omega}]|'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:1/3:2); axis([-pi pi 0 2]);
>>  subplot(122); plot(Omega,angle(Ha(exp(1j*Omega))));
>>  ylabel('\angle H_a[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi); axis([-pi pi -pi pi]);
```


Figure S5.5-2a

(b) By inspection, the transfer function of Fig. P5.5-2b is

$$H_{\mathrm{b}}[z] = \frac{z}{z - 0.4} = \frac{1}{1 - 0.4z^{-1}}.$$

Thus,

$$H_{\mathrm{b}}\left[e^{j\Omega}\right] = \frac{1}{1 - 0.4e^{-j\Omega}} = \frac{1}{1 - 0.4\cos(\Omega) + j0.4\sin(\Omega)}.$$

To determine the magnitude response, we see that

$$\left|H_{\mathrm{b}}\left[e^{j\Omega}\right]\right|^2 = H_{\mathrm{b}}\left[e^{j\Omega}\right]H_{\mathrm{b}}^*\left[e^{j\Omega}\right] = \frac{1}{(1 - 0.4e^{-j\Omega})(1 - 0.4e^{j\Omega})} = \frac{1}{1.16 - 0.8\cos(\Omega)}.$$

Thus,

$$\left|H_{\mathrm{b}}\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{1.16 - 0.8\cos(\Omega)}}.$$

Furthermore,

$$\angle H_{\mathrm{b}}\left[e^{j\Omega}\right] = -\tan^{-1}\left(\frac{0.4\sin(\Omega)}{1 - 0.4\cos(\Omega)}\right).$$

We use MATLAB to compute and plot the system magnitude and phase responses, which are shown in Fig. S5.5-2b. Based on $|H_{\mathrm{b}}[e^{j\Omega}]|$, we see that this system is lowpass (or low-enhance) in nature. Notice that $\left|H_{\mathrm{b}}\left[e^{j\Omega}\right]\right|$ is exactly the same as $\left|H_{\mathrm{a}}\left[e^{j\Omega}\right]\right|$; the only difference between $H_{\mathrm{a}}[e^{j\Omega}]$ and $H_{\mathrm{b}}[e^{j\Omega}]$ is in the phase response.

```
>>  Hb = @(z) z./(z-0.4); Omega = linspace(-pi,pi,1001);
>>  subplot(121); plot(Omega,abs(Hb(exp(1j*Omega))));
>>  ylabel('|H_b[e^{j\Omega}]|'); xlabel('\Omega'); grid on
```

```
>>   set(gca,'xtick',-pi:pi/2:pi,'ytick',0:1/3:2); axis([-pi pi 0 2]);
>>   subplot(122); plot(Omega,angle(Hb(exp(1j*Omega))));
>>   ylabel('\angle H_b[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>   set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi); axis([-pi pi -pi pi]);
```
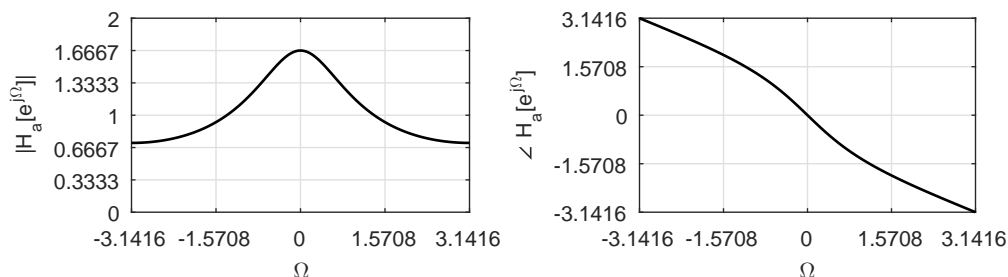

Figure S5.5-2b

(c) By inspection, the transfer function of Fig. P5.5-2c is

$$H_c[z] = \frac{3z^2 + 1.8z}{z^2 - z + 0.16}.$$

Thus,

$$H_c\left[e^{j\Omega}\right] = \frac{3e^{j2\Omega} + 1.8e^{j\Omega}}{e^{j2\Omega} - e^{j\Omega} + 0.16} = \frac{3\cos(2\Omega) + 1.8\cos(\Omega) + j[3\sin(2\Omega) + 1.8\sin(\Omega)]}{\cos(2\Omega) - \cos(\Omega) + 0.16 + j[\sin(2\Omega) - \sin(\Omega)]}.$$

To determine the magnitude response, we see that

$$\left|H_c\left[e^{j\Omega}\right]\right|^2 = H_c\left[e^{j\Omega}\right] H_c^*\left[e^{j\Omega}\right] = \left(\frac{3e^{j2\Omega} + 1.8e^{j\Omega}}{e^{j2\Omega} - e^{j\Omega} + 0.16}\right)\left(\frac{3e^{-j2\Omega} + 1.8e^{-j\Omega}}{e^{-j2\Omega} - e^{-j\Omega} + 0.16}\right)$$

$$= \frac{12.24 + 10.8\cos(\Omega)}{2.0256 - 2.32\cos(\Omega) + 0.32\cos(2\Omega)}.$$

Thus,

$$\left|H_c\left[e^{j\Omega}\right]\right| = \sqrt{\frac{12.24 + 10.8\cos(\Omega)}{2.0256 - 2.32\cos(\Omega) + 0.32\cos(2\Omega)}}.$$

Furthermore,

$$\angle H_c\left[e^{j\Omega}\right] = \tan^{-1}\left(\frac{3\sin(2\Omega) + 1.8\sin(\Omega)}{3\cos(2\Omega) + 1.8\cos(\Omega)}\right) - \tan^{-1}\left(\frac{\sin(2\Omega) - \sin(\Omega)}{\cos(2\Omega) - \cos(\Omega) + 0.16}\right).$$

We use MATLAB to compute and plot the system magnitude and phase responses, which are shown in Fig. S5.5-2c. Based on $|H_c[e^{j\Omega}]|$, we see that this system is lowpass in nature.

```
>>   Hc = @(z) (3*z.^2+1.8*z)./(z.^2-z+0.16); Omega = linspace(-pi,pi,1001);
>>   subplot(121); plot(Omega,abs(Hc(exp(1j*Omega))));
>>   ylabel('|H_c[e^{j\Omega}]|'); xlabel('\Omega'); grid on
>>   set(gca,'xtick',-pi:pi/2:pi,'ytick',0:5:35); axis([-pi pi 0 35]);
>>   subplot(122); plot(Omega,angle(Hc(exp(1j*Omega))));
>>   ylabel('\angle H_c[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>   set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi); axis([-pi pi -pi pi]);
```
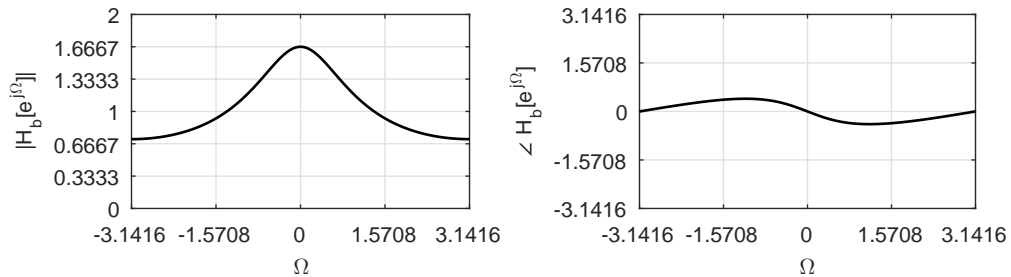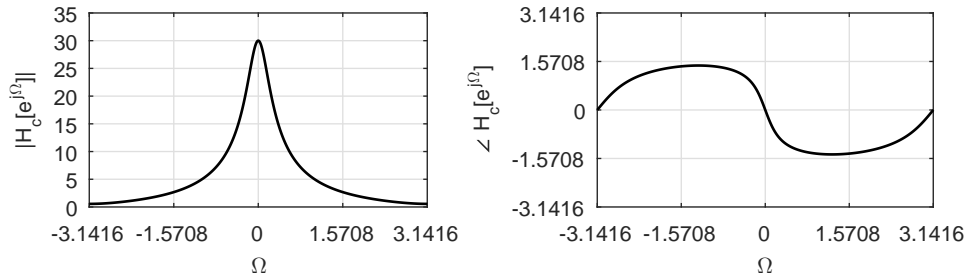
Figure S5.5-2c

Solution 5.5-3

(a) We can easily compute the magnitude response at some key points:

$$|H(\pm j)| = 0, \quad |H(1)| = \frac{21}{16}\left(\frac{1+1}{1+\frac{1}{4}-\frac{3}{8}}\right) = 3, \quad |H(-1)| = \frac{21}{16}\left(\frac{1+1}{1-\frac{1}{4}-\frac{3}{8}}\right) = 7.$$

We use MATLAB to confirm these points and plot the magnitude response over $-2\pi \leq \Omega \leq 2\pi$.

```
>>  H = @(z) 21/16*(z.^2+1)./(z.^2+z/4-3/8); Omega = linspace(-2*pi,2*pi,1001);
>>  plot(Omega,abs(H(exp(1j*Omega))));
>>  ylabel('|H[e^{j\Omega}]|'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-2*pi:pi/2:2*pi,'ytick',0:1:8); axis([-2*pi 2*pi 0 8]);
```


Figure S5.5-3a

(b) We use MATLAB to compute and plot the phase response over $-2\pi \leq \Omega \leq 2\pi$.

```
>>  H = @(z) 21/16*(z.^2+1)./(z.^2+z/4-3/8); Omega = linspace(-2*pi,2*pi,1001);
>>  plot(Omega,angle(H(exp(1j*Omega))));
>>  ylabel('\angle H[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-2*pi:pi/2:2*pi,'ytick',-pi:pi/2:pi); axis([-2*pi 2*pi -pi pi]);
```

(c) To begin, we note that $x[n]$ can be written more compactly as

$$x[n] = \tfrac{3}{2} + \tfrac{1}{2}(-1)^n.$$

Written in this form, we see that $x[n]$ is comprised of two everlasting sinusoids of frequencies $\Omega = 0$ and $\Omega = \pi$. Consequently, we can determine the output using the concept of frequency response as

$$y[n] = \tfrac{3}{2}H[e^{j0}] + \tfrac{1}{2}(-1)^n H[e^{j\pi}].$$

Using the results from (a) and (b), we see that

$$y[n] = \tfrac{9}{2} + \tfrac{7}{2}(-1)^n.$$
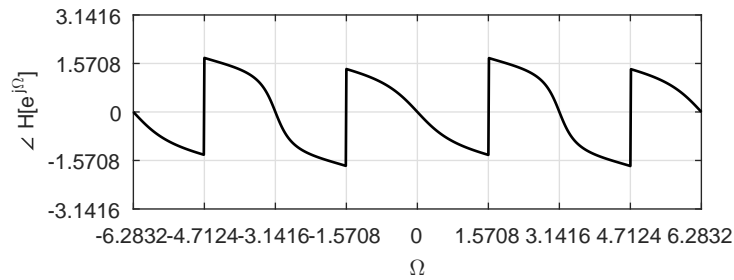
Figure S5.5-3b

That is,

$$y[n] = [\ldots, 1, 8, 1, \overset{\downarrow}{8}, 1, 8, 1, \ldots].$$

Solution 5.5-4

(a) We can easily compute the magnitude response at some key points:

$$|H[e^{j0}]| = \frac{7}{32}\left(\frac{2}{\frac{5}{4}(\frac{5}{4})}\right) = \frac{7}{25}, \quad |H[e^{j\pi/2}]| = \frac{7}{32}\frac{\sqrt{2}}{\frac{1}{4}(\frac{7}{4})} = \frac{1}{\sqrt{2}}, \quad |H[e^{j\pi}]| = 0.$$

We use MATLAB to confirm these points and plot the magnitude response over $-2\pi \leq \Omega \leq 2\pi$.

```
>>  H = @(z) -7/32*(z+1)./(z.^2+9/16); Omega = linspace(-2*pi,2*pi,1001);
>>  plot(Omega,abs(H(exp(1j*Omega))));
>>  ylabel('|H[e^{j\Omega}]|'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-2*pi:pi/2:2*pi,'ytick',0:.1:.8); axis([-2*pi 2*pi 0 .8]);
```


Figure S5.5-4a

(b) We use MATLAB to compute and plot the phase response over $-2\pi \leq \Omega \leq 2\pi$.

```
>>  H = @(z) -7/32*(z+1)./(z.^2+9/16); Omega = linspace(-2*pi,2*pi,1001);
>>  plot(Omega,angle(H(exp(1j*Omega))));
>>  ylabel('\angle H[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-2*pi:pi/2:2*pi,'ytick',-pi:pi/2:pi); axis([-2*pi 2*pi -pi pi]);
```
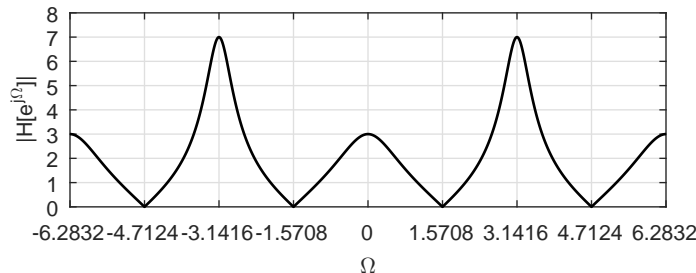
(c) To begin, we note that $x[n]$ can be written more compactly as

$$x[n] = 2 + \cos(\pi n/2).$$

Written in this form, we see that $x[n]$ is comprised of two everlasting sinusoids of frequencies $\Omega = 0$ and $\Omega = \pi/2$. Consequently, we can determine the output using the concept of frequency response as

$$y[n] = 2H[e^{j0}] + |H[e^{j\pi/2}]|\cos\left(\tfrac{\pi}{2}n + \angle H[e^{j\pi/2}]\right).$$
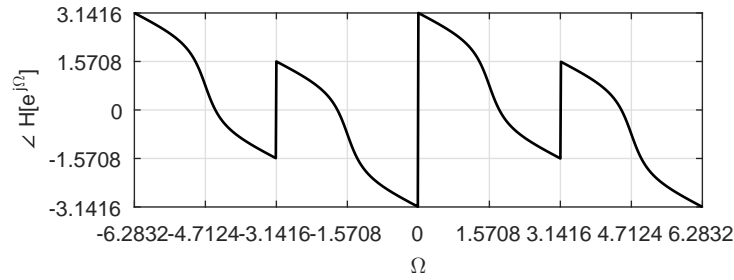
Figure S5.5-4b

Using the results from (a) and (b), we see that

$$y[n] = -\tfrac{14}{25} + \tfrac{1}{\sqrt{2}}\cos\left(\tfrac{\pi}{2}n + \tfrac{\pi}{4}\right).$$

Solution 5.5-5

Since neither system has feedback, both are finite impulse response (FIR) systems.

(a) By inspection, the transfer function of the first FIR filter is

$$H_\mathrm{a}[z] = 1 + 0.5z^{-1} + 2z^{-2} + 2z^{-3} + 0.5z^{-4} + z^{-5}.$$

The frequency response is therefore

$$
\begin{aligned}
H_\mathrm{a}\left(e^{j\Omega}\right) &= 1 + 0.5e^{-j\Omega} + 2e^{-j2\Omega} + 2e^{-j3\Omega} + 0.5e^{-j4\Omega} + e^{-j5\Omega} \\
&= e^{-j2.5\Omega}\left(e^{j2.5\Omega} + 0.5e^{j1.5\Omega} + 2e^{j0.5\Omega} + 2e^{-j0.5\Omega} + 0.5e^{-j1.5\Omega} + e^{-j2.5\Omega}\right) \\
&= e^{-j2.5\Omega}\left(2\cos(2.5\Omega) + \cos(1.5\Omega) + 4\cos(0.5\Omega)\right).
\end{aligned}
$$

The magnitude and phase responses, shown in Fig. S5.5-5a, are

$$\left|H_\mathrm{a}\left(e^{j\Omega}\right)\right| = |2\cos(2.5\Omega) + \cos(1.5\Omega) + 4\cos(0.5\Omega)| \qquad \text{and} \qquad \angle H_\mathrm{a}\left(e^{j\Omega}\right) = -2.5\Omega.$$

```
>>  Ha = @(z) 1+0.5*z.^(-1)+2*z.^(-2)+2*z.^(-3)+0.5*z.^(-4)+z.^(-5);
>>  Omega = linspace(-pi,pi,1001); subplot(121); plot(Omega,abs(Ha(exp(1j*Omega))));
>>  ylabel('|H_a[e^{j\Omega}]|'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:1:8); axis([-pi pi 0 8]);
>>  subplot(122); plot(Omega,angle(Ha(exp(1j*Omega))));
>>  ylabel('\angle H_a[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi); axis([-pi pi -pi pi]);
```
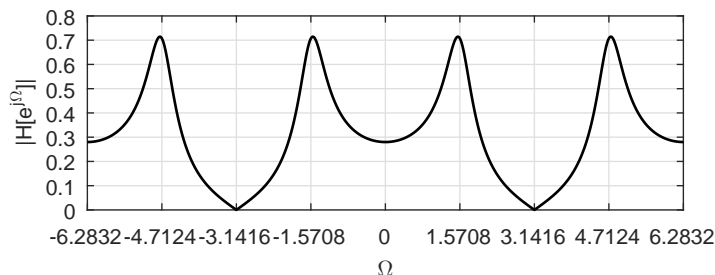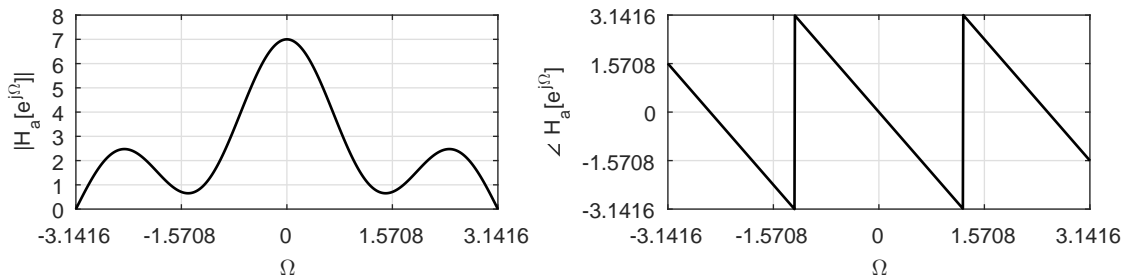

Figure S5.5-5a

(b) By inspection, the transfer function of the second FIR filter is

$$H_b[z] = 1 + 0.5z^{-1} + 2z^{-2} - 2z^{-3} - 0.5z^{-4} - z^{-5}.$$

The frequency response is therefore

$$\begin{aligned}
H_b(e^{j\Omega}) &= 1 + 0.5e^{-j\Omega} + 2e^{-j2\Omega} - 2e^{-j3\Omega} - 0.5e^{-j4\Omega} - e^{-j5\Omega} \\
&= e^{-j2.5\Omega}\left(e^{j2.5\Omega} + 0.5e^{j1.5\Omega} + 2e^{j0.5\Omega} - 2e^{-j0.5\Omega} - 0.5e^{-j1.5\Omega} - e^{-j2.5\Omega}\right) \\
&= e^{j(\frac{\pi}{2} - 2.5\Omega)}\left(2\sin(2.5\Omega) + \sin(1.5\Omega) + 4\sin(0.5\Omega)\right).
\end{aligned}$$

The magnitude response is

$$\left|H_b(e^{j\Omega})\right| = \left|2\sin(2.5\Omega) + \sin(1.5\Omega) + 4\sin(0.5\Omega)\right|.$$

Since $2\sin(2.5\Omega) + \sin(1.5\Omega) + 4\sin(0.5\Omega) > 0$ for $0 < \Omega < \pi$ and $2\sin(2.5\Omega) + \sin(1.5\Omega) + 4\sin(0.5\Omega) < 0$ for $-\pi < \Omega < 0$, the phase response is (over $-\pi \le \Omega < \pi$)

$$\angle H_b(e^{j\Omega}) = \text{sgn}(\Omega)\frac{\pi}{2} - 2.5\Omega.$$

The magnitude and phase responses are shown in Fig. S5.5-5b. Clearly, this FIR filter has linear phase.

```
>>  Hb = @(z) 1+0.5*z.^(-1)+2*z.^(-2)-2*z.^(-3)-0.5*z.^(-4)-z.^(-5);
>>  Omega = linspace(-pi,pi,1001); subplot(121); plot(Omega,abs(Hb(exp(1j*Omega))));
>>  ylabel('|H_b[e^{j\Omega}]|'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:1:6); axis([-pi pi 0 6]);
>>  subplot(122); plot(Omega,angle(Hb(exp(1j*Omega))));
>>  ylabel('\angle H_b[e^{j\Omega}]'); xlabel('\Omega'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi); axis([-pi pi -pi pi]);
```


Figure S5.5-5b

Solution 5.5-6

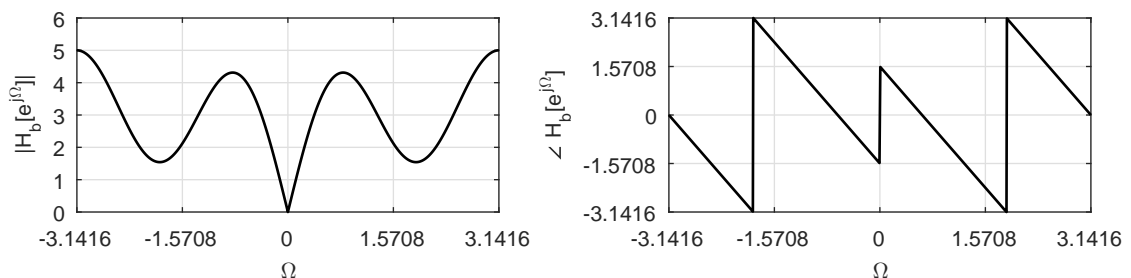In this problem, we consider the 5-point moving-average system given by

$$y[n] = \frac{1}{5}\sum_{k=0}^{4} x[n-k].$$

Taking the z-transform of this difference equation yields

$$Y[z] = \frac{1}{5}\sum_{k=0}^{4} z^{-k}X[z] = X[z]\frac{1}{5}\sum_{k=0}^{4} z^{-k}.$$

Solving for the transfer function yields

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1}{5} \sum_{k=0}^{4} z^{-k}$$

The system frequency response is therefore given as

$$H\left[e^{j\Omega}\right] = \frac{1}{5} \sum_{k=0}^{4} e^{-jk\Omega}$$

Expanding and simplifying yield

$$H\left[e^{j\Omega}\right] = \frac{1}{5} \left(1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega} + e^{-j4\Omega} +\right) = \frac{1}{5} e^{-j2\Omega} \left(1 + 2\cos(\Omega) + 2\cos(2\Omega)\right).$$

Solution 5.5-7

(a) Taking the z-transform of the two difference equations yields

$$Y[z]\left(1 + 0.9z^{-1}\right) = X[z] \quad \text{and} \quad Y[z]\left(1 - 0.9z^{-1}\right) = X[z].$$

The transfer functions of the two filters are therefore

$$H_{\mathrm{i}}[z] = \frac{1}{1 + 0.9z^{-1}} \quad \text{and} \quad H_{\mathrm{ii}}[z] = \frac{1}{1 - 0.9z^{-1}}.$$

The frequency response of the first system is

$$H_{\mathrm{i}}\left[e^{j\Omega}\right] = \frac{1}{1 + 0.9e^{-j\Omega}} = \frac{1}{1 + 0.9\cos(\Omega) - j0.9\sin(\Omega)}.$$

The corresponding magnitude response and phase response are

$$\left|H_{\mathrm{i}}\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{1.81 + 1.8\cos(\Omega)}} \quad \text{and} \quad \angle H_{\mathrm{i}}\left[e^{j\Omega}\right] = -\tan^{-1}\left(\frac{-0.9\sin(\Omega)}{1 + 0.9\cos(\Omega)}\right).$$

The frequency response of the second system is

$$H_{\mathrm{ii}}\left[e^{j\Omega}\right] = \frac{1}{1 - 0.9e^{-j\Omega}} = \frac{1}{1 - 0.9\cos(\Omega) + j0.9\sin(\Omega)}.$$

The corresponding magnitude response and phase response are

$$\left|H_{\mathrm{ii}}\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{1.81 - 1.8\cos(\Omega)}} \quad \text{and} \quad \angle H_{\mathrm{ii}}\left[e^{j\Omega}\right] = -\tan^{-1}\left(\frac{0.9\sin(\Omega)}{1 - 0.9\cos(\Omega)}\right).$$

The first filter $H_{\mathrm{i}}[z]$ has a zero at the origin and a pole at -0.9. Because the pole is near $\Omega = \pi$ ($z = -1$), this is a highpass filter, as verified from the magnitude response plot shown in Fig. S5.5-7a.

The second filter $H_{\mathrm{ii}}[z]$ has a zero at the origin and a pole at 0.9. Because the pole is near $\Omega = 0$ ($z = 1$), this is a lowpass filter, as verified from the magnitude response plot shown in Fig. S5.5-7a.

```
>>  Omega = linspace(-pi,pi,1001);
>>  Hi = @(z) z./(z+0.9); Hii = @(z) z./(z-0.9);
>>  subplot(121); plot(Omega,abs(Hi(exp(1j*Omega)))); grid on;
>>  xlabel('\Omega'); ylabel('|H_i[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:2:10); axis([-pi pi 0 10]);
>>  subplot(122); plot(Omega,abs(Hii(exp(1j*Omega)))); grid on;
>>  xlabel('\Omega'); ylabel('|H_{ii}[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:2:10); axis([-pi pi 0 10]);
```

$$|H_i[e^{j\Omega}]|$$ (plot, left) $$|H_{ii}[e^{j\Omega}]|$$ (plot, right)

Figure S5.5-7a

(b) To determine the filter responses to $x[n] = \cos(0.01\pi n)$, we use MATLAB compute the magnitude and phase responses at $\Omega = 0.01\pi$.

```
>>  [abs(Hi(exp(j*0.01*pi))), angle(Hi(exp(j*0.01*pi)))]
    ans = 0.5264    0.0149
>>  [abs(Hii(exp(j*0.01*pi))), angle(Hii(exp(j*0.01*pi)))]
    ans = 9.5835    -0.2743
```

Thus, the outputs of the first and second filters to input $x[n] = \cos(0.01\pi n)$ are, respectively,

$$y_i[n] = 0.5264\cos(0.01\pi n + 0.0149) \qquad \text{and} \qquad y_{ii}[n] = 9.5835\cos(0.01\pi n - 0.2743).$$

To determine the filter responses to $x[n] = \cos(0.99\pi n)$, we use MATLAB compute the magnitude and phase responses at $\Omega = 0.99\pi$.

```
>>  [abs(Hi(exp(j*0.99*pi))), angle(Hi(exp(j*0.99*pi)))]
    ans = 9.5835    0.2743
>>  [abs(Hii(exp(j*0.99*pi))), angle(Hii(exp(j*0.99*pi)))]
    ans = 0.5264    -0.0149
```

Thus, the outputs of the first and second filters to input $x[n] = \cos(0.99\pi n)$ are, respectively,

$$y_i[n] = 9.5835\cos(0.99\pi n + 0.2743) \qquad \text{and} \qquad y_{ii}[n] = 0.5264\cos(0.01\pi n - 0.0149).$$

Since $\cos(\Omega_0) = -\cos(\pi - \Omega_0)$ we see that

$$\left|H_i\left[e^{j\Omega_0}\right]\right| = \frac{1}{\sqrt{1.81 + 1.8\cos(\Omega_0)}} = \frac{1}{\sqrt{1.81 - 1.8\cos(\pi - \Omega_0)}} = \left|H_{ii}\left[e^{j(\pi-\Omega_0)}\right]\right|.$$

That is, the gain (magnitude response) of the first filter at frequency Ω_0 is the same as the gain of the second filter at frequency $\pi - \Omega_0$. Thus, the gain response of the first system at $\Omega_0 = 0.01\pi$ will be the same as the gain response of the second system at $\Omega_0 = 0.99\pi$. Similarly, the gain response of the first system at $\Omega_0 = 0.99\pi$ will be the same as the gain response of the second system at $\Omega_0 = 0.01\pi$. This is precisely the behavior observed previously.

Solution 5.5-8

By inspection of the difference equation, the system's transfer function is

$$H[z] = \frac{z + 0.8}{z - 0.5}.$$

(a) The frequency response of the system is

$$H\left(e^{j\Omega}\right) = H[z]\big|_{z=e^{j\Omega}} = \frac{e^{j\Omega} + 0.8}{e^{j\Omega} - 0.5} = \frac{\cos(\Omega) + 0.8 + j\sin(\Omega)}{\cos(\Omega) - 0.5 + j\sin(\Omega)}.$$

To determine the magnitude response, we see that

$$\left|H\left[e^{j\Omega}\right]\right|^2 = H\left[e^{j\Omega}\right] H^*\left[e^{j\Omega}\right] = \frac{(e^{j\Omega} + 0.8)(e^{-j\Omega} + 0.8)}{(e^{j\Omega} - 0.5)(e^{-j\Omega} - 0.5)} = \frac{1.64 + 1.6\cos(\Omega)}{1.25 - \cos(\Omega)}.$$

Thus, the magnitude response is

$$\left|H\left[e^{j\Omega}\right]\right| = \sqrt{\frac{1.64 + 1.6\cos(\Omega)}{1.25 - \cos(\Omega)}}.$$

The phase responses is

$$\angle H\left[e^{j\Omega}\right] = \tan^{-1}\left(\frac{\sin(\Omega)}{\cos(\Omega) + 0.8}\right) - \tan^{-1}\left(\frac{\sin(\Omega)}{\cos(\Omega) - 0.5}\right).$$

The magnitude and phase responses are shown in Fig. S5.5-8. H = @[z] (z+0.8)./(z-0.5);

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) (z+0.8)./(z-0.5);
>>  subplot(121); plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:.5:4); axis([-pi pi 0 4]);
>>  subplot(122); plot(Omega,angle(H(exp(1j*Omega)))); grid on;
>>  xlabel('\Omega'); ylabel('\angle H[e^{j\Omega}]');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi); axis([-pi pi -pi pi]);
```
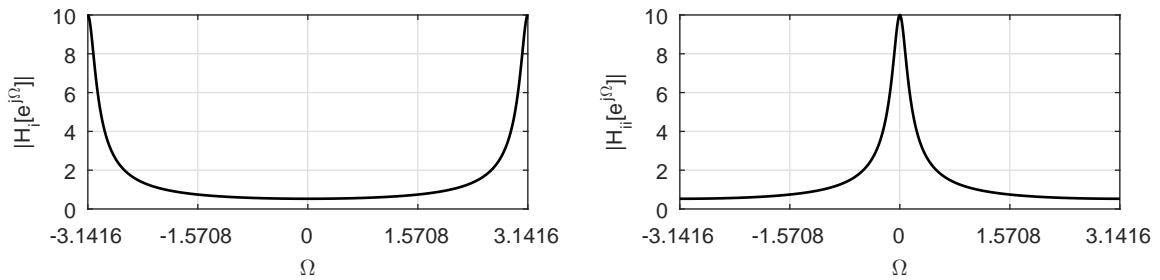

Figure S5.5-8a

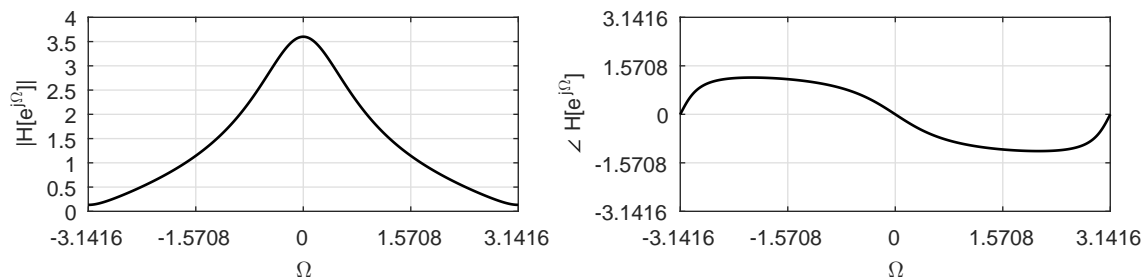**(b)** To determine the response $y[n]$ to input $x[n] = \cos(0.5n - \pi/3)$, we need to evaluate the magnitude response and phase response at $\Omega = 0.5$.

```
>>  [abs(H(exp(1j*0.5))),angle(H(exp(1j*0.5)))]
    ans = 2.8590   -0.6253
```

Thus,

$$y[n] = 2.8590\cos(0.5n - \pi/3 - 0.6253) = 2.8590\cos(0.5n - 1.6725).$$

Solution 5.5-9

From Eq. (5.20), we know that

$$Y[z] = X[z]H[z].$$

For the input $x[n] = e^{j\Omega n}u[n]$, entry 6 of Table 5.1 tells us that $X[z] = \frac{z}{z-e^{j\Omega}}$. Thus,

$$Y[z] = \frac{z}{z - e^{j\Omega}}H[z].$$

Assuming $H[z]$ follows a standard rational form where

$$H[z] = \frac{B[z]}{A[z]} = \frac{B[z]}{(z-p_1)(z-p_2)\cdots(z-p_N)},$$

then

$$Y[z] = \frac{zB[z]}{(z-e^{j\Omega})(z-p_1)(z-p_2)\cdots(z-p_N)}$$

and

$$\frac{Y[z]}{z} = \frac{B[z]}{(z-e^{j\Omega})(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$= \frac{c_0}{z-e^{j\Omega}} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_K}{z-p_N}.$$

The coefficient c_0 is computed as

$$c_0 = \frac{B[z]}{(z-e^{j\Omega})(z-p_1)(z-p_2)\cdots(z-p_N)}(z-e^{j\Omega})\Big|_{z=e^{j\Omega}} = H[z]|_{z=e^{j\Omega}} = H\left(e^{j\Omega}\right).$$

Therefore,

$$Y[z] = H\left[e^{j\Omega}\right]\frac{z}{z-e^{j\Omega}} + \sum_{i=1}^{N} c_i\frac{z}{z-p_i}.$$

and

$$y[n] = \left[H\left[e^{j\Omega}\right]e^{j\Omega n} + \sum_{i=1}^{N} c_i p_i{}^n\right]u[n].$$

For an asymptotically stable system $|p_i| < 1$ ($i = 1, 2, \ldots, N$), and the sum on the right-hand side vanishes as $n \to \infty$. This sum is therefore the *transient* component of the response. The term $H\left[e^{j\Omega}\right]e^{j\Omega n}$, which does not vanish as $n \to \infty$, is the *steady-state* component of the response $y_{ss}[n]$. Thus, the steady-state response of an asymptotically stable LTID system to input $x[n] = e^{j\Omega n}u[n]$ is

$$y_{ss}[n] = H\left[e^{j\Omega}\right]e^{j\Omega n}u[n].$$

Solution 5.5-10

For each of the following, designate Ω_a as the apparent frequency.

(a) Because $\Omega = 0.8\pi$ is in the fundamental range, $\Omega_a = \Omega = 0.8\pi$ and the signal appears unchanged as

$$\cos(0.8\pi n + \theta).$$

(b) Because $\Omega_a = \langle 1.2\pi + \pi\rangle_{2\pi} - \pi = -0.8\pi$,

$$\sin(1.2\pi n + \theta) = \sin(-0.8\pi n + \theta) = -\sin(0.8\pi n - \theta).$$

(c) Because $\Omega_a = \langle 6.9 + \pi\rangle_{2\pi} - \pi = 0.6168$,

$$\cos(6.9n + \theta) = \cos(0.6168n + \theta).$$

(d) Because $\langle 2.8\pi + \pi \rangle_{2\pi} - \pi = 0.8\pi$ and $\langle 3.7\pi + \pi \rangle_{2\pi} - \pi = -0.3\pi$, the apparent frequencies of $\cos(2.8\pi n + \theta)$ and $2\sin(3.7\pi n + \theta)$ are 0.8π and -0.3π, respectively. Hence,

$$\cos(2.8\pi n + \theta) + 2\sin(3.7\pi n + \theta) = \cos(0.8\pi n + \theta) + 2\sin(-0.3\pi n + \theta)$$
$$= \cos(0.8\pi n + \theta) - 2\sin(0.3\pi n - \theta).$$

(e) From the definition of sinc, we know that

$$\operatorname{sinc}\left(\frac{\pi n}{2}\right) = \frac{\sin(\pi n/2)}{(\pi n/2)}.$$

Since $\Omega = \pi/2$ is in the fundamental range, $\Omega_{\mathrm{a}} = \pi/2$ and the signal appears unchanged as

$$\operatorname{sinc}\left(\frac{\pi n}{2}\right).$$

(f) From the definition of sinc, we know that

$$\operatorname{sinc}\left(\frac{3\pi n}{2}\right) = \frac{\sin(3\pi n/2)}{(3\pi n/2)}.$$

Because $\Omega_{\mathrm{a}} = \langle 3\pi/2 + \pi \rangle_{2\pi} - \pi = -\pi/2$, the signal appears as

$$\operatorname{sinc}\left(\frac{3\pi n}{2}\right) = \frac{\sin(-\pi n/2)}{(3\pi n/2)}$$
$$= -\frac{1}{3}\operatorname{sinc}\left(\frac{\pi n}{2}\right).$$

(g) From the definition of sinc, we know that

$$\operatorname{sinc}\left(2\pi n\right) = \frac{\sin(2\pi n)}{(2\pi n)}.$$

Since $\sin(2\pi n) = 0$ for all n, this expression is 0 for all $n \neq 0$. At $n = 0$, we have $\operatorname{sinc}(0) = 1$. Thus, the signal appears as

$$\operatorname{sinc}\left(2\pi n\right) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}.$$

Except for $n = 0$, this is consistent with the apparent frequency $\Omega_{\mathrm{a}} = \langle 2\pi + \pi \rangle_{2\pi} - \pi = 0$, for which sin is always 0.

Solution 5.5-11

Because $\langle 1.4\pi + \pi \rangle_{2\pi} - \pi = -0.6\pi$, frequency $\Omega = 1.4\pi$ appears as $\Omega_{\mathrm{a}} = -0.6\pi$ and $\cos(1.4\pi n + \frac{\pi}{3}) = \cos(-0.6\pi n + \frac{\pi}{3}) = \cos(0.6\pi n - \frac{\pi}{3})$. Also

$$\cos\left(0.6\pi n + \tfrac{\pi}{6}\right) = \cos(0.6\pi n)\cos(\tfrac{\pi}{6}) - \sin(0.6\pi n)\sin(\tfrac{\pi}{6})$$
$$= \tfrac{\sqrt{3}}{2}\cos(0.6\pi n) - \tfrac{1}{2}\sin(0.6\pi n).$$

Similarly,

$$\sqrt{3}\cos\left(0.6\pi n - \tfrac{\pi}{3}\right) = \sqrt{3}\cos(0.6\pi n)\cos(\tfrac{\pi}{3}) + \sqrt{3}\sin(0.6\pi n)\sin(\tfrac{\pi}{3})$$
$$= \tfrac{\sqrt{3}}{2}\cos(0.6\pi n) + \tfrac{3}{2}\sin(0.6\pi n).$$

Adding these results yields

$$\cos\left(0.6\pi n + \tfrac{\pi}{6}\right) + \sqrt{3}\cos\left(1.4\pi n + \tfrac{\pi}{3}\right) = \cos\left(0.6\pi n + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(0.6\pi n - \frac{\pi}{3}\right)$$
$$= \sqrt{3}\cos(0.6\pi n) + \sin(0.6\pi n).$$

From trigonometry, we know that $a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2}\cos(x + \tan^{-1}(-b/a))$. Since $\sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$, we thus obtain the desired result of

$$\cos\left(0.6\pi n + \tfrac{\pi}{6}\right) + \sqrt{3}\cos\left(1.4\pi n + \tfrac{\pi}{3}\right) = 2\cos\left(0.6\pi n - \tfrac{\pi}{6}\right).$$

Solution 5.5-12

(a) For $T = 50$ μs, the sampling frequency is $F_s = \frac{1}{T} = 20$ kHz. From Nyquist, we know that aliasing begins to occur when input frequencies exceed $\frac{F_s}{2}$. Therefore, the maximum frequency that can be processed by this filter without aliasing is $f_{\max} = 10$ kHz.

(b) If the maximum input frequency to a digital system is $f_{\max} = 50$ kHz, then Nyquist requires that the system sampling frequency be no smaller than $F_s = 2f_{\max} = 100$ kHz.

Technically, if an input includes an everlasting sinusoid at exactly the maximum frequency $f_{\max}$, the sampling frequency must be strictly greater than $2f_{\max}$. Thus, part (a) really requires an input frequency just under 10 kHz and part (b) requires a sampling frequency just above 100 kHz. In practice, everlasting sinusoids cannot be generated so this distinction is unimportant.

Solution 5.5-13

Taking the z-transform of $y[n] = \sum_{k=0}^{\infty}(0.5)^k x[n-k]$ yields $Y[z] = \sum_{k=0}^{\infty}(0.5)^k X[z]z^{-k} = X[z]\sum_{k=0}^{\infty}(0.5/z)^k$. For $|z| > 1/2$, this becomes $Y[z] = X[z]\frac{1}{1-0.5z^{-1}}$. Thus, the transfer function is $H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1-0.5z^{-1}}$.

(a) Using $H[z]$ and letting $z = e^{j\Omega}$, the magnitude response is

$$\left|H[e^{j\Omega}]\right| = \left|\frac{1}{1-0.5e^{-j\Omega}}\right| = \frac{1}{\sqrt{(1-0.5\cos(\Omega))^2 + (-0.5\sin(\Omega))^2}}$$
$$= \frac{1}{\sqrt{1-\cos(\Omega) + 0.25(\cos^2(\Omega) + \sin^2(\Omega))}}.$$

Thus,

$$\left|H[e^{j\Omega}]\right| = \frac{1}{\sqrt{5/4 - \cos(\Omega)}}.$$

MATLAB is used to plot $\left|H[e^{j\Omega}]\right|$ in two ways: from the above expression and also by substitution into $H[z]$. As Fig. S5.5-13a shows, both methods yield identical results.

```
>>  Omega = linspace(-pi,pi,501);
>>  Hm = @(Omega) 1./sqrt(5/4-cos(Omega)); H = @(z) 1./(1-0.5*z.^(-1));
>>  subplot(121); plot(Omega,Hm(Omega),'k-'); grid on;
>>  axis([-pi pi 0 2.5]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  subplot(122); plot(Omega,abs(H(exp(1j*Omega))),'k-'); grid on;
>>  axis([-pi pi 0 2.5]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
```

Figure S5.5-13a

(b) Using $H[z]$ and letting $z = e^{j\Omega}$, the phase response is $\angle H(e^{j\Omega}) = \angle \frac{1}{1-0.5e^{-j\Omega}} = -\angle(1 - 0.5\cos(\Omega) - 0.5j\sin(-\Omega))$. Thus,

$$\angle H(e^{j\Omega}) = -\arctan\left(\frac{0.5\sin(\Omega)}{1 - 0.5\cos(\Omega)}\right).$$

MATLAB is used to plot $\angle H(e^{j\Omega})$ in two ways: from the above expression and also by substitution into $H[z]$. As Fig. S5.5-13b shows, both methods yield identical results.

```
>>  Omega = linspace(-pi,pi,501); H = @(z) 1./(1-0.5*z.^(-1));
>>  Ha = @(Omega) -atan2(0.5*sin(Omega),1-0.5*cos(Omega));
>>  subplot(121); plot(Omega,Ha(Omega),'k-'); grid on;
>>  axis([-pi pi -pi/2 pi/2]); xlabel('\Omega'); ylabel('\angle H[e^{j\Omega}]');
>>  subplot(122); plot(Omega,angle(H(exp(1j*Omega))),'k-'); grid on;
>>  axis([-pi pi -pi/2 pi/2]); xlabel('\Omega'); ylabel('\angle H[e^{j\Omega}]');
```


Figure S5.5-13b

(c) Since $H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1-0.5z^{-1}}$, an equivalent difference equation description is $y[n] - 0.5y[n-1] = x[n]$. From this equation, an efficient block representation is found, as shown in Fig. S5.5-13c.

Figure S5.5-13c

Solution 5.6-1

In general, pole-zero plots do not provide the overall gain b_0 of a system. For each of the two cases, we therefore normalize the magnitude response by $|b_0|$ and adjust the phase response by $-\angle b_0$.

(a) Figure S5.6-1a shows sketches of the filter's magnitude and phase responses. The magnitude response is relatively high at frequencies $\Omega = \pm\pi/4$, where the poles are closest to the unit circle. The gain is smallest at $\Omega = \pm\pi$, where the poles are farthest away. The zero at the origin does not affect the magnitude response.

The phase of the zero is zero and the phases of the two poles are equal and opposite at $\Omega = 0$. Thus, the (adjusted) phase response is 0 at $\Omega = 0$. As Ω increases, the phases of the zero and both poles increase toward π. At $\Omega = \pi$, the phase response is therefore $\pi - (\pi + \pi) = -\pi$. The phase response changes most quickly near $\Omega = \pm\pi/4$, where the phase of nearby poles are likewise rapidly changing.

By inspection of the pole-zero plot, we see that the system transfer function is, at least approximately, given by

$$H_a[z] = b_0 \frac{z}{(z - 0.75e^{j\pi/4})(z - 0.75e^{-j\pi/4})}.$$

With this expression, we can use MATLAB to readily confirm the system's frequency response characteristics.

```
>> Omega = linspace(-pi,pi,1001);
>> Ha = @(z) z./((z-0.75*exp(j*pi/4)).*(z-0.75*exp(-j*pi/4)));
>> subplot(121); plot(Omega,abs(Ha(exp(1j*Omega)))); grid on;
>> axis([-pi pi 0 3.5]); xlabel('\Omega'); ylabel('|H_a[e^{j\Omega}]/b_0|');
>> set(gca,'xtick',-pi:pi/2:pi,'ytick',0:.5:3.5);
>> subplot(122); plot(Omega,angle(Ha(exp(1j*Omega))));
>> grid on; axis([-pi pi -pi pi]);
>> xlabel('\Omega'); ylabel('\angle H_a[e^{j\Omega}]-\angle b_0');
>> set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi);
```
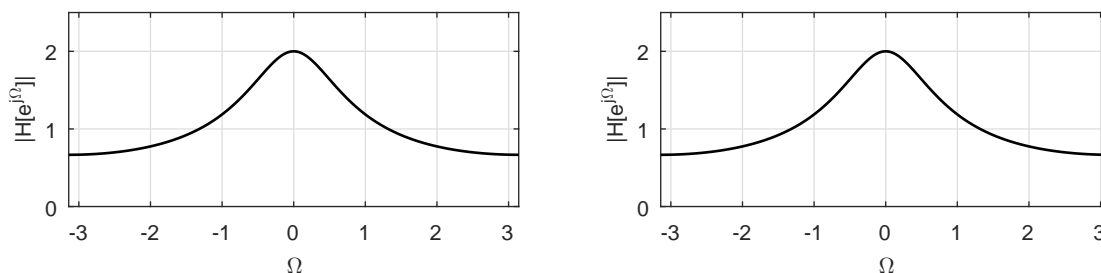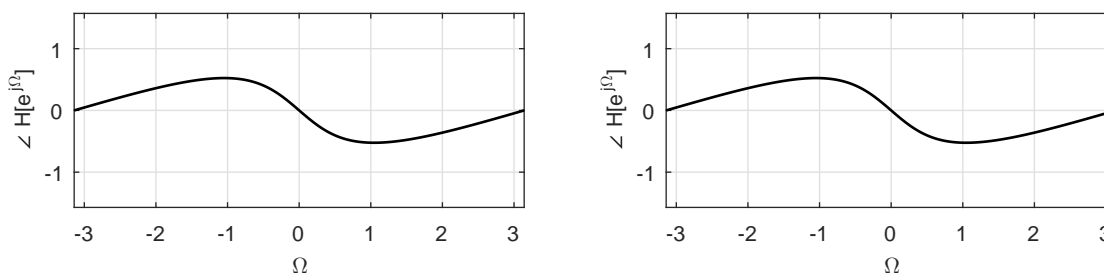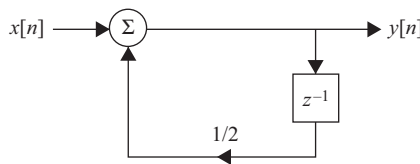

Figure S5.6-1a

(b) Figure S5.6-1b shows sketches of the filter's magnitude and phase responses. The magnitude response is relatively high at frequencies $\Omega = \pm 7\pi/8$, where the poles are closest to the unit circle. The gain is smallest at $\Omega = 0$, where the poles are farthest away. The zeros at the origin do not affect the magnitude response.

The phases of the zero are zero and the phases of the two poles are equal and opposite at $\Omega = 0$. Thus, the (adjusted) phase response is 0 at $\Omega = 0$. At $\Omega = \pi$, the phase response is $2\pi - (\pi + \theta + \pi - \theta) = 0$. As Ω moves between 0 and π, the overall phase bumps up before returning to zero. The phase response changes most quickly near $\Omega = \pm 7\pi/8$, where the phase of nearby poles are likewise rapidly changing.

By inspection of the pole-zero plot, we see that the system transfer function is, at least approximately, given by

$$H_{\mathrm{b}}[z] = b_0 \frac{z^2}{(z - 0.825e^{j7\pi/8})(z - 0.825e^{-j7\pi/8})}.$$

With this expression, we can use MATLAB to readily confirm the system's frequency response characteristics.

```
>>  Omega = linspace(-pi,pi,1001);
>>  Hb = @(z) z.^2./((z-0.825*exp(j*7*pi/8)).*(z-0.825*exp(-j*7*pi/8)));
>>  subplot(121); plot(Omega,abs(Hb(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 8.5]); xlabel('\Omega'); ylabel('|H_b[e^{j\Omega}]/b_0|');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:1:8.5);
>>  subplot(122); plot(Omega,angle(Hb(exp(1j*Omega))));
>>  grid on; axis([-pi pi -pi pi]);
>>  xlabel('\Omega'); ylabel('\angle H_b[e^{j\Omega}]-\angle b_0');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi);
```
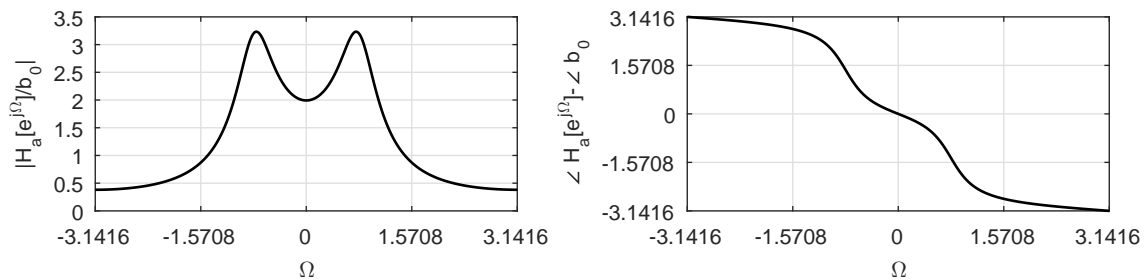

Figure S5.6-1b

Solution 5.6-2

(a) From the pole/zero plot, we see that

$$H[z] = \frac{k(z-1)}{(z-0.5j)(z+0.5j)} = \frac{k(z-1)}{z^2 + \frac{1}{4}}.$$

Further,

$$H[-1] = -1 = \frac{k(-2)}{1 + \frac{1}{4}} \quad \Rightarrow \quad -2k = -\frac{5}{4} \quad \Rightarrow k = \frac{5}{8}.$$

Referenced to $H[z] = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$, we see that

$$b_0 = 0, \quad b_1 = \frac{5}{8}, \quad b_2 = -\frac{5}{8}, \quad a_1 = 0, \quad a_2 = \frac{1}{4}.$$

(b) By inspection, we see that

$$|H[e^{j0}]| = 0 \quad \text{and} \quad |H[e^{-j\pi}]| = 1.$$

At $\Omega = \frac{\pi}{2}$, we see that

$$|H[e^{j\pi/2}]| = \frac{5}{8}\left(\frac{\sqrt{2}}{\frac{1}{2}(\frac{3}{2})}\right) = \frac{5}{3\sqrt{2}} = 1.1785.$$

Figure S5.6-2b

Since the system is real, $|H[e^{j\Omega}]|$ has even symmetry, which when combined with its 2π periodic nature, implies symmetry about $\Omega = -\pi$ as well. Due to the influence of the zero, the peak of the magnitude response occurs just before $\Omega = \pi/2$. Figure S5.6-2b shows the resulting magnitude response (dots for hand-calculated values, curve for actual magnitude response).

(c) From part (b), $|H[e^{j\pi/2}]| = \frac{5}{3\sqrt{2}}$. Further,

$$\angle H[e^{j\pi/2}] = 0 + \frac{3\pi}{4} - \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\frac{\pi}{4}.$$

Thus, the response to $x[n] = \sin\left(\frac{\pi n}{2}\right)$ is

$$y[n] = \frac{5}{3\sqrt{2}} \sin\left(\frac{\pi n}{2} - \frac{\pi}{4}\right).$$
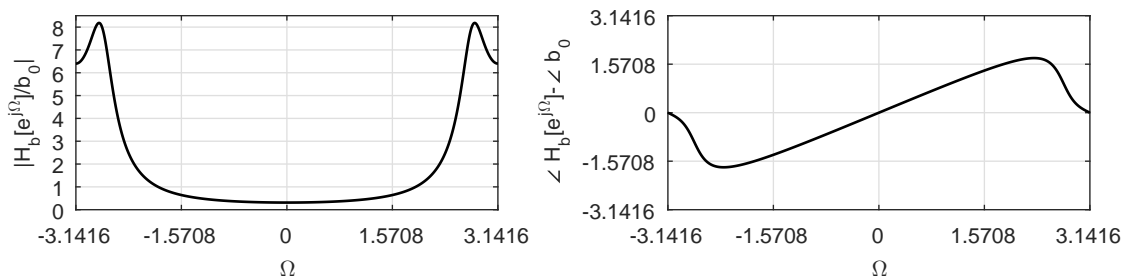
Solution 5.6-3

(a) From the pole/zero plot, we see that

$$H[z] = \frac{k(z+1)}{(z-0.5j)(z+0.5j)} = \frac{k(z-1)}{z^2 + \frac{1}{4}}.$$

Further,

$$H[1] = -1 = \frac{k(2)}{1 + \frac{1}{4}} \quad \Rightarrow \quad 2k = -\frac{5}{4} \quad \Rightarrow k = -\frac{5}{8}.$$

Referenced to $H[z] = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$, we see that

$$b_0 = 0, \quad b_1 = -\frac{5}{8}, \quad b_2 = -\frac{5}{8}, \quad a_1 = 0, \quad a_2 = \frac{1}{4}.$$

(b) By inspection, we see that

$$|H[e^{j0}]| = 1 \quad \text{and} \quad |H[e^{-j\pi}]| = 0.$$

At $\Omega = \frac{\pi}{2}$, we see that

$$|H[e^{j\pi/2}]| = \frac{5}{8} \left(\frac{\sqrt{2}}{\frac{1}{2}(\frac{3}{2})}\right) = \frac{5}{3\sqrt{2}} = 1.1785.$$
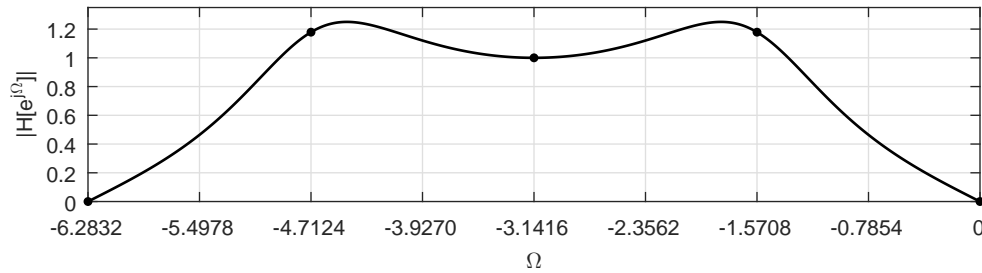
Since the system is real, $|H[e^{j\Omega}]|$ has even symmetry, which when combined with its 2π periodic nature, implies symmetry about $\Omega = -\pi$ as well. Due to the influence of the zero, the peak of the magnitude response occurs just after $\Omega = \pi/2$. Figure S5.6-3b shows the resulting magnitude response (dots for hand-calculated values, curve for actual magnitude response).

Figure S5.6-3b

(c) From part (b), $|H[e^{j\pi/2}]| = \frac{5}{3\sqrt{2}}$. Further,

$$\angle H[e^{j\pi/2}] = \pi + \frac{\pi}{4} - \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{4}.$$

Thus, the response to $x[n] = \sin\left(\frac{\pi n}{2}\right)$ is

$$y[n] = \frac{5}{3\sqrt{2}} \sin\left(\frac{\pi n}{2} + \frac{\pi}{4}\right).$$

Solution 5.6-4

(a) From the pole/zero plot, we see that the zeros can be represented as

$$(z - e^{j\pi/4})(z - e^{-j\pi/4}) = z^2 - 2\frac{e^{j\pi/4} - e^{-j\pi/4}}{2}z + 1 = z^2 - 2\cos(\tfrac{\pi}{4})z + 1 = z^2 - \sqrt{2}z + 1.$$

Similarly, the poles can be represented as

$$(z - \tfrac{1}{\sqrt{2}}e^{j3\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{-j3\pi/4}) = z^2 - \tfrac{2}{\sqrt{2}}\frac{e^{j3\pi/4} - e^{-j3\pi/4}}{2}z + \tfrac{1}{2} = z^2 - \sqrt{2}\cos(\tfrac{3\pi}{4})z + \tfrac{1}{2} = z^2 + z + \tfrac{1}{2}.$$

Further,

$$H[-1] = 1 = k\frac{1 + \sqrt{2} + 1}{1 - 1 - \frac{1}{2}} \quad \Rightarrow \quad k = \frac{1}{4 + 2\sqrt{2}}.$$

Referenced to $H[z] = k\frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$, we see that

$$k = \frac{1}{4 + 2\sqrt{2}}, \quad b_1 = -\sqrt{2}, \quad b_2 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}.$$

(b) By inspection, we see that

$$|H[e^{j\pi}]| = 1 \quad \text{and} \quad |H[e^{\pm j\pi/4}]| = 0.$$

At $\Omega = \pm\frac{3\pi}{4}$, we see that

$$|H[e^{\pm j3\pi/4}]| \approx \frac{1}{4 + 2\sqrt{2}}\left(\frac{(1.4)(2)}{(0.3)(1.25)}\right) = 1.1.$$

At $\Omega = 0$, we see that

$$|H[e^{j0}]| \approx \frac{1}{4 + 2\sqrt{2}}\left(\frac{2 - \sqrt{2}}{2.5}\right) \approx 0.04.$$

Since the system is real, $|H[e^{j\Omega}]|$ has even symmetry. Due to the influence of the zero, the peak of the magnitude response occurs just after $\Omega = 3\pi/4$. Figure S5.6-4b shows the resulting magnitude response (dots for hand-calculated values, curve for actual magnitude response).
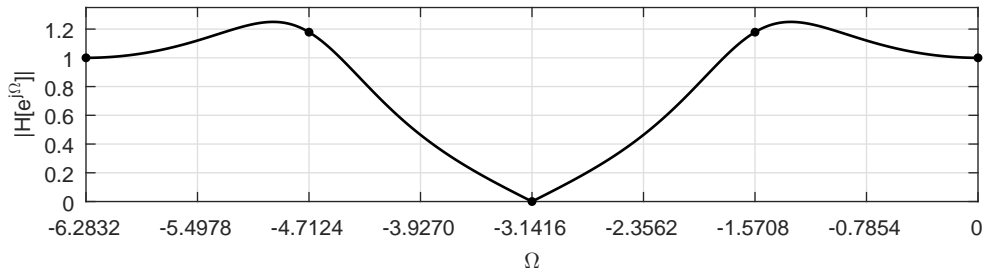
Figure S5.6-4b

(c) To avoid aliasing (i.e., Nyquist is met), we need to sample such that the $\omega = 500\pi$ component hits the system zero at $\Omega = \pi/4$. That is,

$$\Omega = \omega T \quad \Rightarrow \quad \frac{\pi}{4} = 500\pi T \quad \Rightarrow \quad T = \frac{1}{2000} \quad \Rightarrow \quad F_{\mathrm{s}} = \frac{1}{T} = 2000\mathrm{Hz}.$$

Now, $\omega = 100\pi$ is five times small than $\omega = 500\pi$, so

$$\Omega_0 = \frac{1}{5}\left(\frac{\pi}{4}\right) \quad \Rightarrow \quad \Omega_0 = \frac{\pi}{20}.$$

(d) Since the system poles are contained inside the unit circle, the system is asymptotically stable and also BIBO stable. Thus,

yes; the impulse response $h[n]$ is absolutely summable since the system is stable.

Solution 5.6-5

(a) From the pole/zero plot, we see that the zeros can be represented as

$$(z - e^{j3\pi/4})(z - e^{-j3\pi/4}) = z^2 - 2\frac{e^{j3\pi/4} - e^{-j3\pi/4}}{2}z + 1 = z^2 - 2\cos(\tfrac{3\pi}{4})z + 1 = z^2 + \sqrt{2}z + 1.$$

Similarly, the poles can be represented as

$$(z - \tfrac{1}{\sqrt{2}}e^{j\pi/4})(z - \tfrac{1}{\sqrt{2}}e^{-j\pi/4}) = z^2 - \tfrac{2}{\sqrt{2}}\frac{e^{j\pi/4} - e^{-j\pi/4}}{2}z + \tfrac{1}{2} = z^2 - \sqrt{2}\cos(\tfrac{\pi}{4})z + \tfrac{1}{2} = z^2 - z + \tfrac{1}{2}.$$

Further,

$$H[1] = -1 = k\frac{2 + \sqrt{2}}{\frac{1}{2}} \quad \Rightarrow \quad k = \frac{-1}{4 + 2\sqrt{2}}.$$

Referenced to $H[z] = k\frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$, we see that

$$k = \frac{-1}{4 + 2\sqrt{2}}, \quad b_1 = \sqrt{2}, \quad b_2 = 1, \quad a_1 = -1, \quad a_2 = \frac{1}{2}.$$

(b) By inspection, we see that

$$|H[e^{j0}]| = 1 \quad \text{and} \quad |H[e^{\pm j3\pi/4}]| = 0.$$

At $\Omega = \pm\frac{\pi}{4}$, we see that

$$|H[e^{\pm j\pi/4}]| \approx \frac{1}{4 + 2\sqrt{2}}\left(\frac{(1.4)(2)}{(0.3)(1.25)}\right) = 1.1.$$

At $\Omega = \pi$, we see that

$$|H[e^{j\pi}]| \approx \frac{1}{4 + 2\sqrt{2}} \left(\frac{2 - \sqrt{2}}{2.5} \right) \approx 0.04.$$

Since the system is real, $|H[e^{j\Omega}]|$ has even symmetry. Due to the influence of the zero, the peak of the magnitude response occurs just before $\Omega = \pi/4$. Figure S5.6-5b shows the resulting magnitude response (dots for hand-calculated values, curve for actual magnitude response).
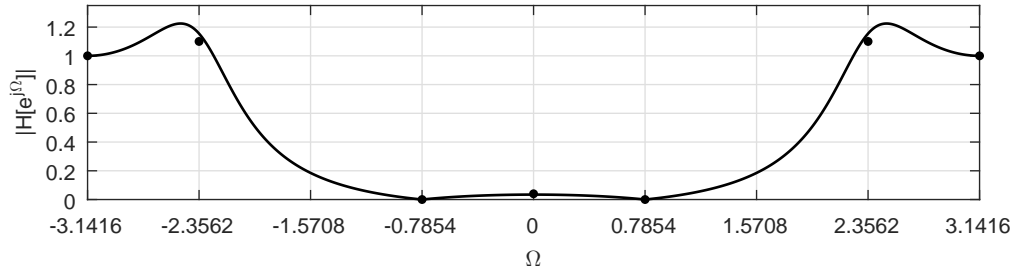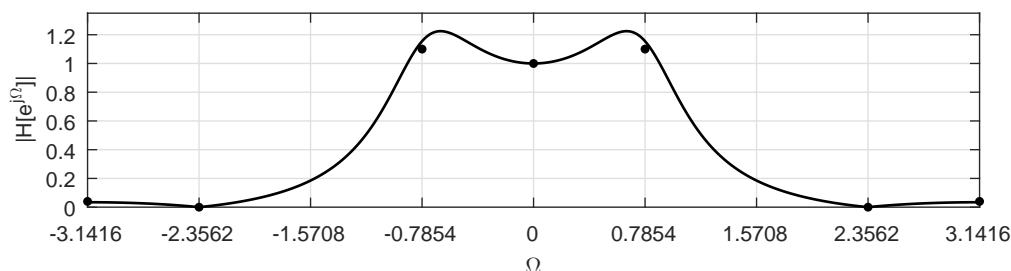

Figure S5.6-5b

(c) We know that $\Omega = \pi$ corresponds to $F_s/2$. Thus, the system zeros at $\Omega = \pm 3\pi/4$ correspond to $\pm 3F_s/8$. Any alias (integer multiple of F_s) also hits these zeros. Thus,

frequencies $f = \pm\frac{3F_s}{8} + kF_s$, where k is any integer, will produce $y[n] = 0$.

For example, if $F_s = 8000$ Hz, sinusoids of frequencies ± 3000, ± 5000, ± 11000, ± 13000, and so forth would produce zero output.

Solution 5.6-6

The two systems are very similar and have identical steady-state characteristics. There is an important difference, however, between the two systems. The system $y[n] - y[n-1] = x[n] - x[n-1]$ is first-order and can support an initial condition; the system $y[n] = x[n]$ is zero-order and cannot support an initial condition. If the initial condition of the first system is non-zero, the output of the two systems can be quite different.

Solution 5.6-7

(a) From the pole/zero plot, we see that the zeros can be represented as

$$(z - 1)(z - 1) = (z^2 - 2z + 1).$$

Similarly, the poles can be represented as

$$(z - \tfrac{1}{2}e^{j3\pi/4})(z - \tfrac{1}{2}e^{-j3\pi/4}) = z^2 - \cos(3\tfrac{\pi}{4})z + \tfrac{1}{4} = z^2 + \tfrac{1}{\sqrt{2}}z + \tfrac{1}{4}.$$

Further,

$$H[-1] = 4 = k\frac{(-1)^2 - 2(-1) + 1}{(-1)^2 + \frac{1}{\sqrt{2}}(-1) + \frac{1}{4}} = k\frac{4}{\frac{5}{4} - \frac{1}{\sqrt{2}}} \quad \Rightarrow \quad k = \frac{5}{4} - \frac{1}{\sqrt{2}} \approx 0.54.$$

Referenced to $H[z] = k\frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$, we see that

$$k = \frac{5}{4} - \frac{1}{\sqrt{2}}, \quad b_1 = -2, \quad b_2 = 1, \quad a_1 = \frac{1}{\sqrt{2}}, \quad a_2 = \frac{1}{4}.$$

(b) By inspection, we see that

$$|H[e^{j0}]| = 0 \quad \text{and} \quad |H[e^{\pm\pi}]| = 4.$$

Using the techniques of Sec. 5.6 (based on distances from poles and zeros to frequencies of interest), we see that

$$|H[e^{\pm j\pi/2}]| \approx 0.54\left(\frac{\sqrt{2}(\sqrt{2})}{\frac{3}{4}^-(\frac{3}{2}^-)}\right) \approx 1$$

and

$$|H[e^{\pm j3\pi/4}]| \approx 0.54\left(\frac{\left(\sqrt{(\frac{7}{4})^2 + (\frac{3}{4})^2}\right)^2}{\frac{1}{2}\sqrt{(\frac{3}{8})^2 + (\frac{9}{8})^2}}\right) = 1.08\frac{\frac{58}{16}}{\frac{\sqrt{90}}{8}} = 3.3.$$

Since the system is real, $|H[e^{j\Omega}]|$ has even symmetry. Figure S5.6-7b shows the resulting magnitude response (dots for hand-calculated values, curve for actual magnitude response).
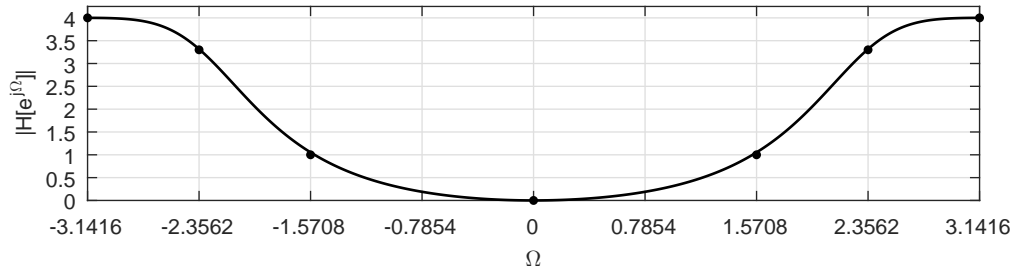
Figure S5.6-7b

**(c)** We already know that $|H[e^{\pm j3\pi/4}]| \approx 3.3$. Further, we see that

$$\angle H[e^{\pm j3\pi/4}] \approx 0 + 2(180° - \text{atan}(\tfrac{3}{7})) - (135° + (90° + \text{atan}(\tfrac{3}{9}))) \approx 2(155°) - (135° + 110°) \approx 65° = 1.14\text{rad}.$$

Thus, the steady-state output to $x[n] = \cos(\frac{3\pi n}{4})u[n]$ is

$$y_{ss}[n] \approx 3.3\cos(\tfrac{3\pi n}{4} + 1.14)u[n].$$

This result (based on somewhat rough calculations) is quite close to the true result of $y_{ss}[n] \approx 3.3157\cos(\frac{3\pi n}{4} + 1.2490)u[n]$.

(d) From the magnitude response in part (b),

the system is HP.

From the magnitude response (as well as the pole locations), we know that the filter cutoff frequency is approximately $\Omega_c = \frac{3\pi}{4} = \omega_c T = 2\pi f_c \frac{1}{F_s}$. Thus, $f_c = \frac{3}{8}(F_s)$. For $F_s = 8000$, we see that

$$f_c \approx \frac{3}{8}(8000) = 3000\text{Hz}.$$

Solution 5.6-8

(a) From the magnitude response plot, it is clear that this is a highpass filter. Low frequencies near $\Omega = 0$ are attenuated, and high frequencies near $\Omega = \pm\pi$ are passed with unity gain.

(b) From the magnitude and phase response plots, $H[e^{j\pi/2}] = \frac{1}{\sqrt{2}}e^{j3\pi/4}$. Thus, the output to $x_1[n] = 2\sin(\frac{\pi}{2}n + \frac{\pi}{4})$ is

$$y_1[n] = \sqrt{2}\sin(\frac{\pi}{2}n + \pi) = -\sqrt{2}\sin(\frac{\pi}{2}n).$$

(c) Notice, $H[e^{j7\pi/4}] = H[e^{-j\pi/4}]$. From the magnitude (even) and phase (odd) response plots, $H[e^{-j\pi/4}] \approx 0.07e^{j3\pi/4}$. Thus, the output to $x_2[n] = \cos(\frac{7\pi}{4}n)$ is

$$y_2[n] \approx 0.07\cos(\frac{7\pi}{4}n + 3\pi/4).$$

Solution 5.6-9

(a) By direct substitution, we see that

$$b_0\frac{e^{-j2\pi} + 1}{e^{-j2\pi} - \frac{9}{16}} = b_0\frac{2}{\frac{7}{16}} = -1.$$

Thus,

$$b_0 = -\frac{7}{32}.$$

(b) As shown in Fig. S5.6-9b, the system has

poles at $z = \frac{3}{4}$ and $z = -\frac{3}{4}$ and zeros at $z = j$ and $z = -j$.
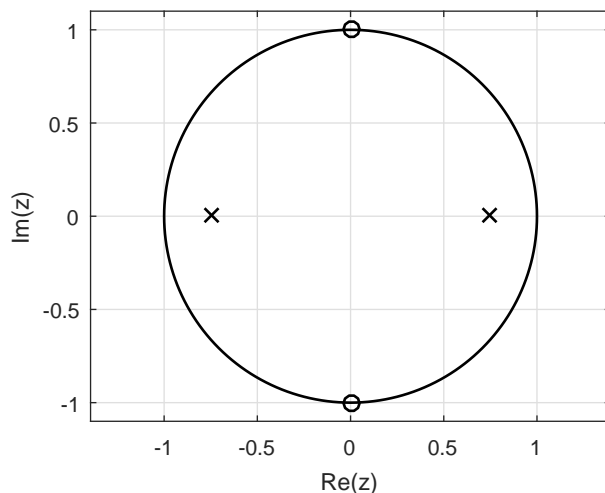
Figure S5.6-9b

(c) By constraint, $|H[e^{-j\pi}]| = 1$. Due to symmetry in the pole-zero plot, we also know that $|H[e^{j0}]| = 1$. Further, the two zeros cause $|H[e^{\pm j\pi/2}]| = 0$. Thus, as shown in Fig. S5.6-9c, the system has bandstop (notch) character (dots for hand-calculated values, curve for actual magnitude response).

(d) Since $H[e^{j0}] = H[e^{j\pi}] = -1$ and $H[e^{j\pi/2}] = H[e^{-j\pi/2}] = 0$, the output in response to $x[n] = (-1 + j) + j^n + (1 - j)\sin(\pi n + 1)$ is

$$y[n] = (1 - j) + (j - 1)\sin(\pi n + 1).$$

(e) Figure S5.6-9e shows a TDFII realization of the system, which generally has the most desirable characteristics of the basic structures (DFI, DFII, TDFI, and TDFII).
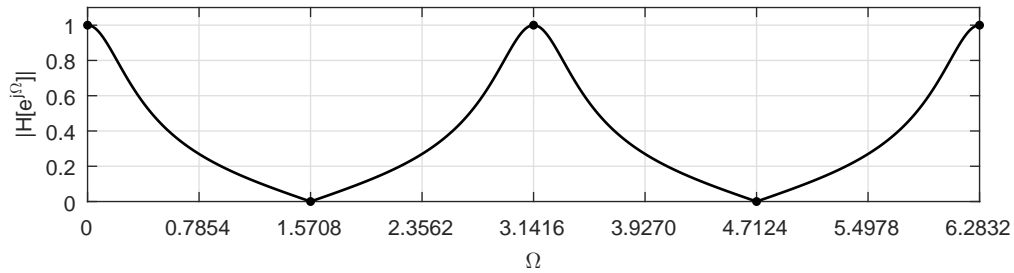
Figure S5.6-9c
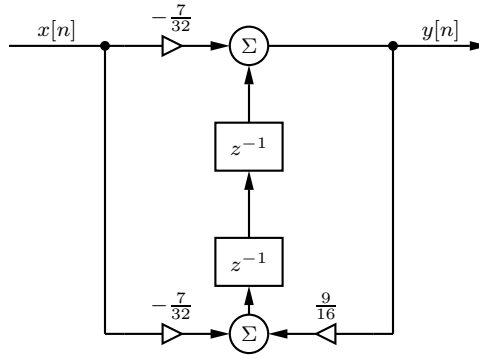
Figure S5.6-9e

Solution 5.6-10

Taking the z-transform of $4y[n+2] - y[n] = x[n+2] + x[n]$ yields $Y[z]\left(4z^2 - 1\right) = X[z]\left(z^2 + 1\right)$. Thus, the system function is

$$H[z] = \frac{Y[z]}{X[z]} = \frac{z^2 + 1}{4z^2 - 1} = 0.25\frac{1 + z^{-2}}{1 - z^{-2}/4}.$$

(a) As shown in Fig. S5.6-10a, the system has

poles at $z = \frac{1}{2}$ and $z = -\frac{1}{2}$ and zeros at $z = j$ and $z = -j$.

(b) Using the techniques of Sec. 5.6 (based on distances from poles and zeros to frequencies of interest), we see that

$$|H[e^{j0}]| = \frac{1}{4}\left(\frac{\sqrt{2}\sqrt{2}}{\frac{1}{2}(\frac{3}{2})}\right) = \frac{2}{3}.$$

Due to symmetry in the pole-zero plot, we also know that

$$|H[e^{j\pi}]| = |H[e^{j0}]| = \frac{2}{3}.$$

The two zeros cause $|H[e^{\pm j\pi/2}]| = 0$. Thus, as shown in Fig. S5.6-10b, the system has bandstop (notch) character (dots for hand-calculated values, curve for actual magnitude response).

(c) The pole-zero plot of Fig. S5.6-10a and the magnitude response plot of Fig. S5.6-10b confirm that this is a band-stop system.

(d) Yes, the system is asymptotically stable. Referring to Fig. S5.6-10a, all the system poles are within the unit circle.
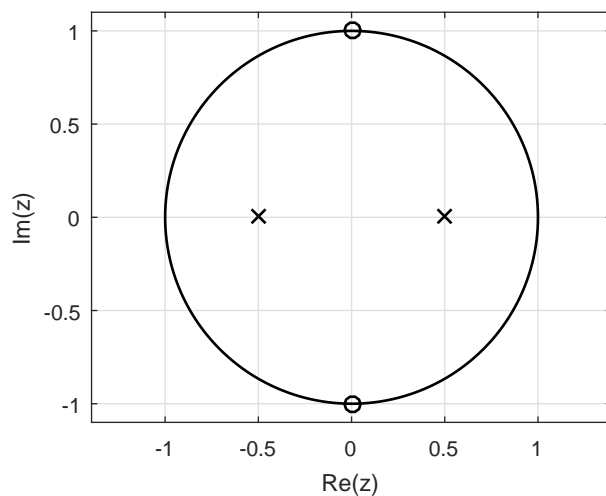
Figure S5.6-10a
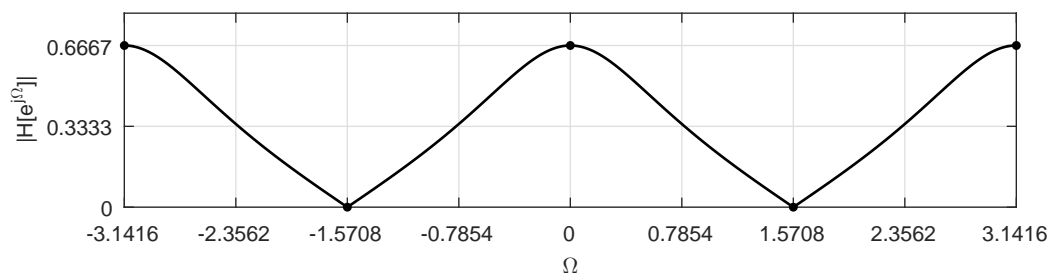
Figure S5.6-10b

(e) Yes, the system is real. Since the system is expressed as a constant-coefficient linear difference equation with real coefficients, the impulse response $h[n]$ and system are both real.

(f) For an input of the form $x[n] = \cos(\Omega n)$, the greatest possible amplitude of the output corresponds to the greatest gain shown in the magnitude response plot of Fig. S5.6-10b. Thus, $\frac{2}{3}$ is the greatest output amplitude given an input of $x[n] = \cos(\Omega n)$. This output amplitude occurs when $\Omega = k\pi$, for any integer k.

(g) Inverting $H[z] = \frac{Y[z]}{X[z]} = 0.25 \frac{1+z^{-2}}{1-z^{-2}/4}$ provides $y[n] - 0.25y[n-2] = 0.25x[n] + 0.25x[n-2]$, which is a convenient form for implementation. Figure S5.6-10g illustrates a TDFII implementation of the system.
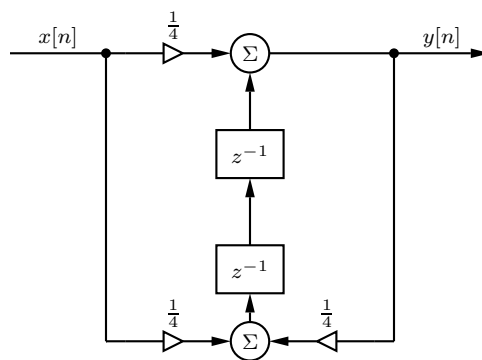
Figure S5.6-10g

Solution 5.6-11

(a) From the block diagram, the corresponding difference equation is written directly.

$$y[n] - 0.5y[n-2] = x[n].$$

(b) Taking the z-transform of $y[n] - 0.5y[n-2] = x[n]$ yields $Y[z](1 - 0.5z^{-2}) = X[z]$. Thus,

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1 - 0.5z^{-2}},$$

and we see that the system has

a repeated zero at $z = 0$ and poles at $z = \pm\frac{1}{\sqrt{2}}$.

Using the techniques of Sec. 5.6 (based on distances from poles and zeros to frequencies of interest), we see that

$$|H[e^{j0}]| = \frac{1(1)}{(1 + \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}})} = \frac{1}{\frac{1}{2}} = 2.$$

Further,

$$|H[e^{\pm j\pi/2}]| = \frac{1(1)}{\sqrt{1^2 + (\frac{1}{\sqrt{2}})^2}\sqrt{1^2 + (\frac{1}{\sqrt{2}})^2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

Due to symmetry in the pole-zero plot, we also know that

$$|H[e^{j\pi}]| = |H[e^{j0}]| = 2.$$

The resulting magnitude response is shown in Fig. S5.6-11b (dots for hand-calculated values, curve for actual magnitude response). Standard filter types do not provide a good description
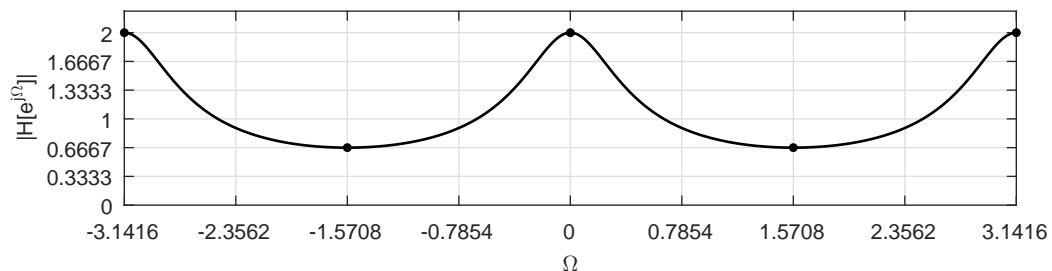
Figure S5.6-11b

of this filter. The system appears most like a bandstop filter, but its stopband attenuation is quite poor. The system boosts the gain of low and high frequencies more than it attenuates the middle frequencies.

(c) Inverting $H[z] = \frac{1}{1 - 0.5z^{-2}} = \frac{1/2}{1 - z^{-1}/\sqrt{2}} + \frac{1/2}{1 + z^{-1}/\sqrt{2}}$ yields

$$h[n] = 0.5\left((1/\sqrt{2})^n + (-1/\sqrt{2})^n\right) u[n].$$

Solution 5.6-12

For this problem, we have $H[z] = K\frac{z+1}{z-a}$.

(a) Figure S5.6-12a illustrates a TDFII implementation of the system.
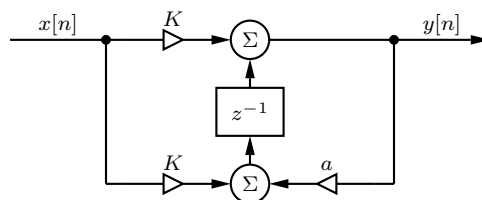
Figure S5.6-12a

(b) This system has a zero at $z = -1$ and a pole at $z = a$. For $|a| < 1$, the pole is closer to $\Omega = 0$ than is the system zero. Hence there is highest gain at dc, and the system is lowpass in nature. Figure S5.6-12b shows the K-normalized magnitude response for $a = \frac{1}{2}$, 0, and $-\frac{1}{2}$.
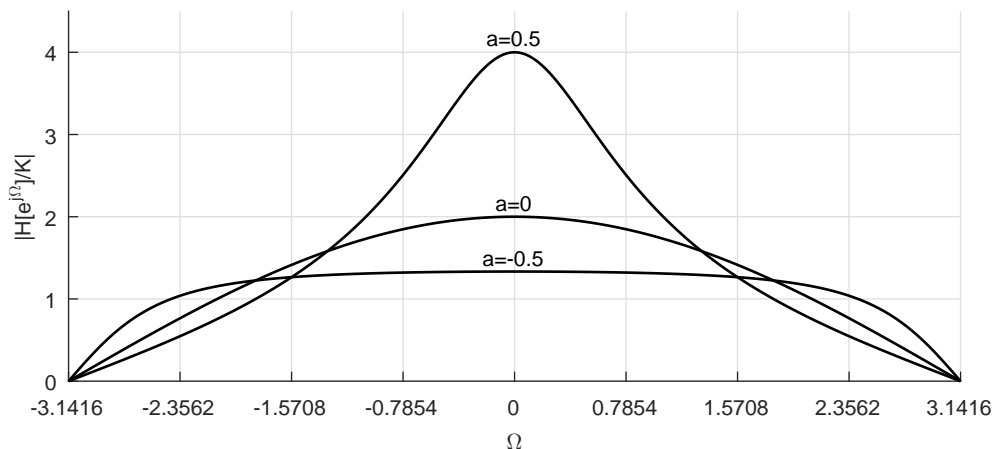
Figure S5.6-12b

(c) To begin, note that

$$H\left[e^{j\Omega}\right] = K\left(\frac{e^{j\Omega} + 1}{e^{j\Omega} - a}\right) = K\left(\frac{\cos\Omega + 1 + j\sin\Omega}{\cos\Omega - a + j\sin\Omega}\right)$$

and

$$|H\left[e^{j\Omega}\right]| = K\sqrt{\frac{2(1 + \cos\Omega)}{1 + a^2 - 2a\cos\Omega}}.$$

For $a = 0.2$

$$|H\left[e^{j\Omega}\right]| = K\sqrt{\frac{2(1 + \cos\Omega)}{1.04 - 0.4\cos\Omega}}.$$

The dc gain is $|H[e^{j0}]| = 2.5K$, and the 3 dB bandwidth occurs when $|H[e^{j\Omega}]|^2 = \frac{1}{2}|H[e^{j0}]|^2 = 3.125K^2$. Hence

$$3.125K^2 = K^2\left[\frac{2(1 + \cos\Omega)}{1.04 - 0.4\cos\Omega}\right] \implies \Omega = 1.176.$$

Hence, the (Hertzian) bandwidth is

$$B = \frac{\Omega}{2\pi T} = \frac{1.176}{2\pi T} = \frac{0.187}{T}\text{Hz}.$$

Solution 5.6-13

Since the highest frequency to be processed is 20 kHz, we can avoid aliasing by satisfying the Nyquist condition of

$$T \leq \frac{1}{2f_{\max}} = \frac{1}{2(20000)} = 25 \ \mu s.$$

Selecting $T = 25 \ \mu s$, the notch frequency of $f_c = 5000$ Hz corresponds to

$$\Omega_c = 2\pi f_c T = (2\pi 5 \cdot 10^3)(25 \cdot 10^{-6}) = \frac{\pi}{4}.$$

Therefore, we must place zeros at $e^{\pm j\pi/4}$. For rapid recovery on either side of 5000 Hz, we need poles at $ae^{\pm j\pi/4}$ where $0 < a < 1$ and a is close to 1. The transfer function is

$$H(z) = K \frac{(z - e^{j\pi/4})(z - e^{-j\pi/4})}{(z - ae^{j\pi/4})(z - ae^{-j\pi/4})} = \frac{K(z^2 - \sqrt{2}z + 1)}{z^2 - \sqrt{2}az + a^2}$$

A canonical TDFII realization of this system is shown in Fig. S5.6-13a.
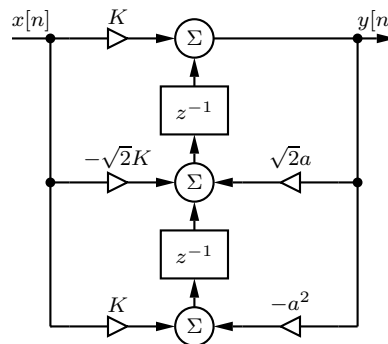
Figure S5.6-13a

The constant K is chosen so that the filter has unity dc $(z = 1)$ gain. That is,

$$H[1] = \frac{K(2 - \sqrt{2})}{1 + a^2 - \sqrt{2}a} = 1.$$

Solving for K yields

$$K = \frac{1 + a^2 - \sqrt{2}a}{2 - \sqrt{2}} = 1.7071(1 + a^2 - \sqrt{2}a).$$

Using K given above, the magnitude response is thus

$$\left| H\left[e^{j\Omega}\right] \right| = \left| K \frac{e^{j2\Omega} - \sqrt{2}e^{j\Omega} + 1}{e^{j2\Omega} - \sqrt{2}ae^{j\Omega} + a^2} \right|.$$

Figure S5.6-13b shows the magnitude response for $a = 0.95$ and confirms the design.

```
>>  a = 0.95; K = (1+a^2-sqrt(2)*a)/(2-sqrt(2));
>>  Omega = linspace(-pi,pi,1001);
>>  H = @(z) K*(z.^2-sqrt(2)*z+1)./(z.^2-sqrt(2)*a*z+a^2);
>>  plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 1.15]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:.25:1);
```
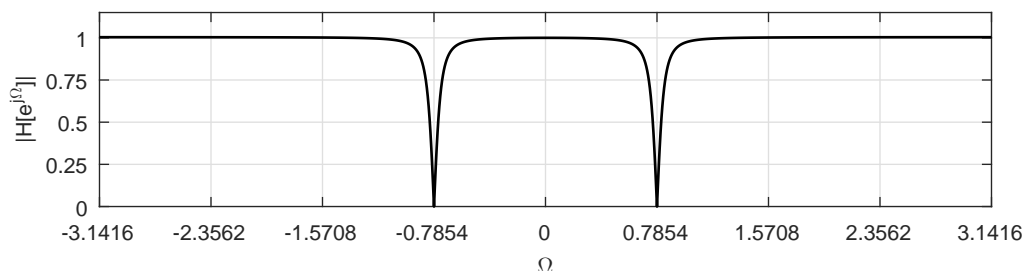
Figure S5.6-13b

Solution 5.6-14

(a) We know that the magnitude response of DT systems are 2π periodic. Periodically replicating the magnitude response of Fig. P5.6-14 and looking over $-\pi \leq \Omega \leq \pi$, we obtain a traditional view of the system magnitude response, as shown in Fig. S5.6-14a.

Clearly from Fig. S5.6-14a, the system is a bandpass filter.
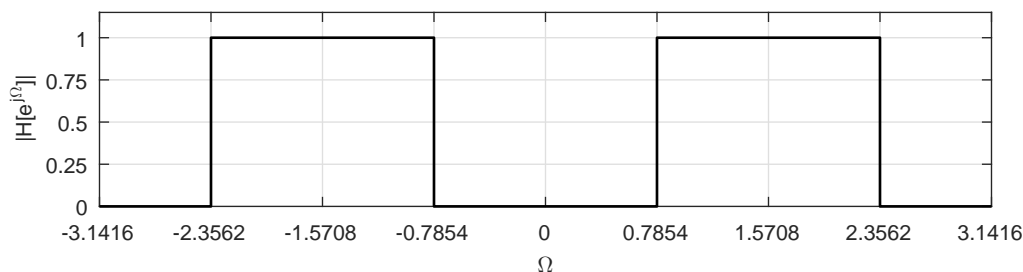
Figure S5.6-14a

(b) We can obtain a reasonable second-order approximation of this response by placing zeros at $z = \pm 1$ and poles at $z = \pm ja$ ($0 \leq a \leq 1$). This results in a transfer function

$$H[z] = b_0 \frac{z^2 - 1}{z^2 + a^2}.$$

To find suitable constants b_0 and a, let us try and set a peak gain of unity and 3 dB (half power) cutoff frequencies. To set the peak gain at unity requires

$$|H[e^{j\pi/2}]| = |b_0| \left| \frac{e^{j2\pi} - 1}{e^{j2\pi} - a^2} \right| = |b_0| \frac{2}{1 - a^2} = 1.$$

Thus, let us set

$$b_0 = \frac{1 - a^2}{2}.$$

We determine a by setting a 3 dB (half power) gain at $\Omega = \pi/4$. That is,

$$|H[e^{j\pi/4}]|^2 = \frac{1}{2} = |b_0|^2 \left| \frac{e^{j\pi} - 1}{e^{j\pi} - a^2} \right|^2$$

$$= \frac{1 - 2a^2 + a^4}{4} \left(\frac{(j-1)(-j-1)}{(j+a^2)(-j+a^2)} \right)$$

$$= \frac{1 - 2a^2 + a^4}{4} \left(\frac{2}{1 - a^4} \right).$$

Thus,
$$1 + a^4 = 1 - 2a^2 + a^4 \quad \Rightarrow \quad a = 0.$$

For $a = 0$, the corresponding coefficient b_0 is just $\frac{1}{2}$. Taken together our system is

$$H[z] = \frac{\frac{1}{2}z^2 - \frac{1}{2}}{z^2}.$$

Figure S5.6-14b shows the resulting pole-zero plot and magnitude response, thereby confirming the design.
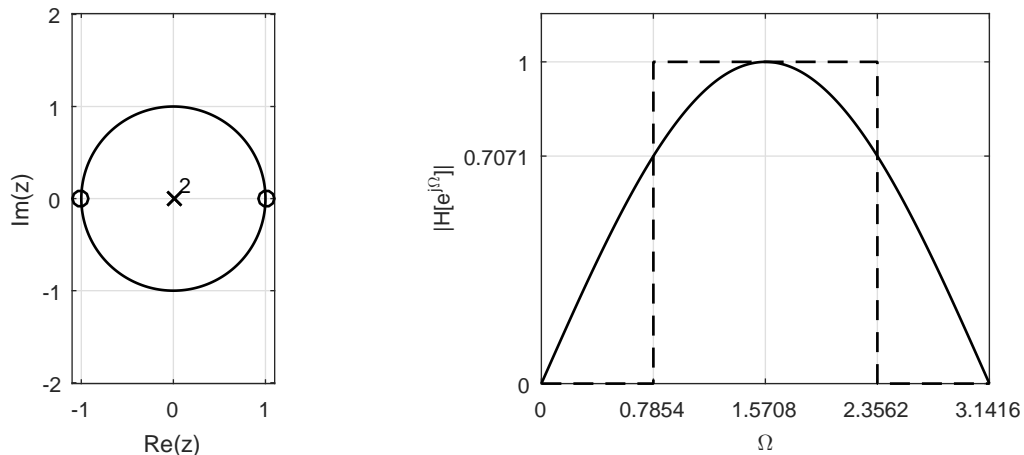
Figure S5.6-14b

Solution 5.6-15

In this problem, we consider a first-order LTID system with a pole at $z = \gamma$ (γ is potentially complex) and a zero at $z = \frac{1}{\gamma^*}$, where $|\gamma| \le 1$. The system transfer function is

$$H_1[z] = \frac{z - \frac{1}{\gamma^*}}{z - \gamma}.$$

The magnitude response squared of this first order system is

$$\left| H_1 \left[e^{j\Omega} \right] \right|^2 = H_1 \left[e^{j\Omega} \right] H_1^* \left[e^{j\Omega} \right] = \left(\frac{e^{j\Omega} - \frac{1}{\gamma^*}}{e^{j\Omega} - \gamma} \right) \left(\frac{e^{-j\Omega} - \frac{1}{\gamma}}{e^{-j\Omega} - \gamma^*} \right)$$

$$= \frac{1 + \frac{1}{|\gamma|^2} - \frac{2}{|\gamma|} \cos(\Omega - \angle\gamma)}{1 + |\gamma|^2 - 2|\gamma| \cos(\Omega - \angle\gamma)} = \frac{1}{|\gamma|^2}.$$

The magnitude response is therefore
$$\left| H_1 \left[e^{j\Omega} \right] \right| = \frac{1}{|\gamma|}.$$

Since the magnitude response is a constant for all frequencies Ω, the filter is clearly allpass.

Letting γ be a real value r ($|r| < 1$), we see that $H_1[z] = \frac{z - \frac{1}{r}}{z - r}$ has a magnitude response

$$\left| H_1 \left[e^{j\Omega} \right] \right| = \frac{1}{|r|}.$$

For a second-order system with poles at $z = re^{\pm j\theta}$ and zeros at $z = (1/r)e^{\pm j\theta}$, we see that

$$H_2[z] = \left(\frac{z - \frac{1}{\gamma_1^*}}{z - \gamma_1}\right)\left(\frac{z - \frac{1}{\gamma_2^*}}{z - \gamma_2}\right),$$

where $\gamma_1 = re^{j\theta}$ and $\gamma_1 = re^{-j\theta}$. Using our earlier result, the magnitude response of this system is

$$\left|H_2\left[e^{j\Omega}\right]\right| = \frac{1}{|\gamma_1||\gamma_2|} = \frac{1}{|r|^2}.$$

Since the magnitude response is a constant for all frequencies Ω, the filter is clearly allpass.

Solution 5.6-16

The impulse responses of two LTID systems are related as

$$h_2[n] = (-1)^n h_1[n].$$

(a) Since $-1 = e^{\mp j\pi}$, we see that

$$h_2[n] = e^{\mp j\pi n} h_1[n].$$

Using the z-domain scaling property of Table ?? with $\gamma = e^{\mp j\pi}$, we see that

$$H_2[z] = H_1[z/e^{\mp j\pi}] = H_1[e^{\pm j\pi}z].$$

Hence

$$H_2\left[e^{j\Omega}\right] = H_1\left[e^{\pm j\pi}e^{j\Omega}\right] = H_1\left[e^{j(\Omega \pm \pi)}\right].$$

In this way, we see that the frequency response of the second filter is just the frequency response of the first filter shifted by π.

(b) Figure S5.6-16b shows the frequency response $H_1\left[e^{j\Omega}\right]$ of an ideal lowpass filter with cutoff frequency Ω_c and the resulting frequency response $H_2\left[e^{j\Omega}\right]$, which is just $H_1\left[e^{j\Omega}\right]$ shifted by π (recall that the frequency response of any DT system is 2π-periodic). It is clear that the shifted response $H_2\left[e^{j\Omega}\right]$ corresponds to an ideal high-pass filter with cutoff frequency $\pi - \Omega_c$.
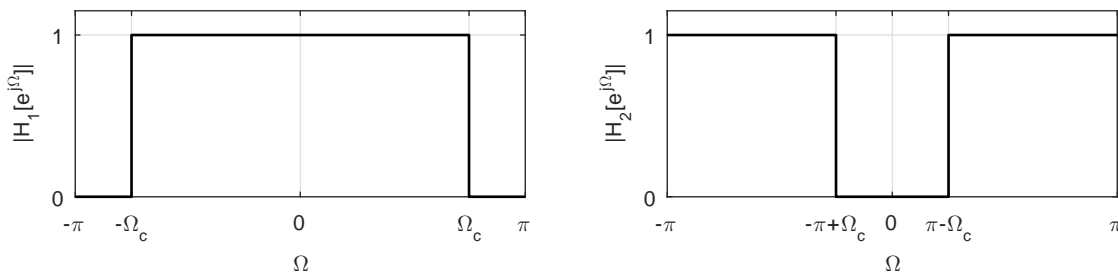
Figure S5.6-16b

Solution 5.6-17

(a) Let $H[z]$ represent the original filter, either highpass or lowpass. The transformed filter has system function $H_T[z] = H[z]|_{z=-z} = H[-z]$. The basic character of the transformed filter can be assessed by its magnitude response, $|H_T[e^{j\Omega}]| = |H[-e^{j\Omega}]| = |H[e^{j\pi}e^{j\Omega}]| = |H[e^{j(\Omega+\pi)}]|$. That is, the magnitude response of a transformed filter is just the magnitude response of the original filter shifted in frequency by π. If the magnitude response of a digital LPF is shifted in

frequency by π, it becomes a highpass filter. Similarly, if the magnitude response of a digital HPF is shifted in frequency by π, it becomes a lowpass filter. Put another way, an original passband centered at $\Omega = 0$ (LPF) is shifted by the transformation to a passband centered at $\Omega = \pi$ (HPF), and vice-versa.

(b) If $H[z] = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ is the original filter, $H_{\mathrm{T}}[z] = H[-z] = \sum_{n=-\infty}^{\infty} h[n](-z)^{-n} = \sum_{n=-\infty}^{\infty}(-1)^n h[n]z^{-n}$. Thus, the transformed filter has impulse response

$$h_{\mathrm{T}}[n] = (-1)^n h[n].$$

Put another way, the same transformation is accomplished by simply negating the values of $h[n]$ for every odd integer n.

Solution 5.6-18

The bilinear transformation states $s = \frac{2(1-z^{-1})}{T(1+z^{-1})}$. Rearranging yields $z = \frac{1+Ts/2}{1-Ts/2}$. Thus, $s = j\omega$ maps to $z = \frac{1+j\omega T/2}{1-j\omega T/2}$.

(a) The magnitude of this transformation is $|z| = \left|\frac{1+j\omega T/2}{1-j\omega T/2}\right| = \frac{|1+j\omega T/2|}{|1-j\omega T/2|} = 1$. Since only the unit circle has $|z| = 1$, the bilinear transform maps $s = j\omega$ onto the unit circle in the z-plane.

(b) Using the results from part (a), we know that $z = \frac{1+j\omega T/2}{1-j\omega T/2}$ describes the unit circle in the complex plane. This allows us to write $z = e^{j\Omega} = \frac{1+j\omega T/2}{1-j\omega T/2}$. Thus, the bilinear transformation maps $(-\infty \leq \omega \leq \infty)$ to $(-\pi/2 \leq \Omega\pi/2)$ in a monotonic, although non-linear, manner according to $\Omega = \angle z = \angle\frac{1+j\omega T/2}{1-j\omega T/2} = \arctan(\omega T/2) - \arctan(-\omega T/2) = 2\arctan(\omega T/2)$.

Solution 5.7-1

For the system in Eq. (3.12), the transfer function $H(s)$ and impulse response $h[n]$ are given by

$$H(s) = \frac{1}{s+c} \quad \text{and} \quad h(t) = e^{-ct}u(t).$$

Using the impulse invariance criterion, we obtain the equivalent digital filter transfer function from Table 5.3 corresponding to $H(s) = \frac{1}{s+c}$ as

$$H[z] = \frac{Tz}{z - e^{-ct}} \simeq \frac{Tz}{z - (1 - cT)} \quad \text{assuming} \quad T \to 0.$$

The approximation of Eq. (3.13) yields

$$\widehat{H}[z] = \frac{\beta z}{z + \alpha}.$$

Substituting $\beta = \frac{T}{1+cT}$ and $\alpha = \frac{-1}{1+cT}$, we obtain

$$\widehat{H}[z] = \frac{\frac{Tz}{1+cT}}{z - \frac{1}{1+cT}}.$$

As $T \to 0$, $\frac{1}{1+cT} \simeq 1 - cT$. Hence

$$\widehat{H}[z] \simeq \frac{T(1 - cT)z}{z - (1 - cT)} \simeq \frac{Tz}{z - (1 - cT)},$$

which is same as that found by impulse-invariance method.

Solution 5.7-2

The impulse invariance method requires that

$$h[n] = Th_{\mathrm{ct}}(nT) = Te^{-nT}u[n].$$

Letting $T = 0.1$ and taking the z-transform yields

$$H[z] = 0.1\frac{z}{z - e^{-0.1}} = \frac{Y[z]}{X[z]}.$$

Cross multiplying and taking the inverse transform yields the system difference equation as

$$y[n] - e^{-0.1}y[n-1] = 0.1x[n].$$

The DFI realization of this system is shown in Fig. S5.7-2.
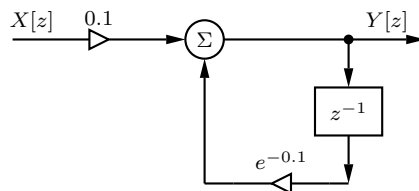
Figure S5.7-2

Solution 5.7-3

(a)

$$H_a(s) = \frac{7s + 20}{2(s^2 + 7s + 10)} = \frac{7s + 20}{2(s + 2)(s + 5)} = \frac{1}{s + 2} + \frac{5/2}{s + 5}.$$

Using Table 5.3, we get

$$H[z] = T\left[\frac{z}{z - e^{-2T}} + \frac{5}{2}\frac{z}{z - e^{-5T}}\right].$$

This is the transfer function of our desired digital filter.

(b) To realize the design, we first select T:

$$H_a(0) = 1 \quad \text{and for} \quad s \gg 5 \Longrightarrow H_a(s) \approx \frac{7}{2s}$$

$$\text{and} \qquad |H_a(j\omega)| \simeq \frac{7}{2\omega} \qquad \omega \gg 5.$$

We shall choose the filter bandwidth to be that frequency where $|H_a(j\omega_0)|$ is 1% of $|H_a(0)|$. Hence,

$$\frac{7}{2\omega_0} = 0.01, \quad \omega_0 = 350, \quad \text{and} \quad T = \frac{\pi}{350}.$$

Substituting this value of T in $H[z]$ yields

$$H[z] = 0.008976\left[\frac{z}{z - 0.9822} + \frac{5}{2}\frac{z}{z - 0.9561}\right] = 0.031416z\left[\frac{z - 0.97475}{z^2 - 1.9383z + 0.9391}\right].$$

For the canonical (DFI) realization, we represent $H[z]$ as

$$H[z] = \frac{0.031416z^2 - 0.03062z}{z^2 - 1.9383z + 0.9391}.$$

For the parallel realization, we represent $H[z]$ as

$$H[z] = \frac{0.008976z}{z - 0.9822} + \frac{0.02244z}{z - 0.9561}.$$

The canonical DFI and parallel realizations are shown in Fig. S5.7-3b.
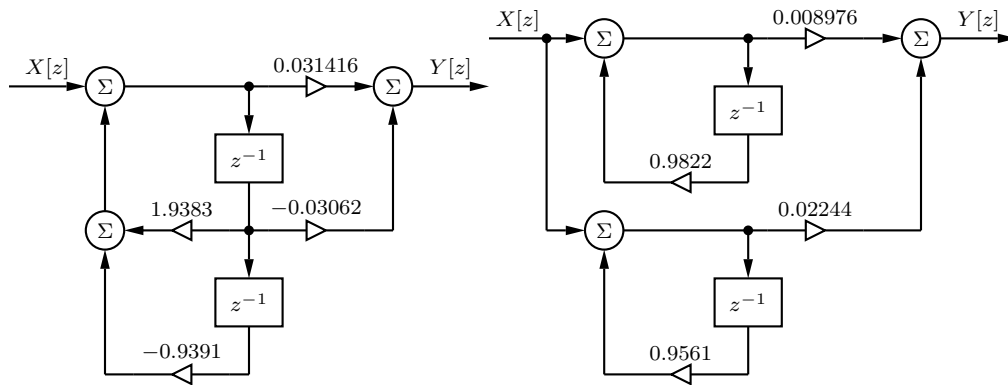
Figure S5.7-3b

Solution 5.7-4

Here, $H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$. Using Table 5.3, we get

$$H[z] = \left[\frac{\sqrt{2}Tze^{-T/\sqrt{2}}\sin\left(\frac{T}{\sqrt{2}}\right)}{z^2 - 2ze^{-T/\sqrt{2}}\cos\left(\frac{T}{\sqrt{2}}\right) + e^{-\sqrt{2}T}} \right].$$

We now select T. At dc, $H_a(0) = 1$. For large s, $H_a(s) \approx \frac{1}{s^2}$, hence $|H_a(j\omega)| \simeq \frac{1}{\omega^2}$ for high ω. For negligible aliasing, we select the frequency ω_0 to be that where $|H_a(j\omega_0)|$ is 1% of $|H_a(0)|$. Hence,

$$\frac{1}{\omega_0^2} = 0.01, \quad \omega_0 = 10, \quad \text{and} \quad T = \frac{\pi}{\omega_0} = \frac{\pi}{10}.$$

Substitution of this value of T in $H[z]$ yields

$$H[z] = \frac{0.0784z}{z^2 - 1.5622z + 0.6413}.$$

Solution 5.7-5

For an ideal integrator

$$H_a(s) = \frac{1}{s}.$$

Using Table 5.3, we find that

$$H[z] = \frac{Tz}{z-1} \quad \text{and} \quad H[e^{j\omega T}] = \frac{Te^{j\omega T}}{e^{j\omega T} - 1}.$$

Therefore,

$$|H[e^{j\omega T}]| = \frac{T}{\left| \sqrt{(\cos\omega T - 1)^2 + \sin^2\omega T} \right|}$$

$$= \frac{T}{\left| \sqrt{2(1 - \cos\omega T)} \right|}$$

$$= \frac{T}{2|\sin\frac{\omega T}{2}|}, \qquad |\omega| \leq \frac{\pi}{T}.$$

The ideal integrator amplitude response is

$$|H_a(j\omega)| = \frac{1}{\omega}.$$

Figure S5.7-5 shows the (T-normalized) digital filter magnitude response $\left|\frac{H[e^{j\omega T}]}{T}\right|$ (solid line) as well as the ideal (T-normalized) integrator magnitude response $\left|\frac{H_a(j\omega)}{T}\right|$. The two responses are quite close for low frequencies (as expected), and, due to aliasing, deviate at higher frequencies. Notice the repeating nature of the DT filter response, as expected.
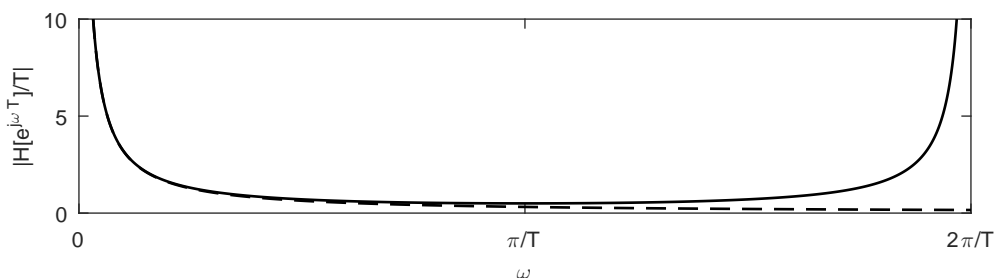
Figure S5.7-5

Solution 5.7-6

(a) Because an oscillator output is basically a system output with no input, a system with zero-input response of the form $\sin \Omega_0 n$ (or $\cos(\Omega_0 n + \theta)$ with any value of θ), where $\Omega_0 = \omega_0 T$ will serve as the desired oscillator. A marginally stable system with impulse response of the above form is a candidate. From Table 5.1, pair 11b, we see that a transfer function

$$H[z] = \frac{z \sin \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}$$

has an impulse response (or zero-input response) of the form $\sin \Omega_0 n$ ($\Omega_0 = \omega_0 T$). The period of the sinusoid is $T_0 = 2\pi/\Omega_0$, and there are 10 samples in each cycle. Therefore, the sampling interval $T = T_0/10 = \pi/5\Omega_0$, and $\Omega_0 T = \pi/5$. Hence,

$$H[z] = \frac{z \sin\left(\frac{\pi}{5}\right)}{z^2 - 2\cos\left(\frac{\pi}{5}\right) z + 1}$$
$$= \frac{0.5878z}{z^2 - 1.618z + 1}.$$

This is one possible solution. By varying the phase in the impulse response, we could obtain variations of this transfer function.

(b) Another approach is to consider an analog system with transfer function $H_a(s)$ such that its impulse response (or zero-input response) is of the form $\sin \omega_0 t$ [or $\cos(\omega_0 t + \theta)$ for any value of θ]. From Table 4.1, pair 8b we find

$$H_a(s) = \frac{\omega_0}{s^2 + \omega_0^2}.$$

Now using Table 5.1, we find the corresponding digital filter using impulse invariance method as

$$H[z] = \frac{Tz \sin \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}.$$

In part (a) we found that $\omega_0 T = \pi/5$. Because $\omega_0 = 2\pi(10,000) = 20,000\pi$, the period $T_0 = 10^{-4}$. There are 10 samples in each period. Hence the sampling interval $T = 10^{-5}$, and

$$H[z] = 10^{-5} \frac{0.5878z}{z^2 - 1.618z + 1}.$$

This is identical to the answer in (a) except for an amplitude scaling by 10^{-5}. Because, we did not specify any amplitude requirement on the oscillator, different answers will differ by a constant multiplier. This is a marginally stable system and will oscillate without input with the response of the form

$$h[n] = 10^{-5} \sin(0.2\pi n).$$

This is a discrete sinusoid with 10 samples for each cycle. Each sample is separated by 10^{-5} seconds. Hence the duration (period) of a cycle is $10 \times 10^{-5} = 10^{-4}$ and the frequency of oscillator is 10^4 Hz or 10 kHz as desired.

(c) The controller canonical form of realization is shown in Fig. S5.7-6c. Note that the multiplier 10^{-5} is not important in this realization, and hence is not shown in the figure. Although the input of $x[n] = 0$ is shown for reference, it would not need to be explicitly included in the final realization.
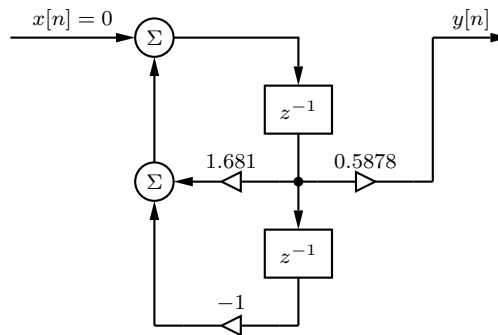
Figure S5.7-6c

Solution 5.7-7

(a) If $g_a(t)$ is the unit step response of the system $H_a(s)$ in Fig. 5.24a, then $g_a(nT)$ should be the response of $H[z]$ to the input $u[n]$. We can use this criterion to design a digital filter to realize a given $H_a(s)$. Consider the filter

$$H_a(s) = \frac{\omega_c}{s + \omega_c}.$$

The unit step response $g_a(t)$ is given by:

$$g_a(t) = \mathcal{L}^{-1}\left[\frac{H_a(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{\omega_c}{s(s + \omega_c)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s + \omega_c}\right].$$

Therefore,

$$g_a(t) = (1 - e^{-\omega_c t})u(t) \quad \text{and} \quad g_a(nT) = (1 - e^{-\omega_c nT})u[n].$$

Also, the response of $H[z]$ to $u[n]$ is given by

$$g[n] = \mathcal{Z}^{-1}\left\{\frac{z}{z - 1}H[z]\right\}.$$

Since $g[n] = g_a(nT)$,

$$\frac{z}{z - 1}H[z] = \mathcal{Z}[(1 - e^{-\omega_c nT})u[n]]$$

$$= \frac{z}{z - 1} - \frac{z}{z - e^{-\omega_c T}} = \frac{z(1 - e^{-\omega_c T})}{(z - 1)(z - e^{-\omega_c T})}.$$

Therefore,
$$H[z] = \frac{1 - e^{-\omega_c T}}{z - e^{-\omega_c T}}.$$

Using the above argument, we can generalize to
$$H[z] = \frac{z-1}{z} \mathcal{Z} \left[\mathcal{L}^{-1} \left(\frac{H_a(s)}{s} \right) \right]_{t=nT}.$$

(b) For $H_a(s) = \frac{\omega_c}{s+\omega_c}$, the unit step invariance method gives
$$H[z] = \frac{1 - e^{-\omega_c T}}{z - e^{-\omega_c T}}.$$

(c) For an integrator, $H_a(s) = 1/s$, and $\mathcal{L}^{-1}[H_s)/s = tu(t)$. Thus,
$$H[z] = \frac{z-1}{z} \mathcal{Z}[nTu[n]] = \frac{T}{z-1}$$

and
$$H[e^{j\omega T}] = \frac{T}{e^{j\omega T} - 1}.$$

Hence,
$$|H[e^{j\omega T}]| = \frac{T}{\left| \sqrt{(\cos\omega T - 1)^2 + \sin^2 \omega T} \right|} = \frac{T}{|\sqrt{2(1 - \cos\omega T)}|} = \frac{T}{2|\sin\frac{\omega T}{2}|} \qquad |\omega| \le \frac{\pi}{T}.$$

The ideal integrator amplitude response is
$$|H_a(j\omega)| = \frac{1}{\omega}.$$

Observe that this amplitude response is identical to that found by the impulse invariance method in Prob. 5.7-5. Hence, this amplitude response and the ideal integrator amplitude response are the same as those in Fig. S5.7-5. The only difference between the answers obtained by these methods is that the phase response of the step invariance response differs from that of the impulse invariance method by a constant $\frac{\pi}{2}$.

Solution 5.7-8

(a) For a differentiator
$$H_a(s) = s.$$

The unit ramp response $r(t)$ is given by
$$r(t) = \mathcal{L}^{-1} F(s) H_a(s) = \mathcal{L}^{-1} \frac{1}{s^2}(s) = u(t).$$

Now we must design $H[z]$ such that its response to input $nTu[n]$ is $u[n]$; that is,
$$\mathcal{Z}[u[n]] = H[z]\mathcal{Z}[nTu[n]]$$

$$\frac{z}{z-1} = \frac{Tz}{(z-1)^2} H[z].$$

Thus,
$$H[z] = \frac{1}{T}(z-1).$$

(b) For an integrator

$$H(s) = \frac{1}{s}.$$

The unit ramp response $r(t)$ is given by

$$r(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\frac{1}{s} = \frac{1}{2}t^2 u(t).$$

Now we design $H[z]$ such that its response to $nTu[n]$ is $\frac{1}{2}n^2T^2u[n]$, that is

$$\mathcal{Z}\left\{\frac{1}{2}n^2T^2u[n]\right\} = H[z]\mathcal{Z}\{nTu[n]\}$$

or

$$\frac{T^2z(z+1)}{2(z-1)^3} = \frac{Tz}{(z-1)^2}H[z].$$

Hence,

$$H[z] = \frac{T}{2}\left(\frac{z+1}{z-1}\right).$$

Solution 5.7-9

$$\text{For} \qquad H_a(s) = \sum_i \frac{k_i}{s-\lambda_i}, \qquad H[z] = T\sum_i \frac{k_iz}{z - e^{\lambda_iT}}.$$

If $\lambda_i = \alpha_i + j\beta_i$, then $e^{\lambda_iT} = e^{\alpha_iT}e^{j\beta_iT}$. When λ_i is in the LHP, $\alpha_i < 0$ and $|e^{\lambda_iT}| = e^{\alpha_iT} < 1$. Hence if λ_i is in the LHP, the corresponding pole of $H[z]$ is within the unit circle. Clearly if $H_a(s)$ is stable, the corresponding $H[z]$ is also stable.

Solution 5.7-10

(a) The ω-axis is given by $s = j\omega$. Rewriting the transformation $s = \frac{1-z^{-1}}{T}$ as $z = \frac{1}{1-sT}$ and substituting $s = j\omega$ yields $z = \frac{1}{1-j\omega T}$. Thus, we need to show that $z = \frac{1}{1-j\omega T}$ describes a circle centered at $(1/2, 0)$ with a radius of $1/2$.

For a complex variable z, the equation $|z-1/2|^2 = (1/2)^2$ describes a circle centered at $(1/2, 0)$ with a radius of $1/2$. The transformation rule $z = \frac{1}{1-j\omega T}$ is substituted into this expression.

$$|z - 1/2|^2 = (z - 1/2)(z^* - 1/2)$$

$$= \left(\frac{1}{1-j\omega T} - \frac{1}{2}\right)\left(\frac{1}{1+j\omega T} - \frac{1}{2}\right)$$

$$= \frac{1}{1+\omega^2T^2} + \frac{-1}{2(1+j\omega T)} + \frac{-1}{2(1-j\omega T)} + \frac{1}{4}$$

$$= \frac{1}{1+\omega^2T^2} + \frac{-(1-j\omega T) - (1+j\omega T)}{2(1+j\omega T)(1-j\omega T)} + \frac{1}{4}$$

$$= \frac{1}{1+\omega^2T^2} + \frac{-2}{2(1+\omega^2T^2)} + \frac{1}{4}$$

$$= 1/4 = (1/2)^2$$

Since the equation is satisfied, the transformation rule $z = \frac{1}{1-j\omega T}$ maps the ω-axis to a circle centered at $(1/2, 0)$ with a radius of $1/2$.

Notice that different values of ω can map to the same value of z (aliasing), which makes an inverse transformation problematic.

(b) First, rewrite the transformation $s = \frac{1-z^{-1}}{T}$ as $z = \frac{1}{1-sT}$. Next, notice

$$|z|^2 = zz^*$$

$$= \frac{1}{1-sT}\frac{1}{1-s^*T}$$

$$= \frac{1}{1-sT-s^*T+ss^*T^2}$$

$$= \frac{1}{1-(\sigma+j\omega)T-(\sigma-j\omega)T+(\sigma^2+\omega^2)T^2}$$

$$= \frac{1}{1-2\sigma T+(\sigma^2+\omega^2)T^2}.$$

For $\sigma < 0$, the denominator $1 - 2\sigma T + (\sigma^2 + \omega^2)T^2 > 1$ and $|z|^2 < 1$. That is, the left-half plane of s ($\sigma < 0$) is guaranteed to map to the interior of the unit circle in the z-plane.

Solution 5.8-1

(a)

$$x[n] = \underbrace{(0.8)^n u[n]}_{x_1[n]} + \underbrace{2^n u[-(n+1)]}_{x_2[n]}$$

$$x_1[n] \Longleftrightarrow \frac{z}{z-0.8} \qquad |z| > 0.8$$

$$x_2[n] \Longleftrightarrow \frac{-z}{z-2} \qquad |z| < 2$$

Hence

$$X[z] = \frac{z}{z-0.8} - \frac{z}{z-2} \qquad 0.8 < |z| < 2$$

$$= \frac{-1.2z}{z^2 - 2.8z + 1.6} \qquad 0.8 < |z| < 2.$$

(b)

$$X_1[z] = \frac{z}{z-2} \qquad |z| > 2$$

$$X_2[z] = \frac{z}{z-3} \qquad |z| < 3$$

Hence

$$X[z] = \frac{z}{z-2} + \frac{z}{z-3} \qquad 2 < |z| < 3$$

$$= \frac{z(2z-5)}{z^2 - 5z + 6} \qquad 2 < |z| < 3.$$

(c) By definition,

$$X[z] = \sum_{n=-\infty}^{0} (-2)^3 \left(-\frac{2}{z}\right)^n + \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k-1} \delta[n-2k]z^{-n}.$$

For $|z| < 2$, the first sum converges, and

$$X[z] = -8\left(\frac{0 - \left(\frac{-2}{z}\right)^1}{1 - \left(\frac{-2}{z}\right)}\right) + \sum_{k=0}^{\infty} 2\left(\frac{1}{2}\right)^k \sum_{n=-\infty}^{\infty} z^{-n}\delta[n - 2k]$$

$$= \frac{\frac{-16}{z}}{1 + \frac{2}{z}} + 2\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^{-2k}$$

$$= \frac{-16}{z + 2} + 2\sum_{k=0}^{\infty}\left(\frac{1}{2z^2}\right)^k.$$

The second sum converges for $|z| > \frac{1}{\sqrt{2}}$, yielding

$$X[z] = \frac{-16}{z + 2} + 2\frac{1 - 0}{1 - \frac{1}{2z^2}} = -\frac{16}{z + 2} + \frac{2z^2}{z^2 - \frac{1}{2}}.$$

Simplifying, we obtain

$$X[z] = \frac{2z^3 - 12z^2 + 8}{z^3 + 2z^2 - \frac{1}{2}z - 1}, \qquad \frac{1}{\sqrt{2}} < |z| < 2.$$

(d)

$$X_1[z] = \frac{z}{z - 0.8} \qquad |z| > 0.8$$

$$X_2[z] = \frac{-z}{z - 0.9} \qquad |z| < 0.9$$

$$\text{Hence} \qquad X[z] = \frac{z}{z - 0.8} - \frac{z}{z - 0.9}$$

$$= \frac{-z}{10(z^2 - 1.7z + 0.72)} \qquad 0.8 < |z| < 0.9.$$

(e)

$$[(0.8)^n + 3(0.4)^n]\,u\,[-(n + 1)] \Longleftrightarrow \left(\frac{-z}{z - 0.8} - \frac{3z}{z - 0.4}\right) \qquad |z| < 0.4$$

$$= \frac{-4z(z - 0.7)}{(z - 0.4)(z - 0.8)} \qquad |z| < 0.4$$

(f)

$$[(0.8)^n + 3(0.4)^n]\,u\,[n] \Longleftrightarrow \frac{z}{z - 0.8} + \frac{3z}{z - 0.4} \qquad |z| > 0.8$$

$$= \frac{4z(z - 0.7)}{(z - 0.4)(z - 0.8)} \qquad |z| > 0.8$$

(g)

$$(0.8)^n u\,[n] + 3(0.4)^n u\,[-(n + 1)]$$

The region of convergence for $(0.8)^n u\,[n]$ is $|z| > 0.8$. The region of convergence for $(0.4)^n u\,[-(n + 1)]$ is $|z| < 0.4$. The common region does not exist. Hence the z-transform for this function does not exist.

(h)

$$x[n] = (0.5)^{|n|} = 0.5^n u[n] + (0.5)^{-n} u[-n-1]$$
$$= 0.5^n u[n] + 2^n u[-n-1].$$

Further,

$$0.5^n u[n] \Longleftrightarrow \frac{z}{z-0.5} \qquad |z| > 0.5$$

$$2^n u[-n-1] \Longleftrightarrow \frac{-z}{z-2} \qquad |z| < 2.$$

Hence,

$$X[z] = \frac{z}{z-o.5} - \frac{z}{z-2} = \frac{-1.5z}{(z-0.5)(z-2)} \qquad 0.5 \le |z| < 2.$$

(i)

$$x[n] = nu[-(n+1)]$$
$$X[z] = \frac{-z}{(z-1)^2} \qquad |z| < 1$$

Solution 5.8-2

(a) Notice that

$$x[n] = 3^n u[-n] = \begin{cases} 3^n & n < 0 \\ 3^0 = 1 & n \ge 0 \end{cases}.$$

By definition,

$$X[z] = \sum_{n=-\infty}^{-1} 3^n z^{-n} + \sum_{n=0}^{\infty} z^{-n}.$$

The first sum converges if $|z| < 3$ and the second sum converges if $|z| > 1$. Thus,

$$X[z] = \frac{0 - \left(\frac{3}{z}\right)^{-1+1}}{1 - \frac{3}{z}} + \frac{\left(\frac{1}{z}\right)^0 - 0}{1 - \frac{1}{z}}$$

$$= \frac{-z}{z-3} + \frac{z}{z-1} = \frac{-z^2 + z + z^2 - 3z}{z^2 - 4z + 3}.$$

Thus,

$$X[z] = \frac{-2z}{z^2 - 4z + 3}, \qquad 1 < |z| < 3.$$

(b) Notice that

$$x[n] = \left(\frac{1}{3}\right)^{nu[n]} = \begin{cases} \left(\frac{1}{3}\right)^0 = 1 & n < 0 \\ \left(\frac{1}{3}\right)^n & n \ge 0 \end{cases}.$$

By definition,

$$X[z] = \sum_{n=-\infty}^{-1} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}.$$

The first sum converges if $|z| < 1$ and the second sum converges if $|z| > \frac{1}{3}$. Thus,

$$X[z] = \frac{0 - \left(\frac{1}{z}\right)^{-1+1}}{1 - \frac{1}{z}} + \frac{\left(\frac{1}{3z}\right)^0 - 0}{1 - \frac{1}{3z}}$$

$$= \frac{-z}{z-1} + \frac{z}{z-\frac{1}{3}} = \frac{z^2 + \frac{z}{3} + z^2 - z}{z^2 - \frac{4}{3}z + \frac{1}{3}}.$$

Thus,

$$X[z] = \frac{-\frac{2}{3}z}{z^2 - \frac{4}{3}z + \frac{1}{3}}, \qquad \frac{1}{3} < |z| < 1.$$

Solution 5.8-3

To begin, we note that

$$\frac{X[z]}{z} = \frac{z - \frac{1}{3}}{(z-1)(z+2)} = \frac{\frac{2}{9}}{z-1} + \frac{\frac{7}{9}}{z+2}.$$

Since $1 < |z| < 2$, the first fraction corresponds to a right-sided signal while the second fraction corresponds to a left-sided signal. Therefore, the inverse bilateral z-transform yields

$$x[n] = \frac{2}{9}u[n] - \frac{7}{9}(-2)^n u[-n-1].$$

Solution 5.8-4

$$\frac{X[z]}{z} = \frac{e^{-2} - 2}{(z - e^{-2})(z-2)} = \frac{1}{z - e^{-2}} - \frac{1}{z-2}$$

$$\text{and} \qquad X[z] = \frac{z}{z - e^{-2}} - \frac{z}{z-2}$$

(a) The region of convergence is $|z| > 2$. Both terms are causal, and

$$x[n] = (e^{-2n} - 2^n)u[n].$$

(b) The region of convergence is $e^{-2} < |z| < 2$. In this case the first term is causal and the second is anticausal. Thus,

$$x[n] = e^{-2n}u[n] + 2^n u[-(n+1)].$$

(c) The region of convergence is $|z| < e^{-2}$. Both terms are anticausal in this case, hence

$$x[n] = (-e^{-2n} + 2^n)u[-(n+1)].$$

Solution 5.8-5

Here,

$$X[z] = \frac{1}{(2z+1)(z+1)(z+\frac{1}{2})} = \frac{1/2}{(z+1/2)^2(z+1)}$$

$$= \frac{1}{(z+1/2)^2} + \frac{-2}{(z+1/2)} + \frac{2}{(z+1)} = -2z^{-1}\frac{-z/2}{(z+1/2)^2} - 2z^{-1}\frac{z}{(z+1/2)} + 2z^{-1}\frac{z}{(z+1)}.$$

Since $\left(|z| < \frac{1}{2}\right)$, the time-domain signal is left-sided. Thus,

$$x[n] = 2(n-1)(-1/2)^{n-1}u[-(n-1)-1] + 2(-1/2)^{n-1}u[-(n-1)-1] - 2(-1)^{n-1}u[-(n-1)-1]$$
$$= -4(n-1)(-1/2)^n u[-n] - 4(-1/2)^n u[-n] + 2(-1)^n u[-n].$$

Simplifying yields

$$x[n] = -4n(-1/2)^n u[-n] + 2(-1)^n u[-n].$$

Solution 5.8-6

To begin, we note that

$$\left(\frac{1}{3}\right)^{n-3} u[n-2] = 3\left(\frac{1}{3}\right)^{n-2} u[n-2] \Longleftrightarrow 3z^{-2}\frac{z}{z - \frac{1}{3}}, \qquad |z| > \frac{1}{3}$$

and
$$(2)^n u[-n] = -2\left(-2^{n-1}u[-(n-1)-1]\right) \Longleftrightarrow -2z^{-1}\frac{z}{z-2}, \qquad |z| < 2.$$

Since convolution in the time domain corresponds to multiplication in the transform domain, we know that

$$y[n] = (\frac{1}{3})^{n-3}u[n-2] * (2)^n u[-n] \Longleftrightarrow Y[z] = 3z^{-2}(-2z^{-1})\left[\frac{z(z)}{(z-\frac{1}{3})(z-2)}\right], \qquad \frac{1}{3} < |z| < 2.$$

Taking a partial fraction expansion, we see that

$$Y[z] = -6z^{-1}\left[\frac{-\frac{3}{5}}{z-\frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$= \frac{18}{5}z^{-2}\frac{z}{z-\frac{1}{3}} - \frac{18}{5}z^{-2}\frac{z}{z-2}.$$

Using $\frac{1}{3} < |z| < 2$, the inverse transform is

$$y[n] = \frac{18}{5}\left(\frac{1}{3}\right)^{n-2} u[n-2] + \frac{18}{5}(2)^{n-2}u[-(n-2)-1]$$

$$= \frac{162}{5}\left(\frac{1}{3}\right)^{n} u[n-2] + \frac{9}{10}(2)^n u[-n+1].$$

Comparing to the form $y[n] = c_1\gamma_1^n u[n+N_1] + c_2\gamma_2^n u[-n+N_2]$, we see that

$$c_1 = \frac{162}{5}, \quad c_2 = \frac{9}{10}, \quad \gamma_1 = \frac{1}{3}, \quad \gamma_2 = 2, \quad N_1 = -2, \quad \text{and} \quad N_2 = 1.$$

Due to a shared boundary, it is also possible to specify $N_1 = -3$ and $N_2 = 2$.

Solution 5.8-7

To begin, we notice that

$$H[z] = z^3\frac{z+1}{(z-2)(z+\frac{1}{2})} = z^3\left(\frac{\frac{6}{5}}{z-2} + \frac{-\frac{1}{5}}{z+\frac{1}{2}}\right) = z^2\left(\frac{\frac{6}{5}z}{z-2} + \frac{-\frac{1}{5}z}{z+\frac{1}{2}}\right).$$

For stable $h[n]$, the ROC must include the unit circle, so $\frac{1}{2} < |z| < 2$. Inverting, we obtain

$$h[n] = -\frac{6}{5}(2)^{n+2}u[-(n+2)-1] = \frac{1}{5}(-\frac{1}{2})^{n+2}u[n+2].$$

Simplifying, we obtain

$$h[n] = -\frac{24}{5}(2)^n u[-n-3] - \frac{1}{20}(-\frac{1}{2})^n u[n+2].$$

Solution 5.8-8

(a) The three poles satisfy $z^3 = \frac{27}{8}$, or $z = 3/2e^{j2\pi k/3}$ for $k = (0,1,2)$. There are two finite zeros at $z = 0$ and $z = 1/2$ as well as a zero at infinity. MATLAB is used to create the corresponding pole-zero plot.

```
>> k = [0:2]; zp = 3/2*exp(j*2*pi*k/3); zz = [0,1/2];
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx');
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis equal; grid; axis([-1.6 1.6 -1.6 1.6]);
```

Figure S5.8-8a

There are two possible regions of convergence, both of which exclude the three system poles:

$$\text{ROC 1: } |z| < 3/2 \quad \text{and} \quad \text{ROC 2: } |z| > 3/2.$$

(b) The poles and zeros of $H^{-1}[z]$ are just the zeros and poles, respectively, of $H[z]$. Thus, the three zeros of $H[z]$ satisfy $z^3 = \frac{27}{8}$, or $z = 3/2e^{j2\pi k/3}$ for $k = (0, 1, 2)$. There are two finite poles at $z = 0$ and $z = 1/2$ as well as a pole at infinity. MATLAB is used to create the corresponding pole-zero plot.

```
>>  k = [0:2]; zz = 3/2*exp(j*2*pi*k/3); zp = [0,1/2];
>>  plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx');
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  axis equal; grid; axis([-1.6 1.6 -1.6 1.6]);
```
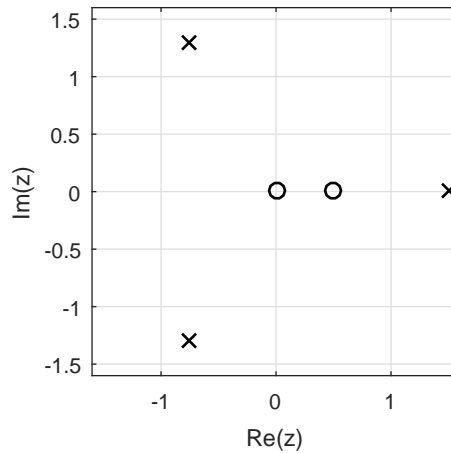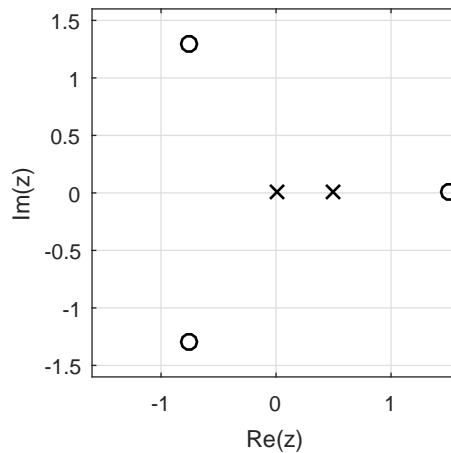

Figure S5.8-8b

There are two possible regions of convergence, both of which exclude the three system poles:

$$\text{ROC 1: } 0 < |z| < 1/2 \quad \text{and} \quad \text{ROC 2: } 1/2 < |z| < \infty.$$

Solution 5.8-9

It is known that $x[n]$ has a mode $(1/2)^n$ and that $x_1[n] = (1/3)^n x[n]$ is absolutely summable. For this to be true, the mode at $(1/2)^n$ must be right-sided. That is, $(1/2)^n (1/3)^n u[n]$ is absolutely summable but $(1/2)^n (1/3)^n u[-n]$ is not. It is also known that $x_2[n] = (1/4)^n x[n]$ is not absolutely summable. For this to be true, there must be a pole somewhere in the annulus $3 < |z| < 4$ that corresponds to a left-sided signal; such a mode when multiplied by $(1/3)^n$ is still absolutely summable but when multiplied by $(1/4)^n$ is not absolutely summable. Thus,

$$x[n] \text{ is a two-sided signal.}$$

Solution 5.8-10

In polar form, the known pole is at $z = \sqrt{18/16}e^{j\pi/4}$. To be absolutely summable, the signal's region of convergence must include the unit circle, $|z| = 1$.

(a) Yes, the signal can be left-sided. Since the known pole is outside the unit circle, a region of convergence that includes the unit circle (needed for absolute summability) implies that the known pole corresponds to a left-sided component of the signal.

(b) No, the signal cannot be right-sided. If the known pole outside the unit circle is right-sided, the region of convergence cannot include the unit circle and the signal cannot be absolutely summable as required.

(c) Yes, the signal can be two-sided. Let the known pole correspond to a left-sided component and let there be another pole within the unit circle that corresponds to a right-sided component. Such a signal is two-sided and has a region of convergence that includes the unit circle, which ensures the signal is absolutely summable as required.

(d) No, the signal cannot be finite duration. Finite duration signals cannot have poles in the region $0 < |z| < \infty$. Such poles, such as the pole known to exist, correspond to time-domain components with infinite duration.

Solution 5.8-11

(a) $X_1[z] = \frac{z}{z-\frac{3}{4}}$. Next, $Y_1[z] = H[z]X_1[z] = \frac{z-\frac{1}{2}}{z+1/2}\left(\frac{z}{z-\frac{3}{4}}\right)$. Thus, $\frac{Y_1[z]}{z} = \frac{z-\frac{1}{2}}{z+\frac{1}{2}}\left(\frac{1}{z-\frac{3}{4}}\right) = \frac{\frac{4}{5}}{z+\frac{1}{2}} + \frac{\frac{1}{5}}{z-\frac{3}{4}}$ or $Y_1[z] = \frac{\frac{4}{5}z}{z+\frac{1}{2}} + \frac{\frac{1}{5}z}{z-\frac{3}{4}}$. Inverting yields

$$y_1[n] = \frac{4}{5}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{5}\left(\frac{3}{4}\right)^n u[n].$$

(b) The idea of frequency response is used to determine the output in response to the everlasting exponential input $x_2[n] = \left(\frac{3}{4}\right)^n$. That is, $H(z = \frac{3}{4}) = \frac{\frac{3}{4}-\frac{1}{2}}{\frac{3}{4}+\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{4}} = \frac{1}{5}$. Thus,

$$y_2[n] = \frac{1}{5}\left(\frac{3}{4}\right)^n.$$

(c) $X_3[z] = -\frac{z}{z-\frac{3}{4}}$ with $|z| < \frac{3}{4}$. Next, $Y_3[z] = H[z]X_3[z] = -\frac{z-\frac{1}{2}}{z+1/2}\left(\frac{z}{z-\frac{3}{4}}\right)$ with $\frac{1}{2} < |z| < \frac{3}{4}$. Thus, $\frac{Y_3[z]}{z} = -\frac{z-\frac{1}{2}}{z+\frac{1}{2}}\left(\frac{1}{z-\frac{3}{4}}\right) = -\frac{\frac{4}{5}}{z+\frac{1}{2}} - \frac{\frac{1}{5}}{z-\frac{3}{4}}$ or $Y_3[z] = -\frac{\frac{4}{5}z}{z+\frac{1}{2}} - \frac{\frac{1}{5}z}{z-\frac{3}{4}}$. Using $\frac{1}{2} < |z| < \frac{3}{4}$ and inverting yields

$$y_1[n] = -\frac{4}{5}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{5}\left(\frac{3}{4}\right)^n u[-n-1].$$

Solution 5.8-12

The given signal is $x[n] = (-1)^n u[n - n_0] + \alpha^n u[-n]$. If $|\alpha| = 2$, the z-transform $X[z] = (-1)^{n_0} z^{-n_0} \frac{z}{z+1} + \frac{-z}{z-\alpha}$ has region of convergence $1 < |z| < |\alpha| = 2$, as desired. Thus, the necessary constraint is

$$|\alpha| = 2.$$

There is no constraint on the integer n_0, other than it be finite.

Solution 5.8-13

(a) $X_1[z] = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left((-j)^{-n} u[-n] + \delta[-n] \right) z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{-1}{jz} \right)^n + \sum_{n=-\infty}^{\infty} \delta[-n]z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{j}{z} \right)^n + 1$. For $|z| < 1$, this becomes

$$X_1[z] = 1 + \frac{0 - j/z}{1 - j/z} = 1 + \frac{-j}{z - j}; \text{ ROC } |z| < 1.$$

(b) $X_2[z] = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \sum_{n=-\infty}^{\infty} (j)^n \cos(n + 1)u[n] = \sum_{n=0}^{\infty} j^n 0.5 \left(e^{j(n+1)} + e^{-jmath(n+1)} \right) z^{-n} = \sum_{n=0}^{\infty} 0.5 e^j \left(\frac{je^j}{z} \right)^n + \sum_{n=0}^{\infty} 0.5 e^{-j} \left(\frac{je^{-j}}{z} \right)^n$. For $|z| > 1$, this becomes

$$X_2[z] = \frac{0.5e^j}{1 - je^j z^{-1}} + \frac{0.5e^{-j}}{1 - je^{-j} z^{-1}}; \text{ ROC } |z| > 1.$$

(c) $X_3[z] = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(j0.5(e^n - e^{-n})u[-n+1] \right) z^{-n} = \sum_{n=-\infty}^{1} j0.5 \left\{ \left(\frac{e}{z} \right)^n - \left(\frac{1}{ez} \right)^n \right\}$. For $|z| < e$ and $|z| < e^{-1}$, this becomes $X_3[z] = \frac{j}{2} \left\{ \frac{0 - (e/z)^2}{1 - e/z} - \frac{0 - (ez)^{-2}}{1 - (ez)^{-1}} \right\}$. Thus,

$$X_3[z] = \frac{jz^{-2}}{2} \left\{ \frac{e^{-2}}{1 - (ez)^{-1}} + \frac{e^2}{1 - ez^{-1}} \right\}; \text{ ROC } |z| < e^{-1}.$$

(d) $X_4[z] = \sum_{n=-\infty}^{\infty} x_4[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{0} (2j)^n \delta[n - 2k] \right) z^{-n} = \sum_{k=-\infty}^{0} \sum_{n=-\infty}^{\infty} \left(\frac{2j}{z} \right)^n \delta[n - 2k] = \sum_{k=-\infty}^{0} \left(\frac{2j}{z} \right)^{2k} = \sum_{k=-\infty}^{0} \left(\frac{-4}{z^2} \right)^k$. For $|z^2| < 4$, this becomes

$$X_4[z] = \frac{0 - (-4z^{-2})}{1 - (-4z^{-2})} = \frac{4z^{-2}}{1 + 4z^{-2}}; \text{ ROC } |z| < 2.$$

Note: $|z^2| = |z|^2$, so the region of convergence is $|z|^2 < 4$ or $|z| < 2$.

Solution 5.8-14

(a) The signal $X_1(z) = \frac{1}{1 + \frac{13}{6}z^{-1} + \frac{1}{6}z^{-2} - \frac{1}{3}z^{-3}}$ has three finite poles, of which at least one must be real. Using the region of convergence, we know that a real root must be either 0.5, -0.5, 2, or -2. Trying these values, we find that $z = -0.5$ and $z = -2$ are both roots of the denominator. The remaining root can be found by noting that the denominator $1 + \frac{13}{6}z^{-1} + \frac{1}{6}z^{-2} - \frac{1}{3}z^{-3}$ must be equal to $(1 + 0.5z^{-1})(1 + 2z^{-1})(1 + Az^{-1})$. Equating the power of z^{-3} on each side yields $-1/3 = A$. Thus, $X_1[z] = \frac{1}{(1+0.5z^{-1})(1+2z^{-1})(1-z^{-1}/3)}$. Expanding yields $X_1[z] = \frac{-1/5}{1+0.5z^{-1}} + \frac{8/7}{1+2z^{-1}} + \frac{2/35}{1-z^{-1}/3}$. Using the region of convergence $(0.5 < |z| < 2)$ and tables, the inverse is

$$x_1[n] = -\frac{1}{5}(-1/2)^n u[n] - \frac{8}{7}(-2)^n u[-n-1] + \frac{2}{35}(1/3)^n u[n].$$

(b) $X_2(z) = \frac{1}{z^{-3}(2-z^{-1})(1+2z^{-1})} = z^3/2\frac{1}{(1-z^{-1}/2)(1+2z^{-1})} = z^3/2\left(\frac{1/5}{1-z^{-1}/2} + \frac{4/5}{1+2z^{-1}}\right)$. Using the region of convergence $(0.5 < |z| < 2)$ and tables, the inverse is

$$x_2[n] = \frac{1}{10}(1/2)^{n+3}u[n+3] - \frac{2}{5}(-2)^{n+3}u[-n-4].$$

Solution 5.8-15

(a) $H_1[z] = \frac{z^{-1}}{(z-\frac{1}{2})(1+\frac{1}{2}z^{-1})} = \frac{z^{-1}}{z^{-1}(z-\frac{1}{2})(z+\frac{1}{2})} = \left(\frac{1}{z-1/2} + \frac{-1}{z+1/2}\right) = z^{-1}\left(\frac{z}{z-1/2} + \frac{-z}{z+1/2}\right)$. Since $h_1[n]$ is stable, the region of convergence for $H_1[z]$ must be $|z| > 1/2$. Using this ROC and z-transform tables, the inverse transform is

$$h_1[n] = (1/2)^{n-1}u[n-1] - (-1/2)^{n-1}u[n-1].$$

(b) $H_2[z] = z^{-3}\left(\frac{z+1}{(z-2)(z+1/2)}\right) = z^{-3}\left(\frac{6/5}{z-2} + \frac{-1/5}{z+1/2}\right) = z^{-4}\left(\frac{6z/5}{z-2} + \frac{-z/5}{z+1/2}\right)$. Since $h_2[n]$ is stable, the region of convergence for $H_2[z]$ must be $1/2 < |z| < 2$. Using this ROC and z-transform tables, the inverse transform is

$$h_2[n] = \frac{-6}{5}(2)^{n-4}u[-n+3] + \frac{-1}{5}(-1/2)^{n-4}u[n-4].$$

Solution 5.8-16

Here,

$$H[z] = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\delta[n-Nk]z^{-n}$$

$$= \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty}\delta[n-Nk]z^{-n} = \sum_{k=0}^{\infty}z^{-Nk} = \sum_{k=0}^{\infty}\left(\frac{1}{z^N}\right)^k.$$

For $|z| > 1$, this becomes

$$H[z] = \frac{1}{1-z^N}; \text{ ROC } |z| > 1.$$

Notice, $H[z]$ has N poles that correspond to the N roots of unity. That is, the poles of $H[z]$ satisfy $z = e^{j2\pi k/N}$ for $k = (0, 1, \ldots, N-1)$.

Solution 5.8-17

By inspection of the system difference equation, we know the transfer function is

$$H[z] = \frac{1}{1-\frac{1}{4}z^{-2}} = \frac{z^2}{z^2-\frac{1}{4}}, \qquad |z| > \frac{1}{2}.$$

Writing the input as $x[n] = 2^3 2^{n-3}u[-(n-3)-1]$, we see that

$$X[z] = -8z^{-3}\frac{z}{z-2}, \qquad |z| < 2.$$

In the transform domain, we know that

$$Y_{\text{zsr}}[z] = H[z]X[z]$$

$$= \frac{-8}{(z-\frac{1}{2})(z+\frac{1}{2})(z-2)} = -8\left[\frac{-\frac{2}{3}}{z-\frac{1}{2}} + \frac{\frac{2}{5}}{z+\frac{1}{2}} + \frac{\frac{4}{15}}{z-2}\right]$$

$$= \frac{16}{3}z^{-1}\frac{z}{z-\frac{1}{2}} - \frac{16}{5}z^{-1}\frac{z}{z+\frac{1}{2}} - \frac{32}{15}z^{-1}\frac{z}{z-2}, \qquad \frac{1}{2} < |z| < 2.$$

Inverting, we obtain

$$y_{\text{zsr}}[n] = \tfrac{16}{3}(\tfrac{1}{2})^{n-1}u[n-1] - \tfrac{16}{5}(-\tfrac{1}{2})^{n-1}u[n-1] + \tfrac{32}{15}(2)^{n-1}\underbrace{u[-(n-1)-1]}_{u[-n]}.$$

Solution 5.8-18

For causal signals, the region of convergence may be ignored. We shall consider it only for noncausal inputs

(a)

$$Y[z] = X[z]H[z] = \frac{z^2}{(z-e)(z+0.2)(z-0.8)}$$

Modified partial fraction expansion of $Y[z]$ yields

$$Y[z] = 0.477\frac{z}{z-e} - 0.068\frac{z}{z+0.2} - 0.412\frac{z}{z-0.8}.$$

Thus,

$$y[n] = [0.477e^n - 0.068(-0.2)^n - 0.412(0.8)^n]\, u[n].$$

(b) Here,

$$X[z] = \frac{-z}{z-2} \qquad |z| < 2,$$

$$H[z] = \frac{z}{(z+0.2)(z-0.8)} \qquad |z| > 0.8,$$

$$Y[z] = \frac{-z^2}{(z+0.2)(z-0.8)(z-2)} \qquad 0.8 < |z| < 2,$$

and

$$\frac{Y[z]}{z} = \frac{-z}{(z+0.2)(z-0.8)(z-2)} = \frac{1/11}{z+0.2} + \frac{2/3}{z-0.8} - \frac{0.758}{z-2}.$$

Therefore, $\quad Y[z] = \frac{1}{11}\frac{z}{z+0.2} + \frac{2}{3}\frac{z}{z-0.8} - 0.758\frac{z}{z-2} \quad 0.8 < |z| < 2$

and $\quad y[n] = \left[\frac{1}{11}(-0.2)^n + \frac{2}{3}(0.8)^n\right]u[n] + 0.758(2)^n u[-(n+1)].$

(c) The input in this case is the sum of the inputs in parts (a) and (b). Hence the response will be the sum of the responses in part (a) and (b). That is,

$$y[n] = (0.477e^n + 0.0229(-0.2)^n + 0.255(0.8)^n)\, u[n] + 0.758(2)^n u[-(n+1)].$$

(d)

$$x[n] = \underbrace{2^n u[n]}_{x_1[n]} + \underbrace{u[-(n+1)]}_{x_2[n]}$$

$$X_1[z] = \frac{z}{z-2} \qquad |z| > 2$$

$$X_2[z] = \frac{-z}{z-1} \qquad |z| < 1$$

There is no region of convergence common to $X_1[z]$ and $X_2[z]$, so we'll need to compute the output of the two components separately. Now,

$$H[z] = \frac{z}{(z + 0.2)(z - 0.8)}, \qquad |z| > 0.8.$$

The output to $x_1[n]$ is determined as

$$Y_1[z] = \frac{z^2}{(z - 2)(z + 0.2)(z - 0.8)} \qquad |z| > 2$$

The modified partial fractions of $Y_1[z]$ yield

$$Y_1[z] = -\frac{1}{11}\frac{z}{z + 0.2} - \frac{2}{3}\frac{z}{z - 0.8} + 0.758\frac{z}{z - 2}.$$

Thus,

$$y_1[n] = \left[-\frac{1}{11}(-0.2)^n - \frac{2}{3}(0.8)^n + 0.758(2)^n \right] u[n].$$

Similarly, the output to $x_2[n]$ is determined as

$$Y_2[z] = \frac{-25}{6}\frac{z}{z - 1} + \frac{1}{6}\frac{z}{z + 0.2} + 4\frac{z}{z - 0.8} \qquad 0.8 < |z| < 1.$$

Thus,

$$y_2[n] = \left[\frac{1}{6}(-0.2)^n + 4(0.8)^n \right] u[n] + \frac{25}{6}u[-(n + 1)].$$

Combining, we see that

$$y[n] = y_1[n] + y_2[n] = \left[\frac{5}{66}(-0.2)^n + \frac{10}{3}(0.8)^n + 0.758(2)^n \right] u[n] + \frac{25}{6}u[-(n + 1)].$$

(e) Here,

$$X[z] = -\frac{z}{z - e^{-2}} \qquad |z| < e^{-2}$$

and

$$H[z] = \frac{z}{(z + 0.2)(z - 0.8)} \qquad |z| > 0.8.$$

No common region of convergence for $X[z]$ and $H[z]$ exists. Hence,

$$y[n] = \infty.$$

Let $f[n] = y[-n]$ so that

$$c_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]f[n - k] = x[n] * f[n] = x[n] * y[-n].$$

Using properties and taking the z-transform yield

$$C_{xy}[z] = X[z]Y[\tfrac{1}{z}], \qquad \text{ROC is at least } R_x \cap \tfrac{1}{R_y}.$$

Solution 5.8-20

To begin, let us perform polynomial division with $\frac{z}{z-1}$.

$$
\begin{array}{r}
1 + z^{-1} + z^{-2} + z^{-3} + \cdots \\
z - 1 \overline{) \; z } \\
\underline{z \; - \; 1 } \\
1 \\
1 \; - \; z^{-1} \\
\underline{ z^{-1} } \\
z^{-1} \; - \; z^{-2} \\
\underline{ z^{-2}} \\
\vdots
\end{array}
$$

Thus,

$$\frac{z}{z-1} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots = \sum_{i=0}^{\infty} z^{-i}.$$

Consequently,

$$\frac{z}{z-1} X[z] = \sum_{i=0}^{\infty} z^{-i} X[z].$$

But $z^{-i} X[z] \Longleftrightarrow x[n-i]$, so

$$\frac{z}{z-1} X[z] \Longleftrightarrow \sum_{i=0}^{\infty} x[n-i].$$

Letting $k = n - i$ (so that limit $i = 0$ becomes $k = n$ and $i = \infty$ becomes $k = -\infty$), we obtain the desired result of

$$\frac{z}{z-1} X[z] \Longleftrightarrow \sum_{k=-\infty}^{n} x[k].$$

Solution 5.10-1

(a) In this case,

$$H_{\mathrm{a}}[z] = \frac{z^4 - \sqrt{2}z^2 + 1}{z^4 + 0.4096}.$$

We use MATLAB to create the system pole-zero plot.

```
>> zz = roots([1,0,-sqrt(2),0,1]); zp = roots([1,0,0,0,0.4096]);
>> Omega = linspace(0,2*pi,1001); ucirc = exp(1j*Omega);
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx',...
>>     real(ucirc),imag(ucirc),'k-');
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis equal; grid; axis([-1.1 1.1 -1.1 1.1]);
```

As shown in Fig. S5.10-1a, all the poles are inside the unit circle, so the system is stable.

(b) In this case,

$$H_{\mathrm{b}}[z] = \frac{-3z^{-1} + \frac{3}{4}z^{-3}}{3 + \frac{3}{2}z^{-2} + 2z^{-4}}.$$

We use MATLAB to create the system pole-zero plot.
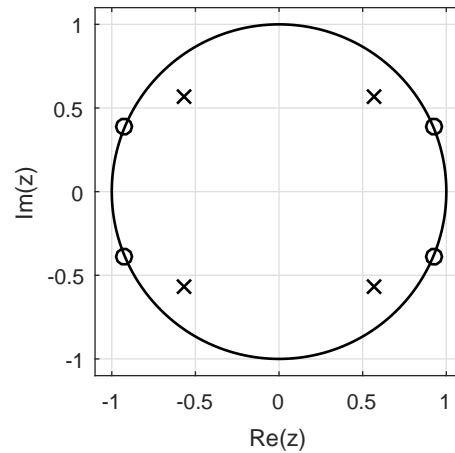
Figure S5.10-1a

```
>>  zz = roots([-3,0,3/4,0]); zp = roots([3,0,3/2,0,2]);
>>  Omega = linspace(0,2*pi,1001); ucirc = exp(1j*Omega);
>>  plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx',...
>>      real(ucirc),imag(ucirc),'k-');
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  axis equal; grid; axis([-1.1 1.1 -1.1 1.1]);
```

As shown in Fig. S5.10-1b, all the poles are inside the unit circle, so the system is stable.
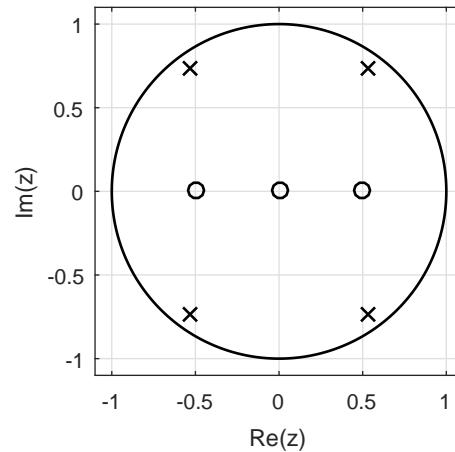
Figure S5.10-1b

Solution 5.10-2

Since the DT systems are stable, frequency response is found from the transfer function by substituting $e^{j\Omega}$ for z.

(a) In this case,

$$H_a[z] = \frac{\cos(z)}{z - 0.5}.$$

We use MATLAB to create the magnitude and phase response plots, as shown in Fig. S5.10-2a.

```
>>  Ha = @(z) cos(z)./(z-0.5); Omega = linspace(-pi,pi,1001);
```

```
>>  subplot(1,2,1); plot(Omega,abs(Ha(exp(1j*Omega))),'k-');
>>  xlabel('\Omega'); ylabel('|H_a[e^{j\Omega}]|'); grid on;
>>  axis([-pi pi 0 1.6]); set(gca,'xtick',-pi:pi/2:pi);
>>  subplot(1,2,2); plot(Omega,angle(Ha(exp(1j*Omega))),'k-');
>>  xlabel('\Omega'); ylabel('\angle H_a[e^{j\Omega}]'); grid on;
>>  axis([-pi pi -pi pi]); set(gca,'xtick',-pi:pi/2:pi);
```
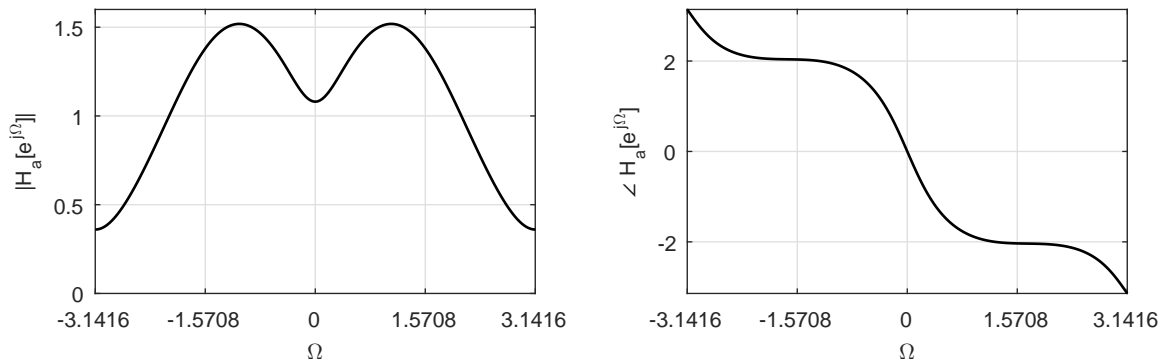

Figure S5.10-2a

(b) In this case,

$$H_b[z] = z^3 \sin(z^{-1}).$$

We use MATLAB to create the magnitude and phase response plots, as shown in Fig. S5.10-2a.

```
>>  Hb = @(z) z.^3.*sin(z.^(-1)); Omega = linspace(-pi,pi,1001);
>>  subplot(1,2,1); plot(Omega,abs(Hb(exp(1j*Omega))),'k-');
>>  xlabel('\Omega'); ylabel('|H_b[e^{j\Omega}]|'); grid on;
>>  axis([-pi pi 0 1.6]); set(gca,'xtick',-pi:pi/2:pi);
>>  subplot(1,2,2); plot(Omega,angle(Hb(exp(1j*Omega))),'k-');
>>  xlabel('\Omega'); ylabel('\angle H_b[e^{j\Omega}]'); grid on;
>>  axis([-pi pi -pi pi]); set(gca,'xtick',-pi:pi/2:pi);
```
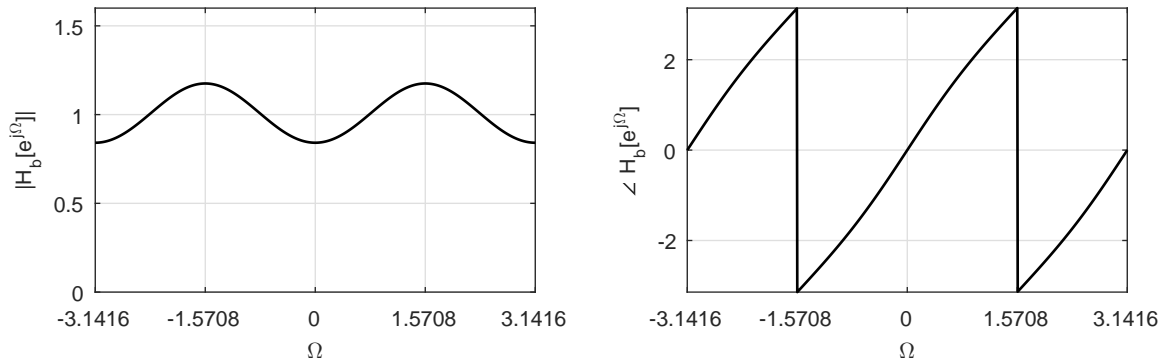

Figure S5.10-2a

Solution 5.10-3

Taking the z-transform of $4y[n+2] - y[n] = x[n+2] + x[n]$ yields $Y[z]\left(4z^2 - 1\right) = X[z]\left(z^2 + 1\right)$. Thus, the system function is

$$H[z] = \frac{Y[z]}{X[z]} = \frac{z^2 + 1}{4z^2 - 1} = 0.25\frac{1 + z^{-2}}{1 - z^{-2}/4}.$$

(a) By inspection of the transfer function, the system has

poles at $z = \frac{1}{2}$ and $z = -\frac{1}{2}$ and zeros at $z = j$ and $z = -j$.

We use MATLAB to create the pole-zero plot shown in Fig. S5.10-3a.

```
>>  zz = [1j,-1j]; pz = [1/2,-1/2]; ucirc = exp(1j*linspace(0,2*pi,201));
>>  plot(real(zz),imag(zz),'ko',real(pz),imag(pz),'kx',...
>>      real(ucirc),imag(ucirc),'k-'); grid on;
>>  xlabel('Re(z)'); ylabel('Im(z)'); axis equal; axis([-1.1 1.1 -1.1 1.1]);
```


Figure S5.10-3a

(b) Next, we use MATLAB to compute and plot the magnitude response, which is shown in Fig. S5.10-3b.

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) (z.^2+1)./(4*z.^2-1);
>>  plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 .8]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:1/3:1);
```
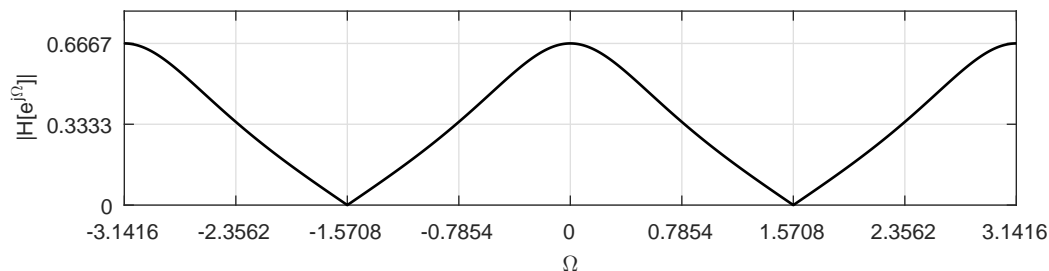

Figure S5.10-3b

(c) The pole-zero plot of Fig. S5.10-3a and the magnitude response plot of Fig. S5.10-3b confirm that this is a band-stop system.

(d) Yes, the system is asymptotically stable. Referring to Fig. S5.10-3a, all the system poles are within the unit circle.

(e) Yes, the system is real. Since the system is expressed as a constant-coefficient linear difference equation with real coefficients, the impulse response $h[n]$ and system are both real.

(f) For an input of the form $x[n] = \cos(\Omega n)$, the greatest possible amplitude of the output corresponds to the greatest gain shown in the magnitude response plot of Fig. S5.10-3b. Thus, $\frac{2}{3}$ is the greatest output amplitude given an input of $x[n] = \cos(\Omega n)$. This output amplitude occurs when $\Omega = k\pi$, for any integer k.

(g) Inverting $H[z] = \frac{Y[z]}{X[z]} = 0.25\frac{1+z^{-2}}{1-z^{-2}/4}$ provides $y[n]-0.25y[n-2] = 0.25x[n]+0.25x[n-2]$, which is a convenient form for implementation. Figure S5.10-3g illustrates a TDFII implementation of the system.
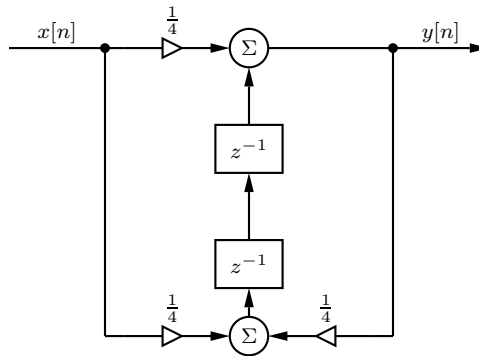
Figure S5.10-3g

Solution 5.10-4

In this problem, the system transfer function is

$$H(z) = b_0\frac{z^2+1}{z^2-4/9}.$$

(a) By direct substitution, we see that

$$b_0\frac{e^{-j2\pi}+1}{e^{-j2\pi}-\frac{4}{9}} = b_0\frac{2}{\frac{5}{9}} = -1.$$

Thus,

$$b_0 = -\frac{5}{18}.$$

(b) We use MATLAB to create the pole-zero plot shown in Fig. S5.10-4b.

```
>>  zz = roots([1 0 1]); pz = roots([1 0 -4/9]);
>>  ucirc = exp(1j*linspace(0,2*pi,201));
>>  plot(real(zz),imag(zz),'ko',real(pz),imag(pz),'kx',...
>>      real(ucirc),imag(ucirc),'k-'); grid on;
>>  xlabel('Re(z)'); ylabel('Im(z)');  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  set(gca,'xtick',-1:1/2:1,'ytick',-1:1/2:1);
```
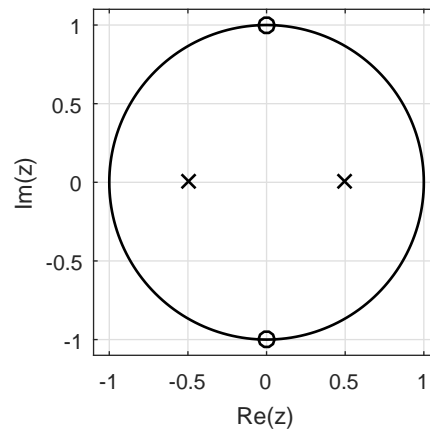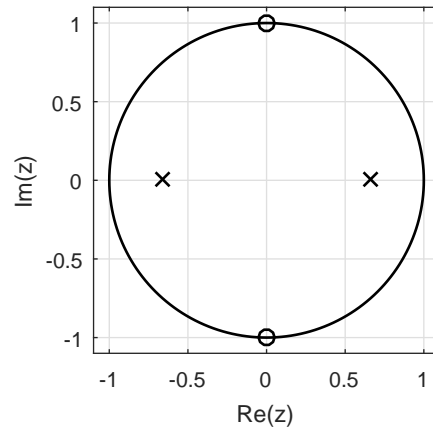
Figure S5.10-4b

(c), (d) Next, we use MATLAB to compute and plot the magnitude and phase responses, which are shown in Figs. S5.10-4c and S5.10-4d.

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) -5/18*(z.^2+1)./(z.^2-4/9);
>>  subplot(1,2,1); plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 1.1]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',0:1/4:1);
>>  subplot(1,2,2); plot(Omega,angle(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi -pi pi]); xlabel('\Omega'); ylabel('\angle H[e^{j\Omega}]');
>>  set(gca,'xtick',-pi:pi/2:pi,'ytick',-pi:pi/2:pi);
```
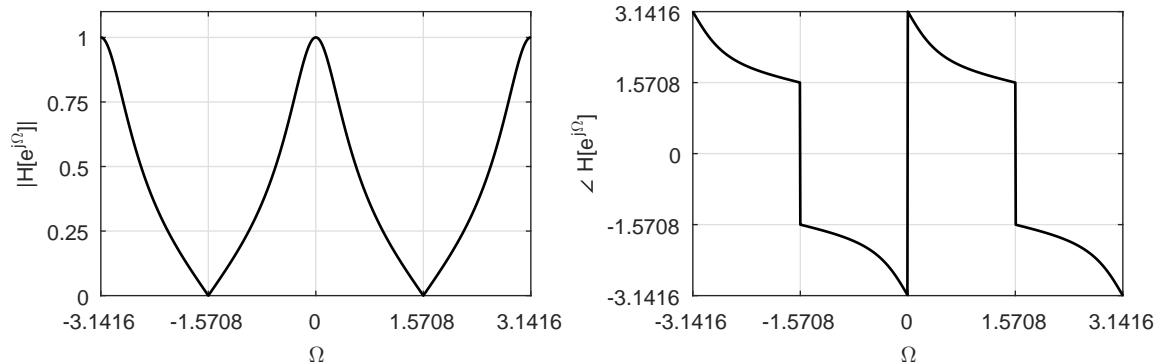

Figures S5.10-4c and S5.10-4d

(c) As shown in Fig. S5.10-4c, the system has bandstop (notch) character.

(d) Since $H[e^{j0}] = H[e^{j\pi}] = -1$ and $H[e^{j\pi/2}] = H[e^{-j\pi/2}] = 0$, the output in response to $x[n] = (-1-j) + (-j)^n + (1-j)\cos(\pi n + \frac{1}{3})$ is

$$y[n] = (1+j) + (j-1)\cos(\pi n + \tfrac{1}{3}).$$

(e) Figure S5.10-4g shows a TDFII realization of the system, which generally has the most desirable characteristics of the basic structures (DFI, DFII, TDFI, and TDFII).
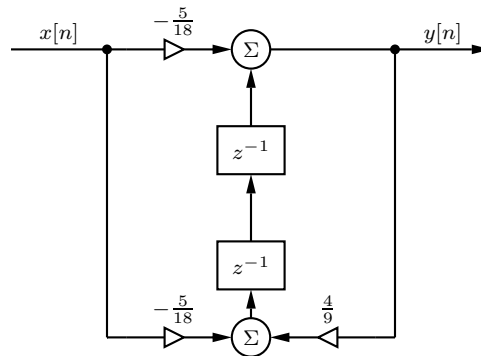
Figure S5.10-4g

Solution 5.10-5

(a) Calling the output of the summing node $v[n]$, we see that

$$v[n] = x[n-1] + c_1 x[n] + c_2 y[n] \quad \text{and} \quad y[n] = v[n-1].$$

Substituting the second expression into the first equation by delayed by 1 yields

$$y[n] = x[n-2] + c_1 x[n-1] + c_2 y[n-1] \quad \text{or} \quad y[n] - c_2 y[n-1] = x[n-2] + c_1 x[n-1].$$

Taking the z-transform yields

$$Y[z]\left(1 - c_2 z^{-1}\right) = X[z]\left(c_1 z^{-1} + z^{-2}\right).$$

Thus,

$$H[z] = \frac{Y[z]}{X[z]} = \frac{c_1 z^{-1} + z^{-2}}{1 - c_2 z^{-1}} = \frac{c_1 z + 1}{z^2 - c_2 z}.$$

(b) As $H[z]$ and the difference equation make clear,

$$\text{the system has order } N = 2.$$

(c) Based on inspection of $H[z]$, we see that the system has

$$\text{poles at } z = 0 \text{ and } z = c_2 \text{ and zeros at } z = -\frac{1}{c_1} \text{ and } z = \infty.$$

To generate the MATLAB pole-zero plot of Fig. S5.10-5c, let us set $c_1 = 1$ and $c_2 = 0.9$ [see part (e)].

```
>>  c1 = 1; c2 = 0.9; zz = roots([c1 1]); pz = roots([1 -c2 0]);
>>  ucirc = exp(1j*linspace(0,2*pi,201));
>>  plot(real(zz),imag(zz),'ko',real(pz),imag(pz),'kx',...
>>      real(ucirc),imag(ucirc),'k-'); grid on;
>>  xlabel('Re(z)'); ylabel('Im(z)'); axis equal;axis([-1.1 1.1 -1.1 1.1]);
>>  set(gca,'xtick',-1:1/2:1,'ytick',-1:1/2:1);
```

(d) For causal systems (such as this one), stability requires that the poles be inside the unit circle. Thus,

$$\text{stability requires } |c_2| < 1; \text{ no restrictions on } c_1.$$
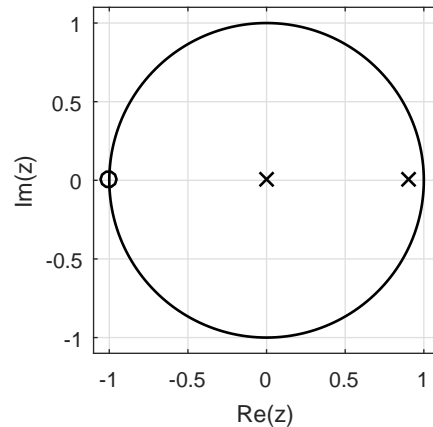
Figure S5.10-5c

(e) To suppress high frequencies, we put a zero at $z = -1 = -\frac{1}{c_1}$. To produce a relatively narrow passband, we desire the pole inside the unit circle near $z = 1$, such as $z = 0.9 = c_2$. Thus,

$$c_1 = 1 \text{ and } c_2 = 0.9 \text{ will produce a LPF with narrow passband.}$$

We use MATLAB to verify filter behavior (Fig. S5.10-5e).

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) (z+1)./(z.^2-0.9*z);
>>  plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 20]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:5:20);
```
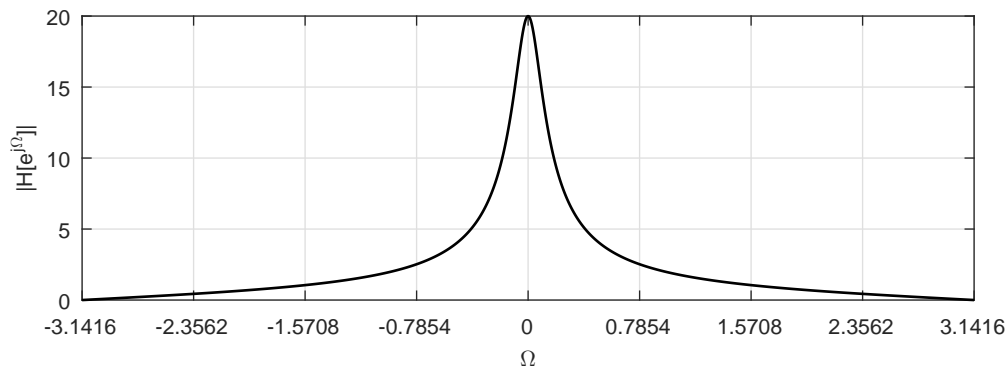

Figure S5.10-5e

Notice, as the LPF becomes more peaked/narrow (c_2 is closer to 1), the dc gain increases.

(f) No; Ms. Zeroine is not correct. Initial conditions in the system (non-zero values held in memory) can produce a nonzero output even though $x[n] = 0$.

(g) No; Dr. Strange is not correct. Although $c_1 = -1$ places a zero at $z = 1$ (dc), as would be expected for a HPF, the pole is outside the unit circle at $z = -2$, which makes the system unstable. An unstable system cannot properly act as any sort of filter.

Solution 5.10-6

(a) $H(z) = \frac{z^2-j}{z-0.9e^{j3\pi/4}} = \frac{(z-e^{j\pi/4})(z-e^{-j3\pi/4})}{z-0.9e^{j3\pi/4}}$. As shown in Fig S5.10-6a, there are zeros at $z = e^{j\pi/4}$ and $z = e^{-j3\pi/4}$, and there are poles at $z = 0.9e^{j3\pi/4}$ and $z = \infty$.

```
>>  zz = roots([1 0 -1j]); pz = roots([1 -0.9*exp(3j*pi/4)]);
>>  ucirc = exp(1j*linspace(0,2*pi,201));
>>  plot(real(zz),imag(zz),'ko',real(pz),imag(pz),'kx',...
>>      real(ucirc),imag(ucirc),'k-'); grid on;
>>  xlabel('Re(z)'); ylabel('Im(z)'); axis equal;axis([-1.1 1.1 -1.1 1.1]);
>>  set(gca,'xtick',-1:1/2:1,'ytick',-1:1/2:1);
```
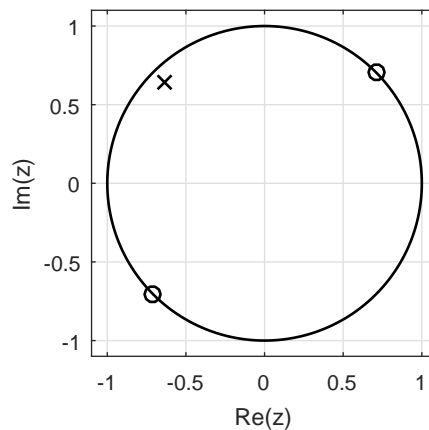

Figure S5.10-6a

Since this is a complex system, poles and zeros need not occur in complex conjugate pairs.

(b) Substituting $z = e^{j\Omega}$ into $H(z) = \frac{z^2-j}{z-0.9e^{j3\pi/4}}$, MATLAB is used to create the magnitude response plot.

```
>>  H = @(z) (z.^2-1j)./(z-0.9*exp(3j*pi/4));
>>  Omega = linspace(-2*pi,2*pi,1001);
>>  plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-2*pi 2*pi 0 22]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-2*pi:pi/2:2*pi,'ytick',0:5:20);
```


Figure S5.10-6b

From Fig. S5.10-6b, the system appears to be a type of bandpass filter. This particular filter tends to pass only positive frequency inputs near $\Omega = 3\pi/4$; the corresponding negative frequencies near $\Omega = -3\pi/4$ are significantly attenuated. Since the magnitude response is not an even function of Ω, the output of the filter will be complex-valued.

Solution 5.10-7

(a) $H(z) = \frac{z^4-1}{2(z^2+0.81j)}$ has four finite zeros and two finite poles. The zeros are the four roots of unity, and the poles are at $z = 0.9e^{j\pi/4}$ and $z = 0.9e^{-j3\pi/4}$. MATLAB is used to compute the pole-zero plot.

```
>>  zz = roots([1 0 0 0 -1]); pz = roots([1 0 0.81j]);
>>  ucirc = exp(1j*linspace(0,2*pi,201));
>>  plot(real(zz),imag(zz),'ko',real(pz),imag(pz),'kx',...
>>      real(ucirc),imag(ucirc),'k-'); grid on;
>>  xlabel('Re(z)'); ylabel('Im(z)'); axis equal;axis([-1.1 1.1 -1.1 1.1]);
>>  set(gca,'xtick',-1:1/2:1,'ytick',-1:1/2:1);
```
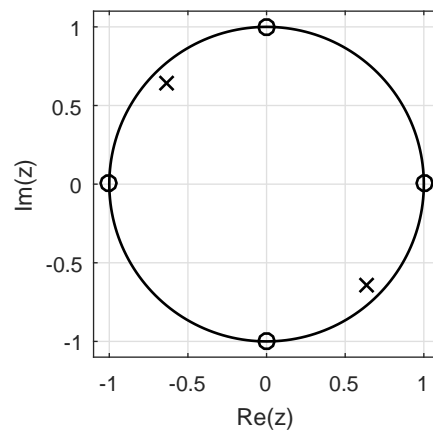

Figure S5.10-7a

(b) MATLAB is used to plot the magnitude response.

```
>>  H = @(z) (z.^4-1)./(2*(z.^2+0.81j));
>>  Omega = linspace(-pi,pi,1001);
>>  plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 6]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:6);
```

(c) Since $H(z)$ is an improper rational function and has poles at infinity, the system is non-causal. By dividing the numerator by the denominator (in such a way as to yield a right-sided impulse response), an improper rational function $H(z)$ will yield an impulse response $h[n]$ that is not zero for all negative n. Right-sided functions with poles at infinity are sometimes called "not-quite causal" since they are not causal only by a finite shift.

(d) Adding two poles at zero ($a = b = 0$) corresponds to a simple right-shift by two and does not affect the magnitude response. That is, $|H_{\text{causal}}(e^{j\Omega})| = \left|\frac{H(e^{j\Omega})}{(e^{j\Omega})^2}\right| = \frac{|H(e^{j\Omega})|}{|(e^{j\Omega})^2|} = |H(e^{j\Omega})|$. By adding two poles at zero to $H(z)$, $H_{\text{causal}}(z)$ becomes a proper rational function with only finite poles, and the corresponding impulse response function becomes causal.
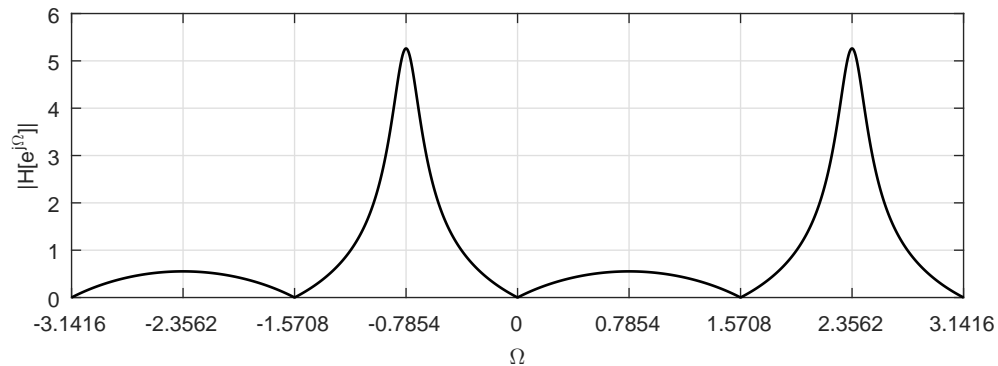
Figure S5.10-7b

(e) Noting that $H_{\text{causal}}(z) = \frac{Y(z)}{X(z)} = \frac{0.5z^4 - 0.5}{z^4 + 0.81jz^2} = \frac{0.5 - 0.5z^{-4}}{1 + 0.81jz^{-2}}$, the corresponding difference equation is $y[n] + 0.81jy[n-2] = 0.5x[n] - 0.5x[n-4]$. Figure S5.10-7e shows the corresponding TDFII implementation.
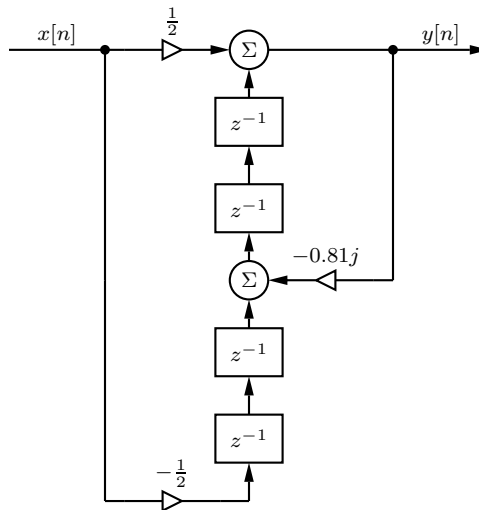
Figure S5.10-7e

Solution 5.10-8

(a) From the block diagram, the corresponding difference equation is written directly.

$$y[n] - 0.5y[n-2] = x[n].$$

(b) Taking the z-transform of $y[n] - 0.5y[n-2] = x[n]$ yields $Y[z](1 - 0.5z^{-2}) = X[z]$. Thus,

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1 - 0.5z^{-2}} = \frac{z^2}{z^2 - 0.5}.$$

MATLAB is used to plot the magnitude response.

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) (z.^2)./(z.^2-1/2);
>>  plot(Omega,abs(H(exp(1j*Omega)))); grid on;
>>  axis([-pi pi 0 2.25]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:1/3:2);
```

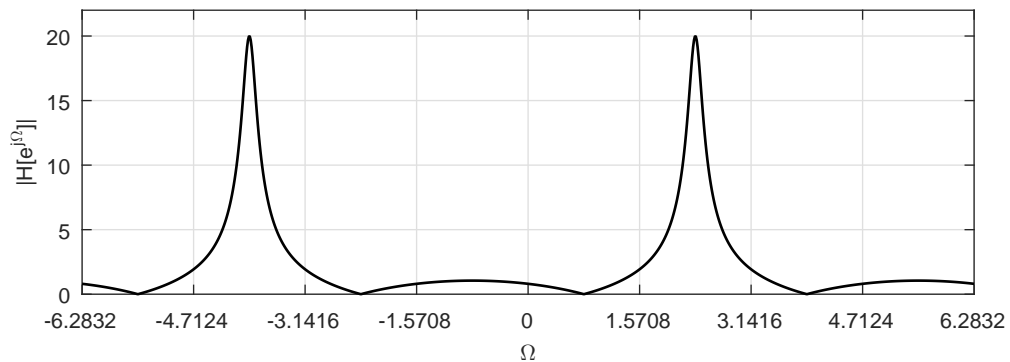Figure S5.10-8b

Standard filter types do not provide a good description of this filter. The system appears most like a bandstop filter, but its stopband attenuation is quite poor. The system boosts the gain of low and high frequencies more than it attenuates the middle frequencies.

(c) Inverting $H[z] = \frac{1}{1-0.5z^{-2}} = \frac{1/2}{1-z^{-1}/\sqrt{2}} + \frac{1/2}{1+z^{-1}/\sqrt{2}}$ yields

$$h[n] = 0.5\left((1/\sqrt{2})^n + (-1/\sqrt{2})^n\right)u[n].$$

Solution 5.10-9

Label the first summation block output as $v[n]$. Thus, $v[n] = x[n-1] + \frac{1}{4}y[n]$ or $V(z) = z^{-1}X(z) + \frac{1}{4}Y(z)$. The output of the second summation block is $y[n+1] = x[n] + v[n-1]$ or $zY(z) = X(z) + z^{-1}V(z)$. Combining the two equations yields $zY(z) = X(z) + z^{-1}\left(z^{-1}X(z) + \frac{1}{4}Y(z)\right)$ or $Y(z)\left(z - \frac{1}{4}z^{-1}\right) = X(z)\left(1 + z^{-2}\right)$. Multiplying both sides by z^{-1} yields $Y(z)\left(1 - \frac{1}{4}z^{-2}\right) = X(z)\left(z^{-1} + z^{-3}\right)$. Thus, $H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}+z^{-3}}{1-\frac{1}{4}z^{-2}} = \frac{1+z^2}{z(z-1/2)(z+1/2)} = \frac{-4}{z} + \frac{5/2}{z-1/2} + \frac{5/2}{z+1/2} = \frac{-4}{z} + z^{-1}\frac{5z/2}{z-1/2} + z^{-1}\frac{5z/2}{z+1/2}$. Taking the inverse transform yields

$$h[n] = -4\delta[n-1] + \frac{5}{2}(1/2)^{n-1}u[n-1] + \frac{5}{2}(-1/2)^{n-1}u[n-1].$$

Since $h[n] = 0$ for $n < 0$, the system is causal. Further, the system has three poles located at $z = 0$, $z = 1/2$, and $z = -1/2$. Since all three poles are inside the unit circle, the system is stable.

Solution 5.10-10

Since $h[n] = \delta[n-1] + \delta[n+1]$, $H(z) = z^{-1} + z$ and $|H(e^{j\Omega})| = 2\left|\frac{e^{-j\Omega}+e^{j\Omega}}{2}\right| = 2|\cos(\Omega)|$.

```
>>  Omega = linspace(-pi,pi,1001); Hm = 2*abs(cos(Omega));
>>  plot(Omega,Hm,'k-'); grid on; axis([-pi pi 0 2.25]);
>>  xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:1/2:2);
```

Using Fig. S5.10-10, this system is best described as a bandstop filter with gain. That is, low and high frequencies have a boosted gain of two and middle frequencies near $\Omega = \pi/2$ are attenuated.
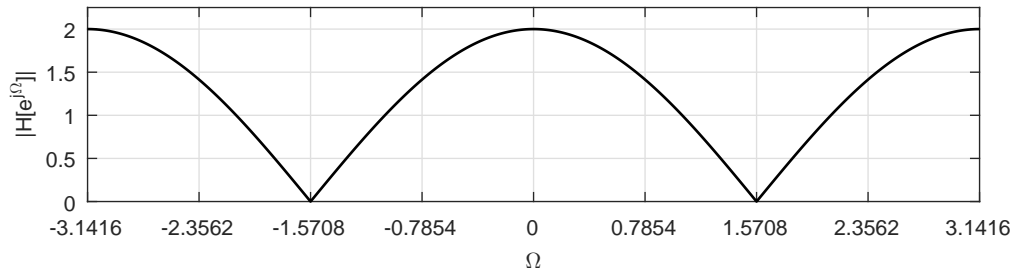
Figure S5.10-10

Solution 5.10-11

(a) MATLAB is used to plot the magnitude response, shown in Fig. S5.10-11a. This system be-haves somewhat like a bandpass filter. Frequencies near $\Omega = \pi/2$ are boosted while frequencies near $\Omega = k\pi$ are somewhat attenuated.

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) cos(z.^(-1));
>>  plot(Omega,abs(H(exp(1j*Omega))),'k-'); grid on;
>>  axis([-pi pi 0 1.7]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:1/4:1.5);
```
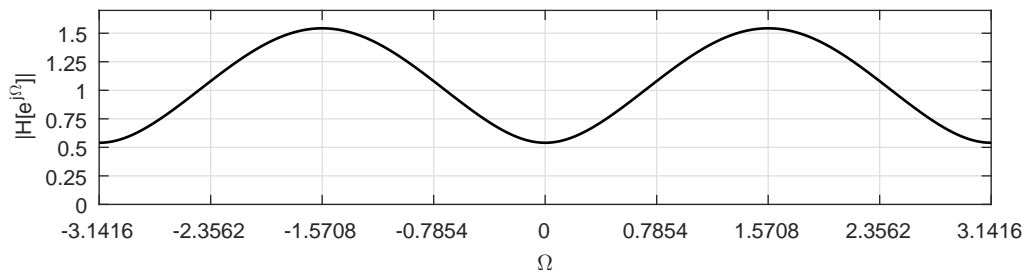

Figure S5.10-11a

(b) A Maclaurin series expansion of $\cos(x)$ is $\sum_{k=0}^{\infty} \frac{(-x^2)^k}{(2k)!}$. Substituting $x = z^{-1}$ yields $H(z) = \cos(z^{-1}) = \sum_{k=0}^{\infty} \frac{(-z^{-2})^k}{(2k)!}$. Inverting yields

$$h[n] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \delta[n - 2k] = \delta[n] - \frac{1}{2!}\delta[n-2] + \frac{1}{4!}\delta[n-4] - \frac{1}{6!}\delta[n-6] + \ldots.$$

We use MATLAB to plot $h[n]$ (Fig. S5.10-11b).

```
>>  delta = @(n) 1.0.*(n==0); K = 10; n = [0:10]; h = @(n) 0;
>>  for k = 0:K, h = @(n) h(n) + (-1).^k/gamma(2*k+1).*delta(n-2*k); end
>>  stem(n,h(n),'k.'); xlabel('n'); ylabel('h[n]');
>>  axis([-.5 10.5 -.6 1.1]); grid on;
```

(c) Figure S5.10-11b shows that the impulse response quickly decays to zero. Thus, only the first few terms from $h[n]$ are needed for a good approximation. Note that $h[n] \approx \delta[n] - \delta[n-2]/2 + \delta[n-4]/24$. An FIR difference equation is found by letting $h[n] = y[n]$ and $\delta[n] = x[n]$. That is,

$$y[n] = x[n] - x[n-2]/2 + x[n-4]/24.$$

This is a fourth-order FIR filter with only three non-zero coefficients. The magnitude response is easily computed using MATLAB.
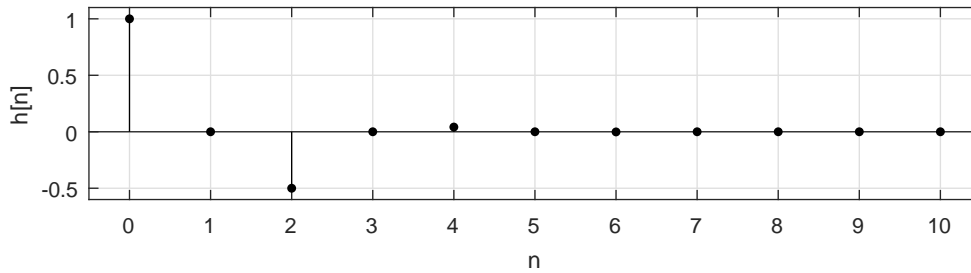
Figure S5.10-11b

```
>>  Omega = linspace(-pi,pi,1001); H = @(z) 1-z.^(-2)/2+z.^(-4)/24;
>>  plot(Omega,abs(H(exp(1j*Omega))),'k-'); grid on;
>>  axis([-pi pi 0 1.7]); xlabel('\Omega'); ylabel('|H[e^{j\Omega}]|');
>>  set(gca,'xtick',-pi:pi/4:pi,'ytick',0:1/4:1.5);
```
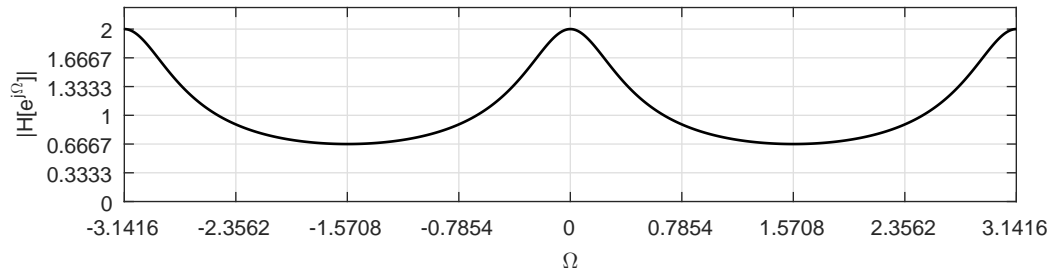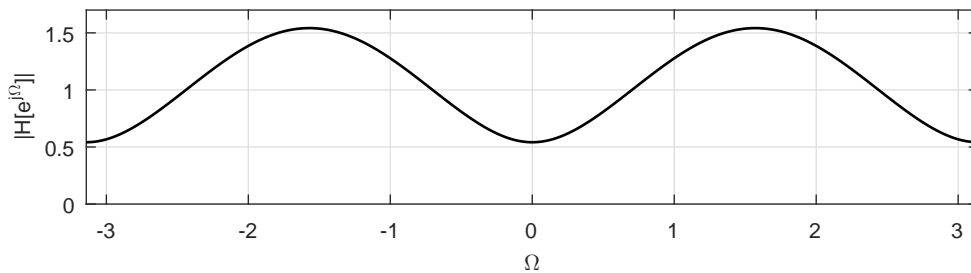

Figure S5.10-11c

Visually, Fig. S5.10-11c is indistinguishable from Fig. S5.10-11b. Thus, this fourth-order FIR filter closely approximates the original system.

Solution 5.10-12

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Butterworth LPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = butter(8,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>     real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = butter(8,Omega_c/pi);
>>  HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{LP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>     {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```

(b) Order-8 Butterworth HPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = butter(8,Omega_c/pi,'high');
```

Figure S5.10-12a

```
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = butter(8,Omega_c/pi,'high');
>>  HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{HP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```


Figure S5.10-12b

(c) Order-8 Butterworth BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command `butter` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.

```
>>  Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = butter(4,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis([-1.1 1.1 -1.1 1.1]); axis equal;
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = butter(4,Omega_c/pi);
>>  HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>>  axis([-pi pi -40 2]); grid;
```

```
>> xlabel('\Omega'); ylabel('|H_{BP}[e^{j\Omega}]|');
>> set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>      'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```


Figure S5.10-12c

(d) Order-8 Butterworth BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command `butter` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = butter(4,Omega_c/pi,'stop');
>> subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = butter(4,Omega_c/pi,'stop');
>> HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BS}[e^{j\Omega}]|');
>> set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>      'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```


Figure S5.10-12d

Solution 5.10-13

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Chebyshev Type I LPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = cheby1(8,3,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = cheby1(8,3,Omega_c/pi);
>>  HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{LP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```


Figure S5.10-13a

(b) Order-8 Chebyshev Type I HPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = cheby1(8,3,Omega_c/pi,'high');
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = cheby1(8,3,Omega_c/pi,'high');
>>  HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{HP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```

(c) Order-8 Chebyshev Type I BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command **cheby1** requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.

Figure S5.10-13b

```
>>   Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>>   [z,p,k] = cheby1(4,3,Omega_c/pi);
>>   subplot(121),plot(real(p),imag(p),'kx',...
>>       real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>   axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>   xlabel('Re(z)'); ylabel('Im(z)');
>>   [B,A] = cheby1(4,3,Omega_c/pi);
>>   HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>   subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>>   axis([-pi pi -40 2]); grid;
>>   xlabel('\Omega'); ylabel('|H_{BP}[e^{j\Omega}]|');
>>   set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>       'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```
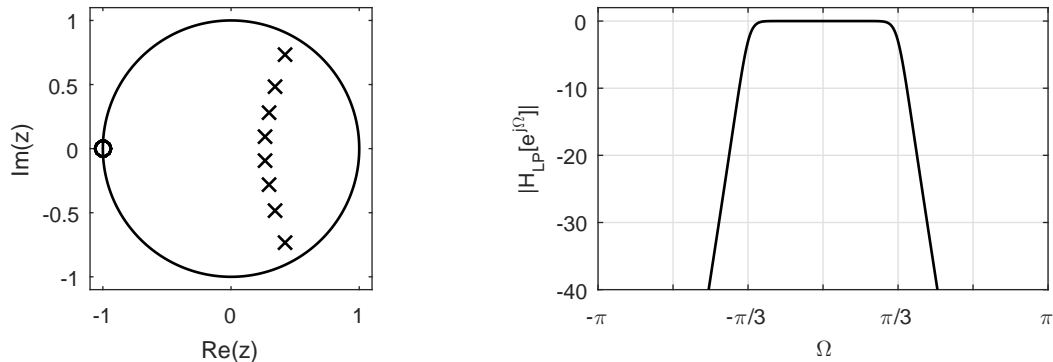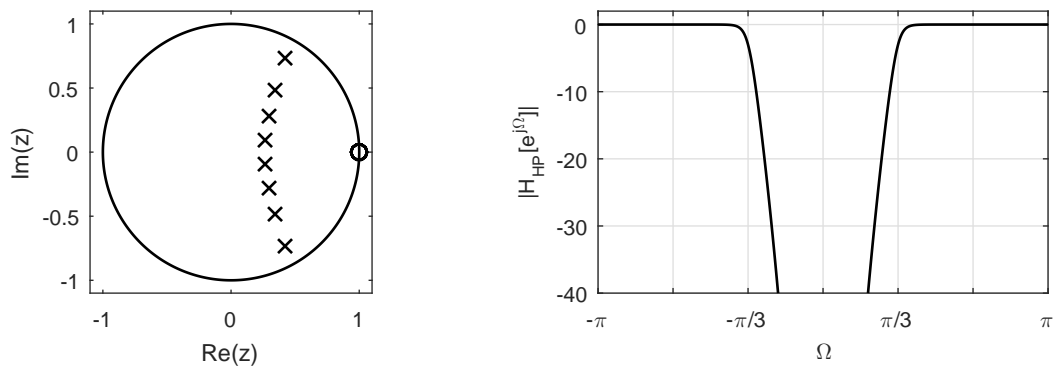

Figure S5.10-13c

(d) Order-8 Chebyshev Type I BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command `cheby1` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.

```
>>   Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>>   [z,p,k] = cheby1(4,3,Omega_c/pi,'stop');
>>   subplot(121),plot(real(p),imag(p),'kx',...
>>       real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>   axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>   xlabel('Re(z)'); ylabel('Im(z)');
>>   [B,A] = cheby1(4,3,Omega_c/pi,'stop');
>>   HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
```

```
>>   subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>>   axis([-pi pi -40 2]); grid;
>>   xlabel('\Omega'); ylabel('|H_{BS}[e^{j\Omega}]|');
>>   set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>       'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```
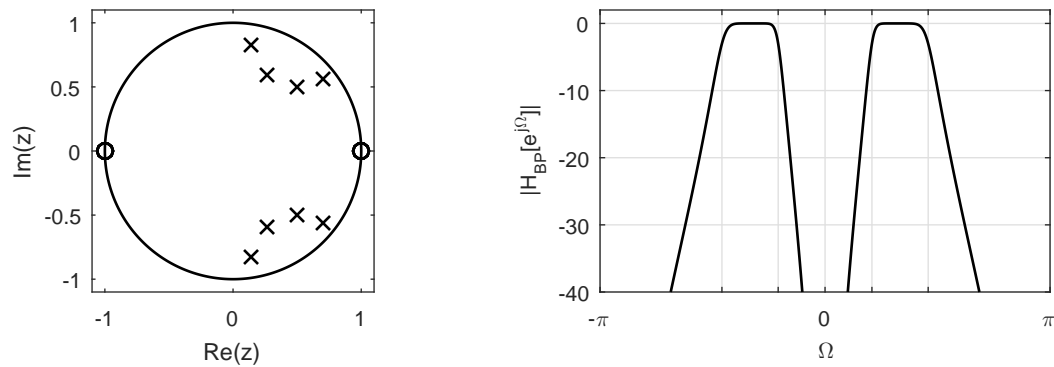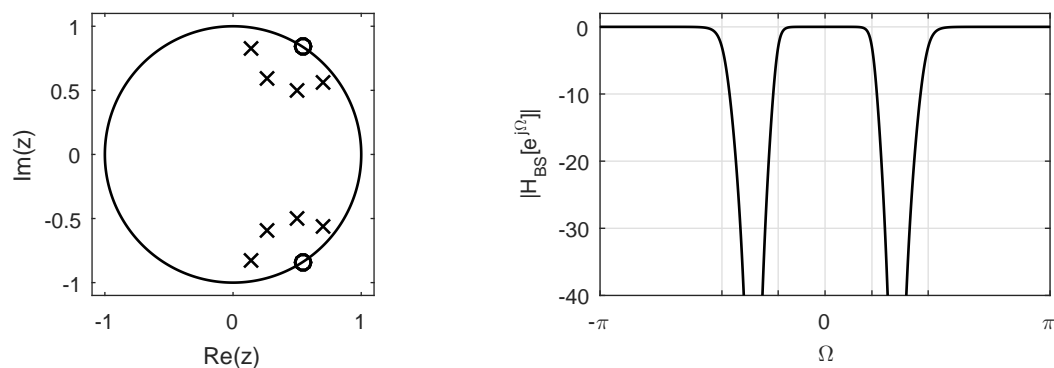

Figure S5.10-13d

To demonstrate the effect of decreasing the passband ripple, consider magnitude response plots for Chebyshev Type I LPFs with $R_p = \{0.1, 1.0, 3.0\}$.

```
>>   LSS3eSMMATLABFigFormat(7,2,6);
>>   Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>   [B,A] = cheby1(8,.1,Omega_c/pi);
>>   HLP1 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>   [B,A] = cheby1(8,1,Omega_c/pi);
>>   HLP2 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>   [B,A] = cheby1(8,3,Omega_c/pi);
>>   HLP3 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>   plot(Omega,20*log10(abs(HLP1)),'k-',...
>>           Omega,20*log10(abs(HLP2)),'k--',...
>>           Omega,20*log10(abs(HLP3)),'k:');
>>   axis([-pi pi -15 2]); grid;
>>   xlabel('\Omega'); ylabel('|H_{LP}[e^{j\Omega}]|');
>>   legend('R_p = 0.1','R_p = 1.0','R_p = 3.0');
>>   set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>       {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```


Figure S5.10-13e

Thus, reducing the allowable passband ripple R_p tends to broaden the transition bands of the filter.

Solution 5.10-14

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Chebyshev Type II LPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = cheby2(8,20,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = cheby2(8,20,Omega_c/pi);
>>  HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{LP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```
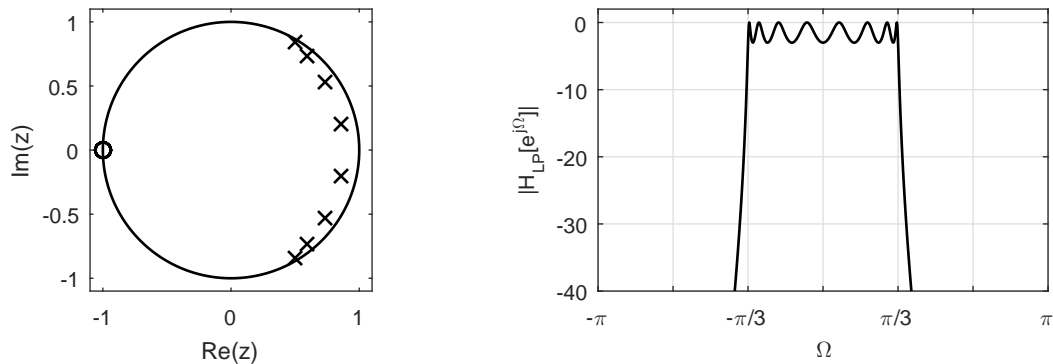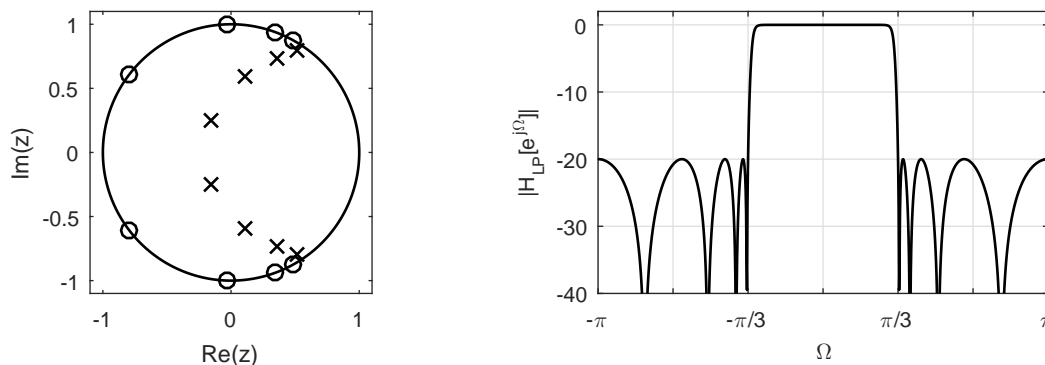

Figure S5.10-14a

(b) Order-8 Chebyshev Type II HPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = cheby2(8,20,Omega_c/pi,'high');
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = cheby2(8,20,Omega_c/pi,'high');
>>  HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{HP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```

(c) Order-8 Chebyshev Type II BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command `cheby2` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.
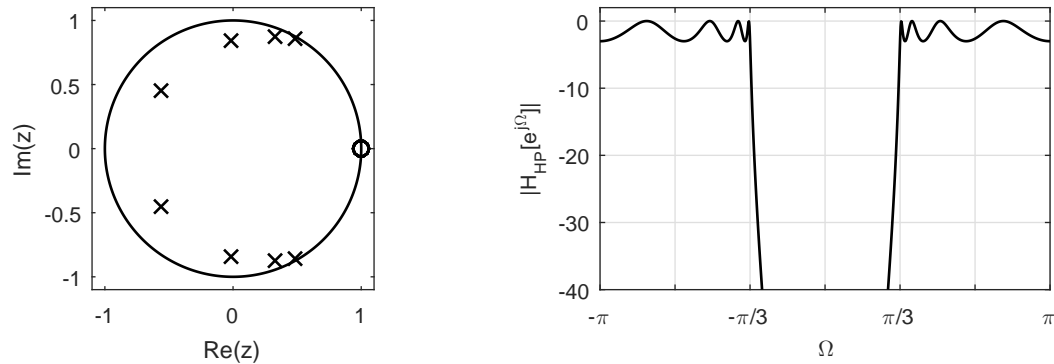
Figure S5.10-14b

```
>>  Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = cheby2(4,20,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = cheby2(4,20,Omega_c/pi);
>>  HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{BP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>      'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```
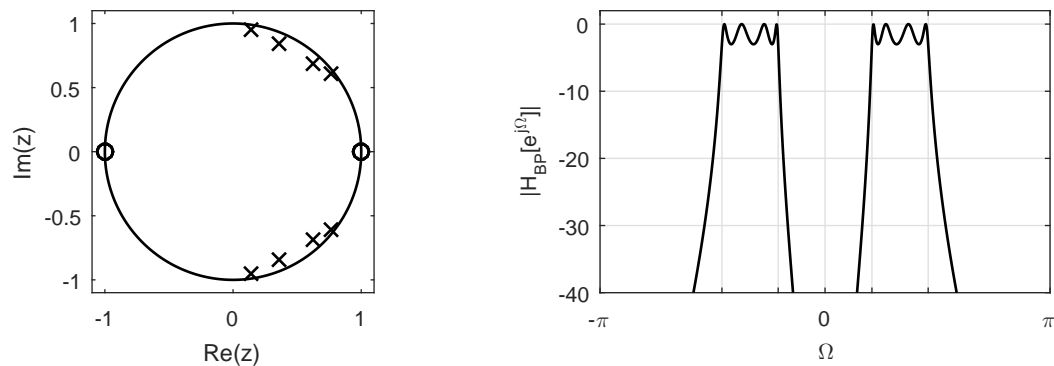

Figure S5.10-14c

(d) Order-8 Chebyshev Type II BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command `cheby2` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.

```
>>  Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = cheby2(4,20,Omega_c/pi,'stop');
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = cheby2(4,20,Omega_c/pi,'stop');
>>  HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
```

```
>>  subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{BS}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>      'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```
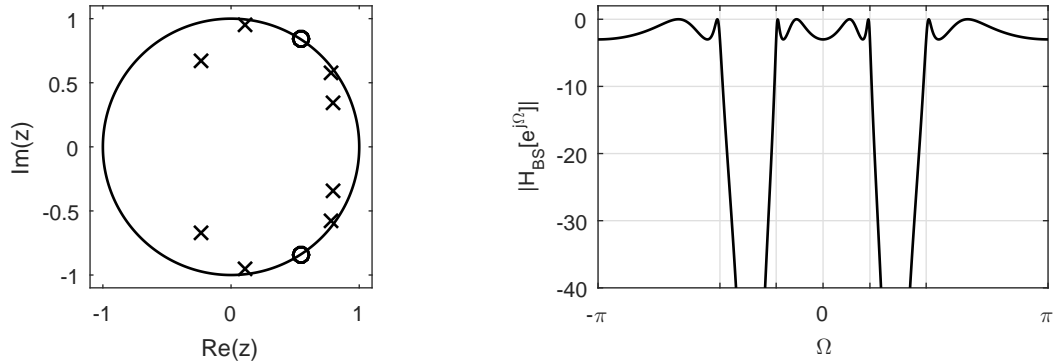

Figure S5.10-14d

To demonstrate the effect of increasing R_s, consider magnitude response plots for Chebyshev Type II LPFs with $R_s = \{10, 20, 30\}$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [B,A] = cheby2(8,10,Omega_c/pi);
>>  HLP1 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  [B,A] = cheby2(8,20,Omega_c/pi);
>>  HLP2 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  [B,A] = cheby2(8,30,Omega_c/pi);
>>  HLP3 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  plot(Omega,20*log10(abs(HLP1)),'k-',...
>>          Omega,20*log10(abs(HLP2)),'k--',...
>>          Omega,20*log10(abs(HLP3)),'k:');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{LP}[e^{j\Omega}]|');
>>  legend('R_s = 10','R_s = 20','R_s = 30');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```
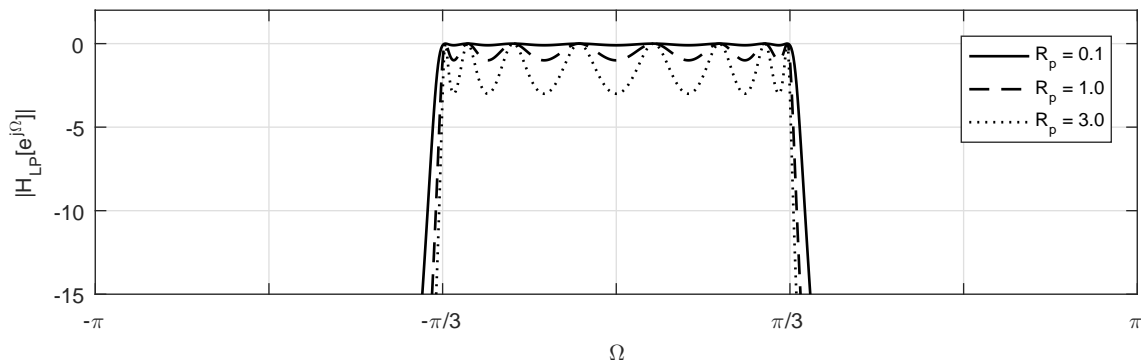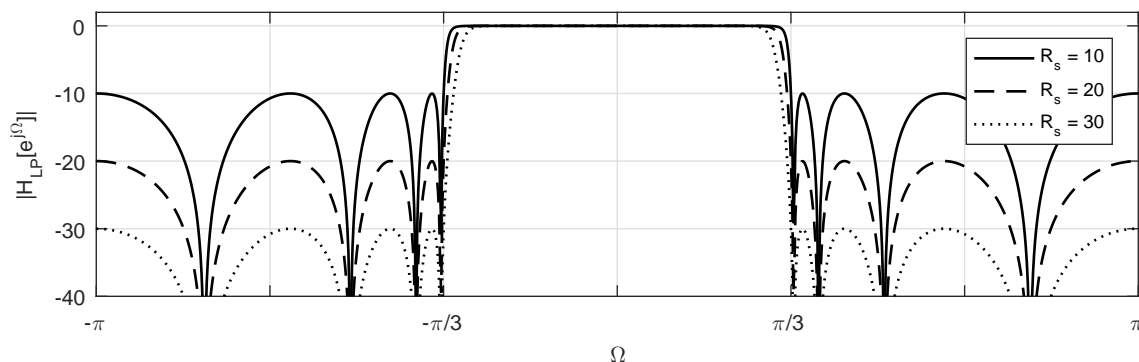

Figure S5.10-14e

Thus, increasing R_s tends to broaden the transition bands of the filter.

Solution 5.10-15

Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Elliptic LPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = ellip(8,3,20,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = ellip(8,3,20,Omega_c/pi);
>>  HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{LP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```
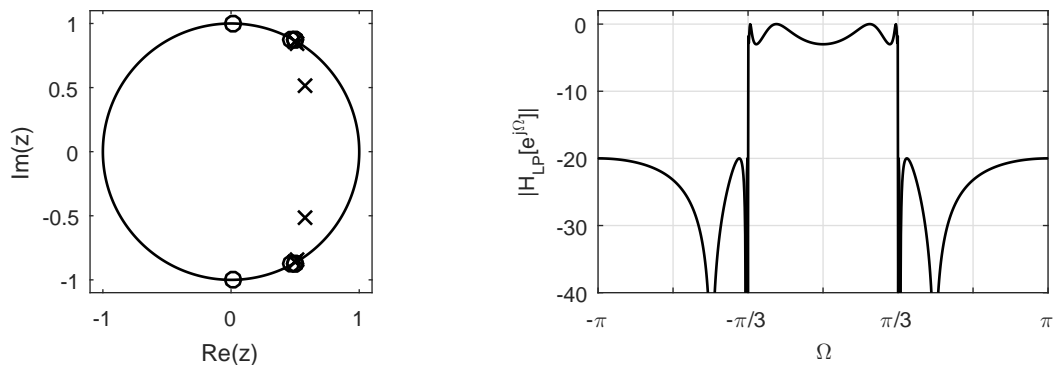

Figure S5.10-15a

(b) Order-8 Elliptic HPF with $\Omega_c = \pi/3$.

```
>>  Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = ellip(8,3,20,Omega_c/pi,'high');
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = ellip(8,3,20,Omega_c/pi,'high');
>>  HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{HP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',...
>>      {'-\pi',' ','-\pi/3',' ','\pi/3',' ','\pi'})
```

(c) Order-8 Elliptic BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command `ellip` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.

```
>>  Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
```
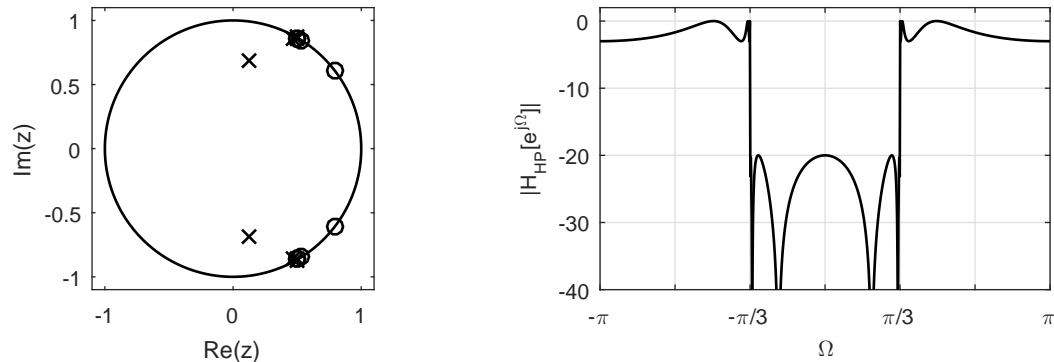
Figure S5.10-15b

```
>>  [z,p,k] = ellip(4,3,20,Omega_c/pi);
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = ellip(4,3,20,Omega_c/pi);
>>  HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>>  axis([-pi pi -40 2]); grid;
>>  xlabel('\Omega'); ylabel('|H_{BP}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>      'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```
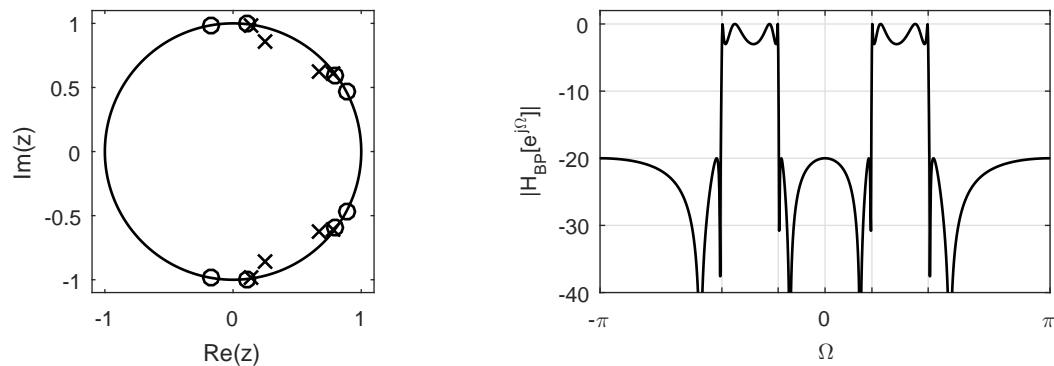

Figure S5.10-15c

(d) Order-8 Elliptic BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command `ellip` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.
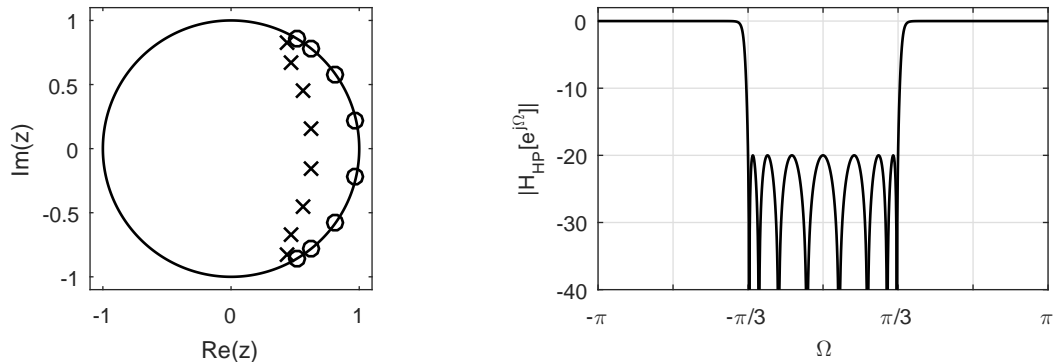
```
>>  Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>>  [z,p,k] = ellip(4,3,20,Omega_c/pi,'stop');
>>  subplot(121),plot(real(p),imag(p),'kx',...
>>      real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>>  axis equal; axis([-1.1 1.1 -1.1 1.1]);
>>  xlabel('Re(z)'); ylabel('Im(z)');
>>  [B,A] = ellip(4,3,20,Omega_c/pi,'stop');
>>  HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>>  subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>>  axis([-pi pi -40 2]); grid;
```

```
>>  xlabel('\Omega'); ylabel('|H_{BS}[e^{j\Omega}]|');
>>  set(gca,'xtick',[-pi,-11*pi/24,-5*pi/24,0,5*pi/24,11*pi/24,pi],...
>>     'xticklabel',{'-\pi',' ',' ','0',' ',' ','\pi'})
```
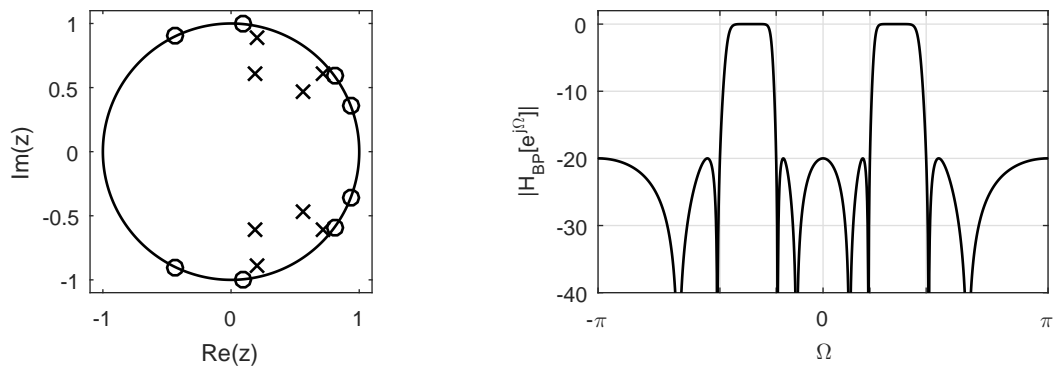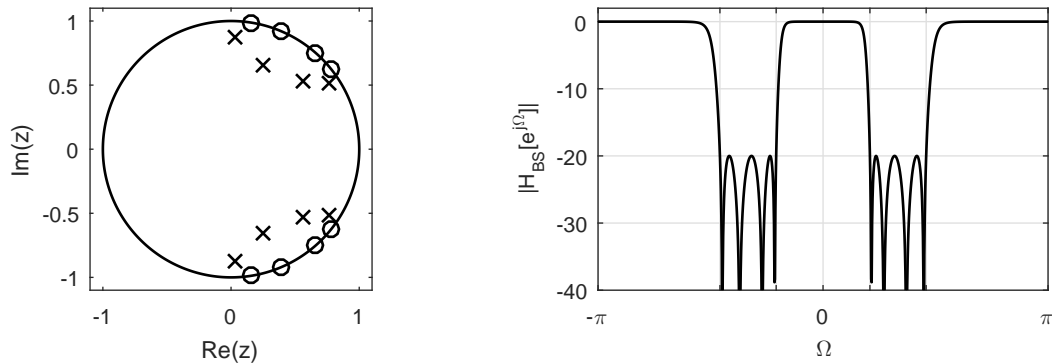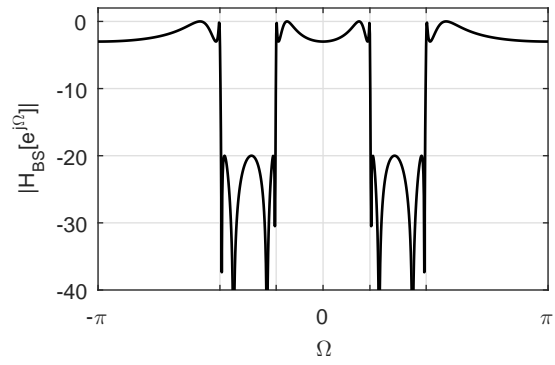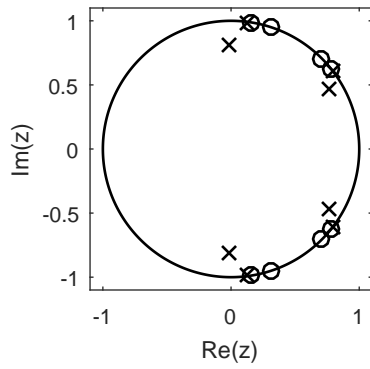

Figure S5.10-15d

Chapter 6 Solutions

In the following problems when $C_n = 0$, MATLAB phase computations may not be meaningful and can rightfully be ignored; see, for example, part (a) and the phase computed at $n = 4$. While not done here, phase plots often ignore computed phase and use 0 when $C_n = 0$ (or is nearly 0).

(a) $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Because of even symmetry, all sine terms are zero.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right)$$

$$a_0 = 0 \text{ (by inspection)}$$

$$a_n = \frac{4}{4}\left[\int_0^1 \cos\left(\frac{n\pi}{2}t\right)\,dt - \int_1^2 \cos\left(\frac{n\pi}{2}t\right)\,dt\right] = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

Therefore, the Fourier series for $x(t)$ is

$$x(t) = \frac{4}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \frac{1}{7}\cos\frac{7\pi t}{2} + \cdots\right).$$

Since $b_n = 0$, $C_n = |a_n|$ and $\theta_n = \angle a_n$, both shown in Fig. S6.1-1a. The corresponding frequency ω is easily computed as $\omega_0 n = \frac{\pi n}{2}$.

```
>> n = 1:15; an = 4./(pi*n).*sin(pi*n/2); bn = zeros(size(n));
>> Cn = [0,sqrt(an.^2+bn.^2)]; thetan = [atan(0),atan2(-bn,an)];n = [0,n];
>> subplot(121); stem(n,Cn,'k.'); xlabel('n'); ylabel('|C_n|');
>> axis([-.5 15.5 0 1.4]); grid on
>> subplot(122); stem(n,thetan,'k.'); xlabel('n'); ylabel('\theta_n');
>> axis([-.5 15.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```


Figure S6.1-1a

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(b) Here, $T_0 = 10\pi$, $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$. Because of even symmetry, all the sine terms are zero.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right)$$

$$a_0 = \frac{1}{5} \qquad \text{(by inspection)}$$

$$a_n = \frac{2}{10\pi} \int_{-\pi}^{\pi} \cos\left(\frac{n}{5}t\right) dt = \frac{1}{5\pi}(\frac{5}{n}) \sin\left(\frac{n}{5}t\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi n} \sin\left(\frac{n\pi}{5}\right)$$

$$b_n = \frac{2}{10\pi} \int_{-\pi}^{\pi} \sin\left(\frac{n}{5}t\right) dt = 0 \qquad \text{(integrand is an odd function of } t\text{)}$$

Since $b_n = 0$, $C_n = |a_n|$ and $\theta_n = \angle a_n$, both shown in Fig. S6.1-1b. The corresponding frequency ω is easily computed as $\omega_0 n = \frac{n}{5}$.

```
>>  n = 1:15; an = 2./(pi*n).*sin(pi*n/5); bn = zeros(size(n));
>>  Cn = [1/5,sqrt(an.^2+bn.^2)]; thetan = [atan(0),atan2(-bn,an)];n = [0,n];
>>  subplot(121); stem(n,Cn,'k.'); xlabel('n'); ylabel('|C_n|');
>>  axis([-.5 15.5 0 0.5]); grid on
>>  subplot(122); stem(n,thetan,'k.'); xlabel('n'); ylabel('\theta_n');
>>  axis([-.5 15.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```


Figure S6.1-1b

(c) In this case, $T_0 = 2\pi$, $\omega_0 = 1$. Thus,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{with} \quad a_0 = 0.5 \quad \text{(by inspection)},$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{t}{2\pi} \cos nt \, dt = 0, \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{t}{2\pi} \sin nt \, dt = -\frac{1}{\pi n}$$

and

$$x(t) = 0.5 - \frac{1}{\pi}\left(\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t + \cdots\right)$$

$$= 0.5 + \frac{1}{\pi}\left[\cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2t + \frac{\pi}{2}\right) + \frac{1}{3}\cos\left(3t + \frac{\pi}{2}\right) + \cdots\right].$$

The reason the cosine terms vanish is that when 0.5 (the dc component) is subtracted from $x(t)$, the remaining function has odd symmetry. Hence, the Fourier series contains dc and sine terms only. Since $a_n = 0$, $C_n = |b_n|$ and $\theta_n = \angle b_n$, both shown in Fig. S6.1-1c. The corresponding frequency ω is easily computed as $\omega_0 n = n$.

```
>>  n = 1:15; bn = -1./(pi*n); an = zeros(size(n));
>>  Cn = [1/2,sqrt(an.^2+bn.^2)]; thetan = [atan(0),atan2(-bn,an)];n = [0,n];
>>  subplot(121); stem(n,Cn,'k.'); xlabel('n'); ylabel('|C_n|');
>>  axis([-.5 15.5 0 0.6]); grid on
>>  subplot(122); stem(n,thetan,'k.'); xlabel('n'); ylabel('\theta_n');
>>  axis([-.5 15.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```
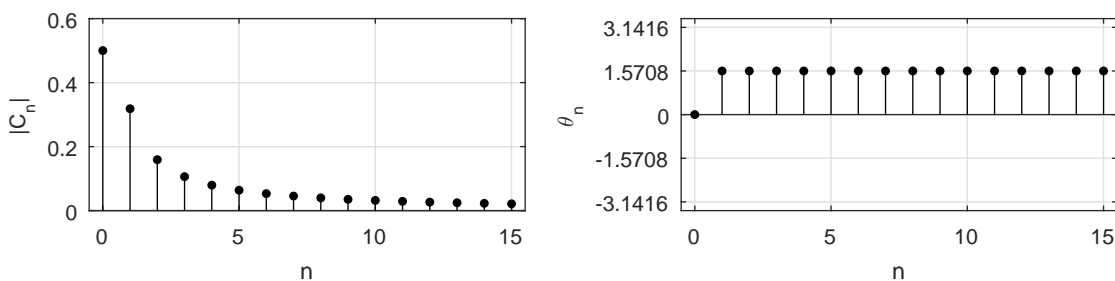

Figure S6.1-1c

(d) Now, $T_0 = \pi$, $\omega_0 = 2$ and $x(t) = \frac{4}{\pi}t$. $a_0 = 0$　　(by inspection). Thus,

$$a_n = 0 \quad (n > 0) \quad \text{because of odd symmetry,}$$

$$b_n = \frac{4}{\pi} \int_0^{\pi/4} \frac{4}{\pi}t \sin 2nt \, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right),$$

and

$$x(t) = \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots$$

$$= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdots.$$

Since $a_n = 0$, $C_n = |b_n|$ and $\theta_n = \angle b_n$, both shown in Fig. S6.1-1d. The corresponding frequency ω is easily computed as $\omega_0 n = 2n$.

```
>>  n = 1:15; bn = 2./(pi*n).*(2./(pi*n).*sin(pi*n/2)-cos(pi*n/2));
>>  an = zeros(size(n)); Cn = [0,sqrt(an.^2+bn.^2)];
>>  thetan = [atan(0),atan2(-bn,an)];n = [0,n];
>>  subplot(121); stem(n,Cn,'k.'); xlabel('n'); ylabel('|C_n|');
>>  axis([-.5 15.5 0 0.45]); grid on
>>  subplot(122); stem(n,thetan,'k.'); xlabel('n'); ylabel('\theta_n');
>>  axis([-.5 15.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```
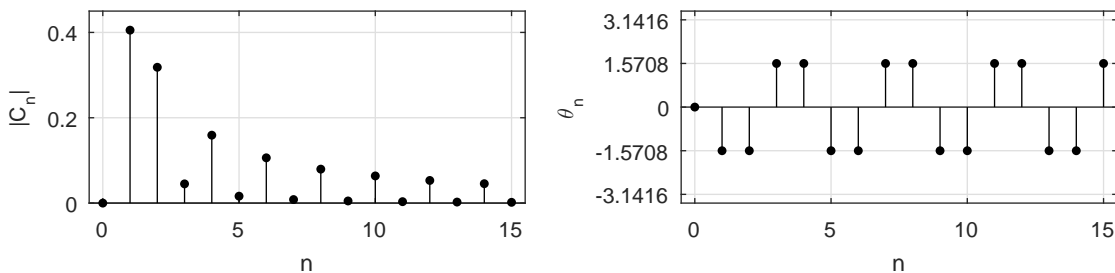

Figure S6.1-1d

(e) Here, $T_0 = 3$, $\omega_0 = 2\pi/3$. Thus,

$$a_0 = \frac{1}{3} \int_0^1 t\, dt = \frac{1}{6}$$

$$a_n = \frac{2}{3} \int_0^1 t \cos \frac{2n\pi}{3} t\, dt = \frac{3}{2\pi^2 n^2} [\cos \frac{2\pi n}{3} + \frac{2\pi n}{3} \sin \frac{2\pi n}{3} - 1]$$

$$b_n = \frac{2}{3} \int_0^1 t \sin \frac{2n\pi}{3} t\, dt = \frac{3}{2\pi^2 n^2} [\sin \frac{2\pi n}{3} - \frac{2\pi n}{3} \cos \frac{2\pi n}{3}].$$

Therefore, $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2} \left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos \frac{2\pi n}{3} - \frac{4\pi n}{3} \sin \frac{2\pi n}{3}} \right]$$

and

$$\theta_n = \tan^{-1} \left(\frac{\frac{2\pi n}{3} \cos \frac{2\pi n}{3} - \sin \frac{2\pi n}{3}}{\cos \frac{2\pi n}{3} + \frac{2\pi n}{3} \sin \frac{2\pi n}{3} - 1} \right).$$

Both C_n and θ_n are shown in Fig. S6.1-1e. The corresponding frequency ω is easily computed as $\omega_0 n = \frac{2\pi n}{3}$.

```
>>  n = 1:15; an = 3./(2*pi^2*n.^2).*(cos(2*pi*n/3)+2*pi*n/3.*sin(2*pi*n/3)-1);
>>  bn = 3./(2*pi^2*n.^2).*(sin(2*pi*n/3)-2*pi*n/3.*cos(2*pi*n/3))
>>  Cn = [1/6,sqrt(an.^2+bn.^2)]; thetan = [atan(0),atan2(-bn,an)];n = [0,n];
>>  subplot(121); stem(n,Cn,'k.'); xlabel('n'); ylabel('|C_n|');
>>  axis([-.5 15.5 0 0.45]); grid on
>>  subplot(122); stem(n,thetan,'k.'); xlabel('n'); ylabel('\theta_n');
>>  axis([-.5 15.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```
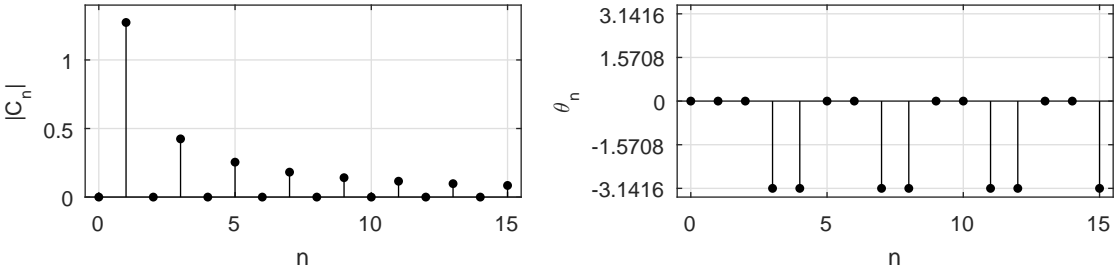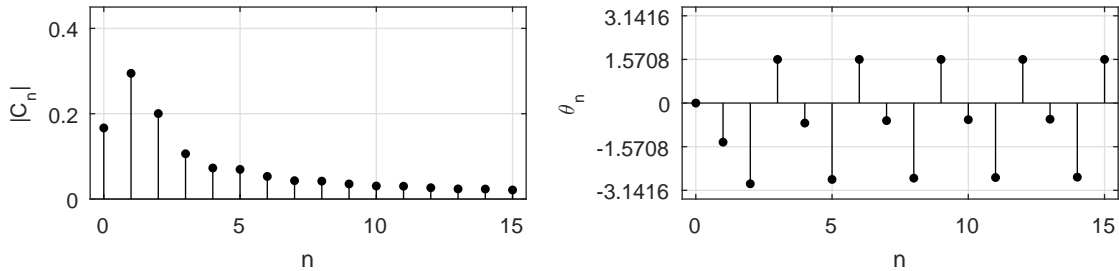

Figure S6.1-1e

(f) For this case, $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 0.5$ (by inspection). Since the signal has even symmetry, $b_n = 0$. Further,

$$a_n = \frac{4}{6} \int_0^3 x(t) \cos \frac{n\pi}{3} dt$$

$$= \frac{2}{3} \left[\int_0^1 \cos \frac{n\pi}{3} dt + \int_1^2 (2 - t) \cos \frac{n\pi}{3} t\, dt \right]$$

$$= \frac{6}{\pi^2 n^2} \left[\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3} \right]$$

and

$$x(t) = 0.5 + \frac{6}{\pi^2} \left(\cos \frac{\pi}{3} t - \frac{2}{9} \cos \pi t + \frac{1}{25} \cos \frac{5\pi}{3} t + \frac{1}{49} \cos \frac{7\pi}{3} t + \cdots \right).$$

Observe that this signal does not have even harmonics. The reason is that if the dc component (0.5) is subtracted from $x(t)$, the resulting function has half-wave symmetry (see Prob. 6.1-6). Since $b_n = 0$, $C_n = |a_n|$ and $\theta_n = \angle a_n$, both shown in Fig. S6.1-1f. The corresponding frequency ω is easily computed as $\omega_0 n = \frac{\pi n}{3}$.

```
>> n = 1:15; an = 6./(pi^2*n.^2).*(cos(pi*n/3)-cos(2*n*pi/3)); bn = zeros(size(n));
>> Cn = [1/2,sqrt(an.^2+bn.^2)]; thetan = [atan(0),atan2(-bn,an)];n = [0,n];
>> subplot(121); stem(n,Cn,'k.'); xlabel('n'); ylabel('|C_n|');
>> axis([-.5 15.5 0 0.75]); grid on
>> subplot(122); stem(n,thetan,'k.'); xlabel('n'); ylabel('\theta_n');
>> axis([-.5 15.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```
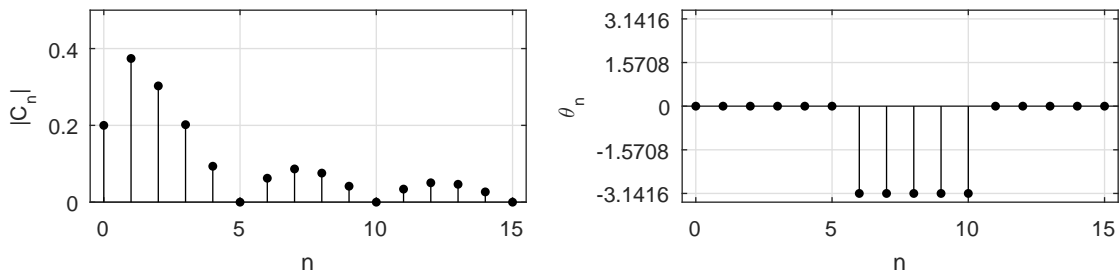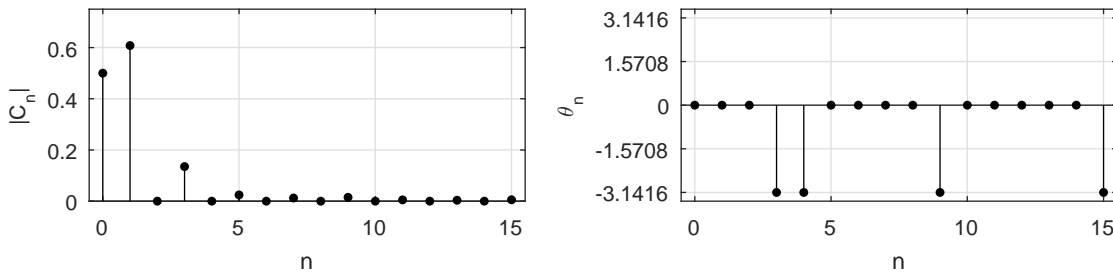

Figure S6.1-1f

Solution 6.1-2

(a) Here, $T_0 = \pi$ and $\omega_0 = \frac{2\pi}{T_0} = 2$. Therefore,

$$y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + b_n \sin 2nt.$$

To compute the coefficients, we shall use the interval π to 0 for integration. Thus,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{0} e^{t/2} \, dt = 0.504$$

$$a_n = \frac{2}{\pi} \int_{-\pi}^{0} e^{t/2} \cos 2nt \, dt = 0.504 \left(\frac{2}{1 + 16n^2} \right)$$

$$b_n = \frac{2}{\pi} \int_{-\pi}^{0} e^{t/2} \sin 2nt \, dt = -0.504 \left(\frac{8n}{1 + 16n^2} \right).$$

Therefore,

$$C_0 = a_0 = 0.504$$

$$C_n = \sqrt{a_n^2 + b_n^2} = 0.504 \left(\frac{2}{\sqrt{1 + 16n^2}} \right)$$

$$\theta_n = \tan^{-1} \left(\frac{-b_n}{a_n} \right) = \tan^{-1} 4n$$

$$y(t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1 + 16n^2}} \cos \left(2nt + \tan^{-1} 4n \right).$$

(b) This Fourier series is identical to that in Eq. (6.11) with t replaced by $-t$. That is,

$$y(t) = x(-t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1 + 16n^2}} \cos \left(-2nt - \tan^{-1} 4n \right).$$

Since cosine is an even function, this is equivalent to

$$y(t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos\left(2nt + \tan^{-1} 4n\right).$$

This matches the Fourier series derived in part (a).

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(-t) = C_0 + \sum C_n \cos(-n\omega_0 t + \theta_n) = C_0 + \sum C_n \cos(n\omega_0 t - \theta_n).$$

Thus, time inversion of a signal merely changes the sign of the phase θ_n. Everything else remains unchanged. This result is consistent with the earlier results of parts (a) and (b).

<div style="background-color:#dddddd"></div>

Solution 6.1-3

(a) Here, $T_0 = \pi/2$ and $\omega_0 = \frac{2\pi}{T_0} = 4$. Therefore,

$$y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 4nt + b_n \sin 4nt,$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} e^{-t}\, dt = 0.504,$$

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} e^{-t} \cos 4nt\, dt = 0.504 \left(\frac{2}{1+16n^2} \right),$$

and

$$b_n = \frac{4}{\pi} \int_0^{\pi/2} e^{-t} \sin 4nt\, dt = 0.504 \left(\frac{8n}{1+16n^2} \right).$$

Therefore $C_0 = a_0 = 0.504$, $C_n = \sqrt{a_n^2 + b_n^2} = 0.504 \left(\frac{2}{\sqrt{1+16n^2}} \right)$, and $\theta_n = -\tan^{-1} 4n$.

(b) This Fourier series is identical to that in Eq. (6.11) with t replaced by $2t$. That is,

$$y(t) = x(2t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos\left(4nt - \tan^{-1} 4n\right).$$

Notice that the Fourier series coefficients themselves are unchanged when a periodic signal is compressed or expanded.

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(at) = C_0 + \sum C_n \cos(n(a\omega_0)t + \theta_n)$$

Thus, time scaling by a factor a merely scales the fundamental frequency by the same factor a. Everything else remains unchanged. If we time scale (compress or expand) a periodic signal by a factor a, its fundamental frequency increases by the same factor a. This result is consistent with the earlier results of parts (a) and (b).

Solution 6.1-4

(a) Here, $T_0 = 2$ and $\omega_0 = \frac{2\pi}{T_0} = \pi$. Also $g(t)$ is an even function of t. Therefore,

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t,$$

where, by inspection, $a_0 = 0$ and [see Eq. (6.14)]

$$a_n = \frac{4}{2} \int_0^1 A(-2t + 1) \cos n\pi t \, dt = -\frac{4}{\pi^2 n^2} \left(\cos n\pi t - 1 \right)\big|_0^1 = \begin{cases} 0 & n \text{ even} \\ \frac{8A}{n^2\pi^2} & n \text{ odd} \end{cases}.$$

Therefore,

$$g(t) = \frac{8A}{\pi^2} \left[\cos \pi t + \frac{1}{9} \cos 3\pi t + \frac{1}{25} \cos 5\pi t + \frac{1}{49} \cos 7\pi t + \cdots \right].$$

(b) We know that $g(t) = x(t + 0.5)$. From Eq. (6.12), we know that

$$x(t) = \frac{8A}{\pi^2} \left[\sin \pi t - \frac{1}{9} \sin 3\pi t + \frac{1}{25} \sin 5\pi t - \frac{1}{49} \sin 7\pi t + \cdots \right]$$

Thus,

$$g(t) = x(t + 0.5)$$
$$= \frac{8A}{\pi^2} \left[\sin \pi(t + 0.5) - \frac{1}{9} \sin 3\pi(t + 0.5) + \frac{1}{25} \sin 5\pi(t + 0.5) - \frac{1}{49} \sin 7\pi(t + 0.5) + \cdots \right].$$

Using trigonometry properties, this simplifies to

$$g(t) = \frac{8A}{\pi^2} \left[\cos \pi t + \frac{1}{9} \cos 3\pi t + \frac{1}{25} \cos 5\pi t + \frac{1}{49} \cos 7\pi t + \cdots \right].$$

Clearly, this result matches that in part (a).

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(t + T) = C_0 + \sum C_n \cos[n\omega_0(t + T) + \theta_n] = C_0 + \sum C_n \cos[n\omega_0 t + (\theta_n + n\omega_0 T)].$$

Thus, time shifting by T merely changes the phase of the nth harmonic by $n\omega_0 T$.

Solution 6.1-5

Recall that the trigonometric form of the Fourier series is

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t.$$

(a) For $x_a(t) = \cos(3\pi t)$, we see that

$$a_n = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad b_n = 0, \qquad \text{and} \qquad \omega_0 = 3\pi.$$

(b) For $x_b(t) = \sin(7\pi t)$,

$$a_n = 0, \qquad b_n = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \text{and} \qquad \omega_0 = 7\pi.$$

(c) For $x_c(t) = 2 + 4\cos(3\pi t) - 2j\sin(7\pi t)$,

$$a_n = \begin{cases} 2 & n = 0 \\ 4 & n = 3 \\ 0 & \text{otherwise} \end{cases} \quad, \quad b_n = \begin{cases} -2j & n = 7 \\ 0 & \text{otherwise} \end{cases} \quad, \quad \text{and} \quad \omega_0 = \pi.$$

(d) For $x_d(t) = (1+j)\sin(3\pi t) + (2-j)\cos(7\pi t)$

$$a_n = \begin{cases} 2 - j & n = 7 \\ 0 & \text{otherwise} \end{cases} \quad, \quad b_n = \begin{cases} 1 + j & n = 3 \\ 0 & \text{otherwise} \end{cases} \quad, \quad \text{and} \quad \omega_0 = \pi.$$

(e) To begin, we express $x_e(t) = \sin(3\pi t + 1) + 2\cos(7\pi t - 2)$ as

$$x_e(t) = \sin(3\pi t)\cos(1) + \cos(3\pi t)\sin(1) + 2\cos(7\pi t)\cos(2) + 2\sin(7\pi t)\sin(2).$$

Thus,

$$a_n = \begin{cases} \sin(1) & n = 3 \\ 2\cos(2) & n = 7 \\ 0 & \text{otherwise} \end{cases} \quad, \quad b_n = \begin{cases} \cos(1) & n = 3 \\ 2\sin(2) & n = 7 \\ 0 & \text{otherwise} \end{cases} \quad, \quad \text{and} \quad \omega_0 = \pi.$$

(f) For $x_f(t) = \sin(6\pi t) + 2\cos(14\pi t)$,

$$a_n = \begin{cases} 2 & n = 7 \\ 0 & \text{otherwise} \end{cases} \quad, \quad b_n = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases} \quad, \quad \text{and} \quad \omega_0 = 2\pi.$$

Solution 6.1-6

For half wave symmetry

$$x(t) = -x\left(t \pm \frac{T_0}{2}\right)$$

and

$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t)\cos n\omega_0 t\, dt = \frac{2}{T_0} \int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt + \int_{T_0/2}^{T_0} x(t)\cos n\omega_0 t\, dt.$$

Let $\tau = t - T_0/2$ in the second integral. This gives

$$a_n = \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt + \int_0^{T_0/2} x\left(\tau + \frac{T_0}{2}\right)\cos n\omega_0\left(\tau + \frac{T_0}{2}\right) d\tau\right]$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt + \int_0^{T_0/2} -x(\tau)[-\cos n\omega_0 \tau]\, d\tau\right]$$

$$= \frac{4}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt\right].$$

In a similar way, we can show that

$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t)\sin n\omega_0 t\, dt.$$

Next, we apply these results to the waveforms in Fig. P6.1-6.

(a) For the waveform in Fig. P6.1-6a, $T_0 = 8$, $\omega_0 = \frac{\pi}{4}$, and $a_0 = 0$ (by inspection). Since the waveform has half-wave symmetry,

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \cos \frac{n\pi}{4}t + b_n \sin \frac{n\pi}{4}t,$$

where

$$a_n = \frac{4}{8}\left[\int_0^4 x(t)\cos\frac{n\pi}{4}t\,dt\right] = \frac{1}{2}\left[\int_0^2 \frac{t}{2}\cos\frac{n\pi}{4}t\,dt\right]$$

$$= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd})$$

$$= \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd}),$$

or

$$a_n = \begin{cases} \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \ldots \\ -\frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \ldots \end{cases},$$

and

$$b_n = \frac{1}{2}\int_0^2 \frac{t}{2}\sin\frac{n\pi}{4}t\,dt = \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad (n \text{ odd}).$$

(b) For the waveform in Fig. P6.1-6b, $T_0 = 2\pi$, $\omega_0 = 1$, and $a_0 = 0$ (by inspection). Since the waveform has half-wave symmetry,

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n \cos nt + b_n \sin nt,$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi e^{-t/10}\cos nt\,dt$$

$$= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + 0.01}(-0.1\cos nt + n\sin nt)\right]_0^\pi \quad (n \text{ odd})$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2 + 0.01}(0.1) - \frac{1}{n^2 + 0.01}(-0.1)\right]$$

$$= \frac{2}{10\pi(n^2 + 0.01)}(e^{-\pi/10} - 1) = \frac{0.0465}{n^2 + 0.01}$$

and

$$b_n = \frac{2}{\pi}\int_0^\pi e^{-t/10}\sin nt\,dt$$

$$= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + 0.01}(-0.1\sin nt - n\cos nt)\right]_0^\pi \quad (n \text{ odd})$$

$$= \frac{2n}{(n^2 + 0.01)}(e^{-\pi/10} - 1) = \frac{1.461n}{n^2 + 0.01}.$$

Solution 6.1-7

(a) Here, we need only cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it is an even function of t, has a value t over the interval $0 \leq t \leq 1$, and repeats every 4 seconds as shown in Fig. S6.1-7a. We selected the pulse width $W = 2$ seconds. But it can be anywhere from 2 to 4, and still satisfy these conditions. Each value of W results in different series. Yet all of them converge to t over 0 to 1, and satisfy the other requirements. Clearly, there are infinite number of Fourier series that will satisfy the given requirements. The present choice yields

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right) t.$$

By inspection, we find $a_0 = 1/4$. Because of symmetry $b_n = 0$ and

$$a_n = \frac{4}{4} \int_0^1 t \cos \frac{n\pi}{2} t\, dt = \frac{4}{n^2\pi^2} \left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2} \sin\left(\frac{n\pi}{2}\right) - 1\right].$$
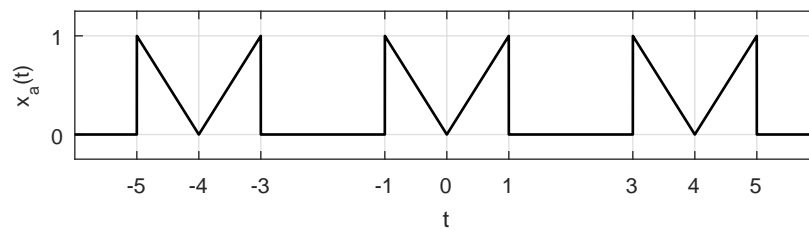
Figure S6.1-7a

(b) Here, we need only sine terms and $\omega_0 = 2$. Hence, we must construct a pulse with odd symmetry, which has a value t over the interval $0 \leq t \leq 1$, and repeats every π seconds as shown in Fig. S6.1-7b. Similar to the case (a), the pulse width can be anywhere from 1 to π. For the present case

$$x(t) = \sum_{n=1}^{\infty} b_n \sin 2nt.$$

Because of odd symmetry, $a_n = 0$ and

$$b_n = \frac{4}{\pi} \int_0^1 t \sin 2nt\, dt = \frac{1}{\pi n^2} (\sin 2n - 2n \cos 2n).$$
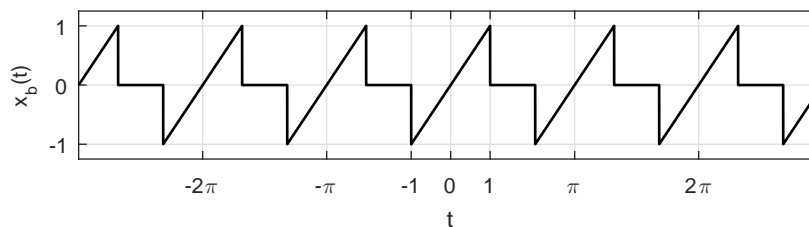
Figure S6.1-7b

(c) Here, we need both sine and cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it has no symmetry of any kind, has a value t over the interval $0 \leq t \leq 1$, and repeats every 4 seconds as shown in Fig. S6.1-7c. As usual, the pulse width can be have any value in the range 1 to 4.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right) t + b_n \sin\left(\frac{n\pi}{2}\right) t.$$

By inspection, $a_0 = 1/8$ and

$$a_n = \frac{2}{4} \int_0^1 t \cos\frac{n\pi}{2} t \, dt = \frac{2}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right]$$

$$b_n = \frac{2}{4} \int_0^1 t \sin\frac{n\pi}{2} t \, dt = \frac{2}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right].$$
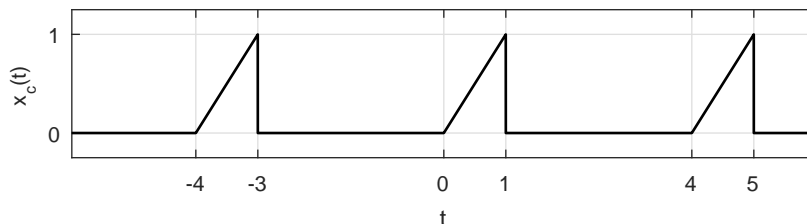
Figure S6.1-7c

(d) Here, we need only cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it is an even function of t, has a value t over the interval $0 \leq t \leq 1$, repeats every 2π seconds and has half-wave symmetry as shown in Fig. S6.1-7d. Observe that the first half cycle (from 0 to π) and the second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. This will cause even harmonics to vanish. The pulse has an even and half-wave symmetry. This yields

$$x(t) = a_0 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos nt.$$

By inspection, $a_0 = 0$. Because of even symmetry $b_n = 0$. Because of half-wave symmetry (see Prob. 6.1-6),

$$a_n = \frac{4}{2\pi}\left[\int_0^{\pi/2} t \cos nt \, dt - \int_{\pi/2}^{\pi} (t - \pi) \cos nt \, dt\right] = \frac{2}{\pi n^2}(\cos n\pi - 1) + \frac{2}{n}\sin\frac{n\pi}{2} \qquad n \text{ odd}.$$
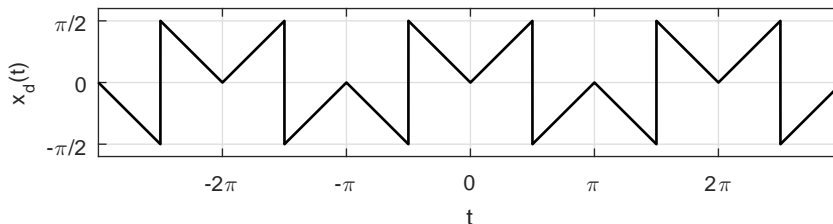
Figure S6.1-7d

(e) Here, we need only sine terms with $\omega_0 = \pi$ and odd harmonics only. Hence, we must construct a pulse such that it is an odd function of t, has a value t over the interval $0 \leq t \leq 1$, repeats every 4 seconds and has half-wave symmetry as shown in Fig. S6.1-7e. Observe that the first half cycle (from 0 to 2) and the second half cycle (from 2 to 4) are negatives of each other as required in half-wave symmetry. This will cause even harmonics to vanish. The pulse has an odd and half-wave symmetry. This yields

$$x(t) = \sum_{\substack{n=1 \\ n \ odd}}^{\infty} b_n \sin \frac{n\pi}{2}t.$$

By inspection, $a_0 = 0$. Because of odd symmetry $a_n = 0$. Because of half-wave symmetry (see Prob. 6.1-6),

$$b_n = \frac{4}{4} \int_0^1 t \sin \frac{n\pi}{2}t \, dt + \int_1^2 (-t+2) \sin \frac{n\pi}{2}t \, dt = \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \qquad n \text{ odd}.$$
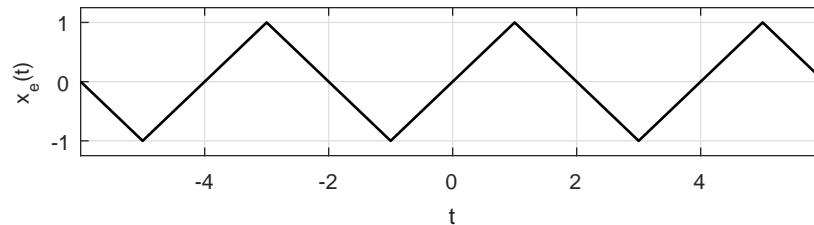
Figure S6.1-7e

(f) Here, we need both sine and cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it has half-wave symmetry, but neither odd nor even symmetry, has a value t over the interval $0 \leq t \leq 1$, and repeats every 2π seconds as shown in Fig. S6.1-7f. Observe that the first half cycle from 0 to π) and the second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. By inspection, $a_0 = 0$. This yields

$$x(t) = \sum_{\substack{n=1 \\ n \ odd}}^{\infty} a_n \cos nt + b_n \sin nt.$$

Because of half-wave symmetry (see Prob. 6.1-6),

$$a_n = \frac{4}{2\pi} \int_0^1 t \cos nt \, dt = \frac{2}{\pi n^2} (\cos n + n \sin n - 1)$$

$$b_n = \frac{4}{2\pi} \int_0^1 t \sin nt \, dt = \frac{2}{\pi n^2} (\sin n - n \cos n) \qquad n \text{ odd}.$$
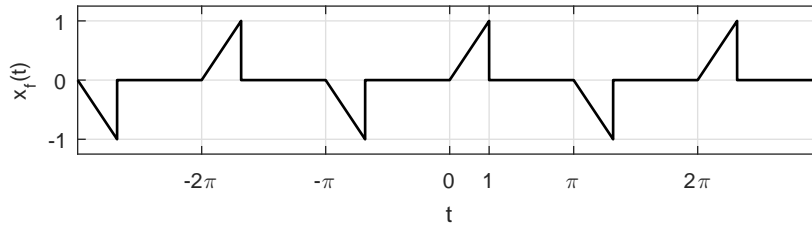
Figure S6.1-7f

In each case the signal is periodic, it is because $x(t) = x(t+T)$ for all t. If $x(t) \neq x(t+T)$ for all t, then the signal is not periodic.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
Periodic?	Yes	Yes	No	Yes	No	Yes	Yes	Yes	Yes
ω_0	1	1		π		$\frac{1}{70}$	$\frac{3}{4}$	1	2
Period T	2π	2π		2		140π	$\frac{8\pi}{3}$	2π	π

In each case, the exponential Fourier series is given as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}.$$

(a) Here, $T_0 = 4$, $\omega_0 = \pi/2$, and $D_0 = 0$ (by inspection). For $|n| \geq 1$, we have

$$D_n = \frac{1}{2\pi} \int_{-1}^{1} e^{-j(n\pi/2)t}\, dt - \int_{1}^{3} e^{-j(n\pi/2)t}\, dt = \frac{2}{\pi n} \sin \frac{n\pi}{2}.$$

We use MATLAB to plot the spectrum, which is shown in Fig. S6.3-1a.

```
>> n = -10:10; Dn = 2./(pi*n).*sin(n*pi/2); Dn(n==0) = 0;
>> subplot(121); stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|');
>> axis([-10 10 0 .7]); set(gca,'xtick',-10:5:10); grid on
>> subplot(122); stem(n,angle(Dn),'k.'); xlabel('n'); ylabel('\angle D_n');
>> axis([-10 10 -1.1*pi 1.1*pi]); set(gca,'xtick',-10:5:10,'ytick',-pi:pi/2:pi);
>> set(gca,'yticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'}); grid on
```
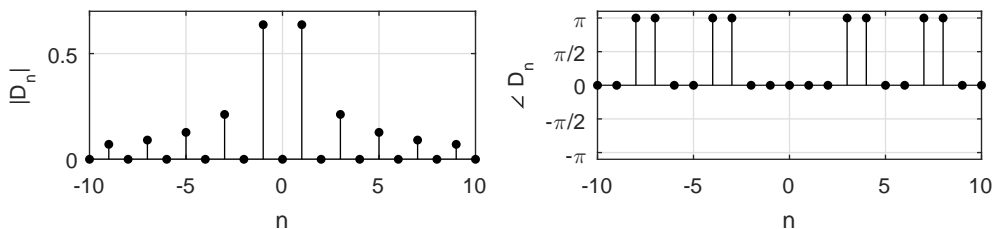

Figure S6.3-1a

(b) In this case, $T_0 = 10\pi$, $\omega_0 = 2\pi/10\pi = 1/5$, and $D_0 = \frac{1}{5}$. For $|n| \geq 1$, we have

$$D_n = \frac{1}{10\pi} \int_{\pi}^{\pi} e^{-j\frac{n}{5}t}\, dt = \frac{j}{2\pi n}\left(-2j \sin \frac{n\pi}{5}\right) = \frac{1}{\pi n} \sin \left(\frac{n\pi}{5}\right).$$

More compactly, we see that (for all n)

$$D_n = \frac{1}{5}\text{sinc}\left(\frac{n\pi}{5}\right).$$

We use MATLAB to plot the spectrum, which is shown in Fig. S6.3-1b.

```
>> n = -20:20; Dn = 1/5*sinc(n/5);
>> subplot(121); stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|');
>> axis([-20 20 0 .22]); set(gca,'xtick',-20:10:20); grid on
>> subplot(122); stem(n,angle(Dn),'k.'); xlabel('n'); ylabel('\angle D_n');
>> axis([-20 20 -1.1*pi 1.1*pi]); set(gca,'xtick',-20:10:20,'ytick',-pi:pi/2:pi);
>> set(gca,'yticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'}); grid on
```
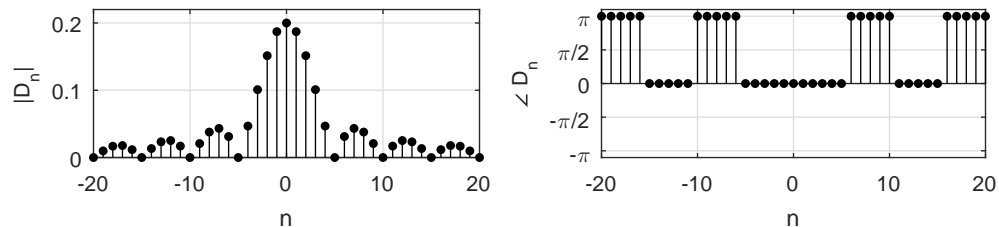

Figure S6.3-1b

(c) By inspection, $T_0 = 2\pi$, $\omega_0 = 1$, and $D_0 = 0.5$. For $|n| \geq 1$, we have

$$D_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}e^{-jnt}\,dt = \frac{j}{2\pi n},$$

so that

$$|D_n| = \frac{1}{2\pi n} \quad\text{and}\quad \angle D_n = \begin{cases} \frac{\pi}{2} & n>0 \\ \frac{-\pi}{2} & n<0 \end{cases}.$$

We use MATLAB to plot the spectrum, which is shown in Fig. S6.3-1c.

```
>> n = -10:10; Dn = 1j./(2*pi*n); Dn(n==0) = 0.5;
>> subplot(121); stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|');
>> axis([-10 10 0 .55]); set(gca,'xtick',-10:5:10); grid on
>> subplot(122); stem(n,angle(Dn),'k.'); xlabel('n'); ylabel('\angle D_n');
>> axis([-10 10 -1.1*pi 1.1*pi]); set(gca,'xtick',-10:5:10,'ytick',-pi:pi/2:pi);
>> set(gca,'yticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'}); grid on
```
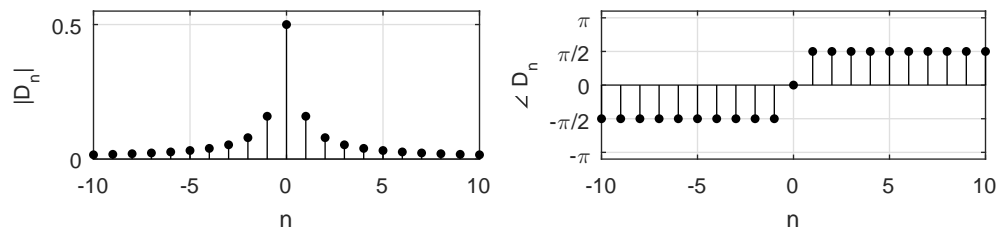

Figure S6.3-1c

(d) Here, $T_0 = \pi$, $\omega_0 = 2$, and $D_n = 0$. For $|n| \geq 1$,

$$D_n = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{4t}{\pi}e^{-j2nt}\,dt = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right).$$

We use MATLAB to plot the spectrum, which is shown in Fig. S6.3-1d.

```
>>  n = -10:10; Dn = -1j./(pi*n).*(2./(pi*n).*sin(pi*n/2)-cos(pi*n/2)); Dn(n==0) = 0;
>>  subplot(121); stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|');
>>  axis([-10 10 0 .25]); set(gca,'xtick',-10:5:10); grid on
>>  subplot(122); stem(n,angle(Dn),'k.'); xlabel('n'); ylabel('\angle D_n');
>>  axis([-10 10 -1.1*pi 1.1*pi]); set(gca,'xtick',-10:5:10,'ytick',-pi:pi/2:pi);
>>  set(gca,'yticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'}); grid on
```
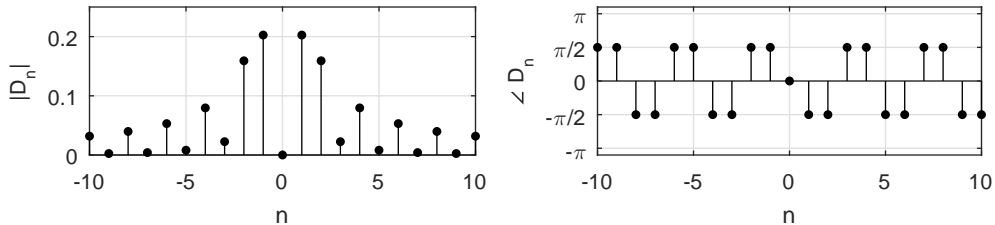

Figure S6.3-1d

(e) In this case, $T_0 = 3$, $\omega_0 = \frac{2\pi}{3}$, and $D_0 = \frac{1}{6}$. For $|n| \geq 1$, we have

$$D_n = \frac{1}{3} \int_0^1 t\, e^{-j\frac{2\pi n}{3}t}\, dt = \frac{3}{4\pi^2 n^2} \left[e^{-j\frac{2\pi n}{3}} \left(\frac{j2\pi n}{3} + 1 \right) - 1 \right].$$

We use MATLAB to plot the spectrum, which is shown in Fig. S6.3-1e.

```
>>  n = -10:10; Dn = 3./(4*pi.^2.*n.^2).*(exp(-1j*2*pi*n/3).*(1j*2*pi*n/3+1)-1);
>>  Dn(n==0) = 1/6; subplot(121); stem(n,abs(Dn),'k.'); xlabel('n');
>>  ylabel('|D_n|'); axis([-10 10 0 .2]); set(gca,'xtick',-10:5:10); grid on
>>  subplot(122); stem(n,angle(Dn),'k.'); xlabel('n'); ylabel('\angle D_n');
>>  axis([-10 10 -1.1*pi 1.1*pi]); set(gca,'xtick',-10:5:10,'ytick',-pi:pi/2:pi);
>>  set(gca,'yticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'}); grid on
```
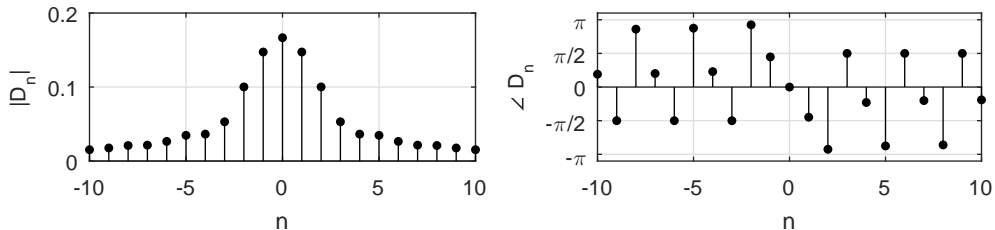

Figure S6.3-1e

(f) By inspection, $T_0 = 6$, $\omega_0 = \pi/3$, and $D_0 = 0.5$. For $|n| \geq 1$, we have

$$D_n = \frac{1}{6} \left[\int_{-2}^{-1} (t+2)e^{-\frac{j\pi n t}{3}}\, dt + \int_{-1}^{1} e^{-\frac{j\pi n t}{3}}\, dt + \int_{1}^{2} (-t+2)e^{-\frac{j\pi n t}{3}}\, dt \right]$$

$$= \frac{3}{\pi^2 n^2} \left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3} \right).$$

We use MATLAB to plot the spectrum, which is shown in Fig. S6.3-1f.

```
>>  n = -10:10; Dn = 3./(pi^2*n.^2).*(cos(n*pi/3)-cos(2*pi*n/3)); Dn(n==0) = 0.5;
>>  subplot(121); stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|');
>>  axis([-10 10 0 .55]); set(gca,'xtick',-10:5:10); grid on
>>  subplot(122); stem(n,angle(Dn),'k.'); xlabel('n'); ylabel('\angle D_n');
>>  axis([-10 10 -1.1*pi 1.1*pi]); set(gca,'xtick',-10:5:10,'ytick',-pi:pi/2:pi);
>>  set(gca,'yticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'}); grid on
```
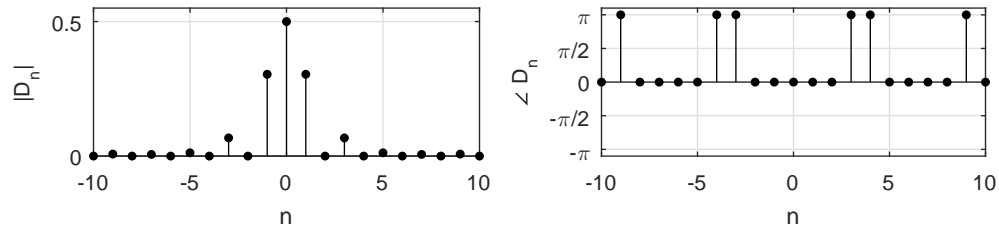
Figure S6.3-1f

Solution 6.3-2

Note that the signal $x(t)$ is defined as

$$x(t) = \begin{cases} \frac{1}{A}t & 0 \le t < A \\ 1 & A \le t < \pi \\ 0 & \pi \le t < 2\pi \\ x(t+2\pi) & \text{otherwise} \end{cases}.$$

The exponential Fourier series coefficients are determined by

$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt.$$

Since $T_0 = 2\pi$, $\omega_0 = \frac{2\pi}{T_0} = 1$. For $n = 0$,

$$D_0 = \frac{1}{T_0} \int_{T_0} x(t) dt = \frac{1}{2\pi} \left(\int_0^A \frac{t}{A} dt + \int_A^\pi dt \right)$$

$$= \frac{1}{2\pi} \left(\frac{t^2}{2A} \bigg|_{t=0}^A \quad t|_{t=A}^\pi \right) = \frac{1}{2\pi} \left(\frac{A}{2} + \pi - A \right)$$

$$= \frac{2\pi - A}{4\pi}.$$

For $n \ne 0$,

$$D_n = \frac{1}{T_0} \int_{T_0} e^{-jn\omega_0 t} x(t) dt$$

$$= \frac{1}{2\pi} \left(\int_0^A \frac{t}{A} e^{-jn\omega_0 t} dt + \int_A^\pi e^{-jn\omega_0 t} dt \right)$$

$$= \frac{1}{2\pi} \left(\frac{te^{-jnt}}{-jAn} \bigg|_{t=0}^A - \int_0^A \frac{e^{-jnt}}{jAn} dt + \frac{e^{-jnt}}{-jn} \bigg|_{t=A}^\pi \right)$$

$$= \frac{1}{2\pi} \left(\frac{e^{-jnA}}{-jn} - \frac{e^{-jnt}}{-An^2} \bigg|_{t=0}^A + \frac{e^{-jn\pi} - e^{-jnA}}{-jn} \right)$$

$$= \frac{1}{2\pi} \left(\frac{je^{-jn\pi}}{n} + \frac{e^{-jnA} - 1}{An^2} \right)$$

$$= \frac{1}{2\pi n} \left(\frac{e^{-jnA} - 1}{An} + je^{-jn\pi} \right).$$

Thus,

$$D_n = \begin{cases} \frac{2\pi - A}{4\pi} & n = 0 \\ \frac{1}{2\pi n} \left(\frac{e^{-jnA} - 1}{nA} + je^{-jn\pi} \right) & \text{otherwise} \end{cases}.$$

Solution 6.3-3

(a) Here,

$$x(t) = 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right).$$

For a compact trigonometric form, all terms must have cosine form and amplitudes must be positive. For this reason, we rewrite $x(t)$ as

$$x(t) = 3\cos t + \cos\left(5t - \frac{\pi}{6} - \frac{\pi}{2}\right) + 2\cos\left(8t - \frac{\pi}{3} - \pi\right)$$

$$= 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right).$$

In the preceding expression, we could have expressed the term $2\cos\left(8t - \frac{4\pi}{3}\right)$ as $2\cos\left(8t + \frac{2\pi}{3}\right)$. Fig. S6.3-3a shows amplitude and phase spectra.
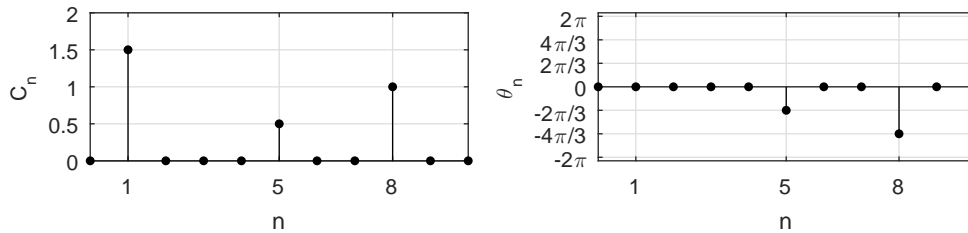
Figure S6.3-3a

(b) By inspection of the trigonometric spectra in Fig. S6.3-3a, we plot the exponential spectra as shown in Fig. S6.3-3b.
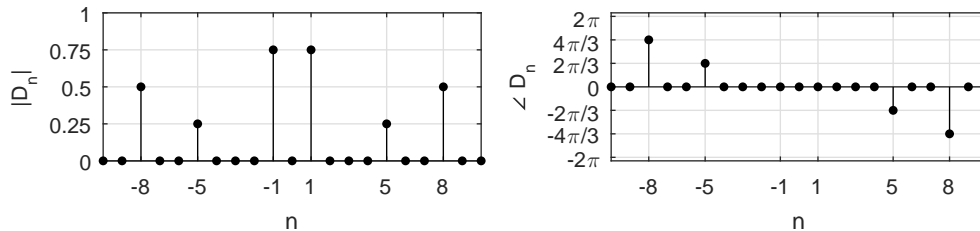
Figure S6.3-3b

(c) By inspection of exponential spectra in Fig. S6.3-3a, we obtain

$$x(t) = \frac{3}{2}(e^{jt} + e^{-jt}) + \frac{1}{2}\left[e^{j(5t - \frac{2\pi}{3})} + e^{-j(5t - \frac{2\pi}{3})}\right] + \left[e^{j(8t - \frac{4\pi}{3})} + e^{-j(8t - \frac{4\pi}{3})}\right]$$

$$= \left(e^{j\frac{4\pi}{3}}\right)e^{-j8t} + \left(\frac{1}{2}e^{j\frac{2\pi}{3}}\right)e^{-j5t} + \frac{3}{2}e^{-jt} + \frac{3}{2}e^{jt} + \left(\frac{1}{2}e^{-j\frac{2\pi}{3}}\right)e^{j5t} + \left(e^{-j\frac{4\pi}{3}}\right)e^{j8t}.$$

(d) By inspection of the first line in part (c), we can immediately write $x(t)$ in the trigonometric form as

$$x(t) = 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right)$$

$$= 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right).$$

Clearly, this result matches the trigonometric series for $x(t)$ given in part (a).

Solution 6.3-4

(a) In compact trigonometric form, all terms are of cosine form and amplitudes are positive. We can express $x(t)$ as

$$x(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right)$$

$$= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right).$$

From this expression we sketch the trigonometric Fourier spectra as shown in Fig. S6.3-4a.
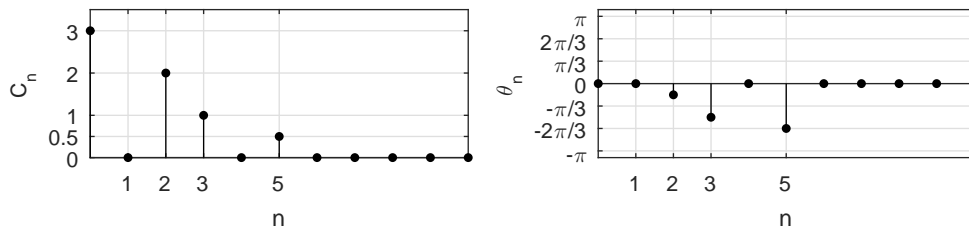
Figure S6.3-4a

(b) By inspection of trigonometric spectra, we sketch the exponential Fourier spectra shown in Fig. S6.3-4b.
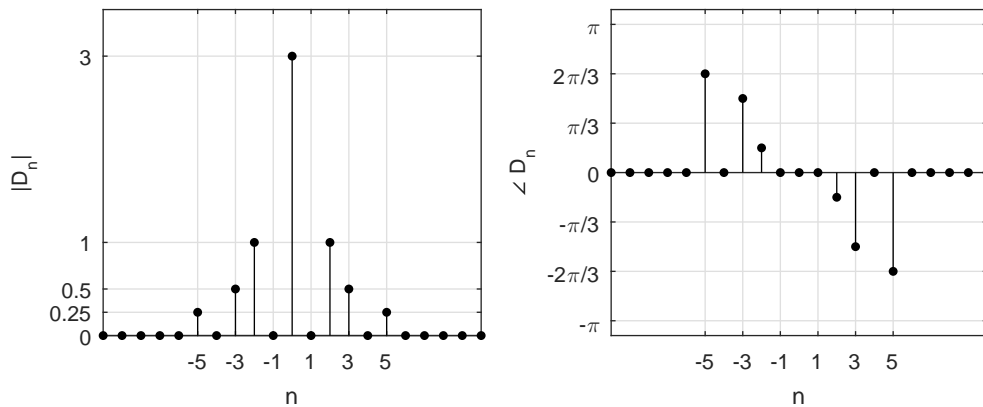
Figure S6.3-4b

(c) From these exponential spectra, we can now write the exponential Fourier series as

$$x(t) = 3 + e^{j(2t - \frac{\pi}{6})} + e^{-j(2t - \frac{\pi}{6})} + \frac{1}{2}e^{j(3t - \frac{\pi}{2})} + \frac{1}{2}e^{-j(3t - \frac{\pi}{3})} + \frac{1}{4}e^{j(5t - \frac{2\pi}{3})} + \frac{1}{4}e^{-j(5t - \frac{2\pi}{3})}.$$

(d) By inspection of the first line in part (c), we can immediately write $x(t)$ in the trigonometric form as

$$x(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$

$$= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \sin 3t - \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right).$$

Clearly, this result matches the trigonometric series for $x(t)$ given in the original problem statement.

Solution 6.3-5

(a) The exponential Fourier series can be expressed with coefficients in polar form as

$$x(t) = (2\sqrt{2}e^{j\pi/4})e^{-j3t} + 2e^{j\pi/2}e^{-jt} + 3 + 2e^{-j\pi/2}e^{jt} + (2\sqrt{2}e^{-j\pi/4})e^{j3t}.$$

From this expression the exponential spectra are sketched as shown in Fig. S6.3-5a.
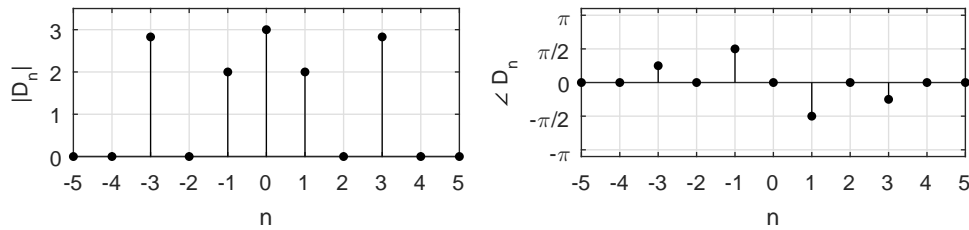
Figure S6.3-5a

(b) By inspection of the exponential spectra in Fig. S6.3-5a, we sketch the trigonometric spectra as shown in Fig. S6.3-5b. From these spectra, we can write the compact trigonometric Fourier series as

$$x(t) = 3 + 4\cos\left(t - \frac{\pi}{2}\right) + 4\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right).$$
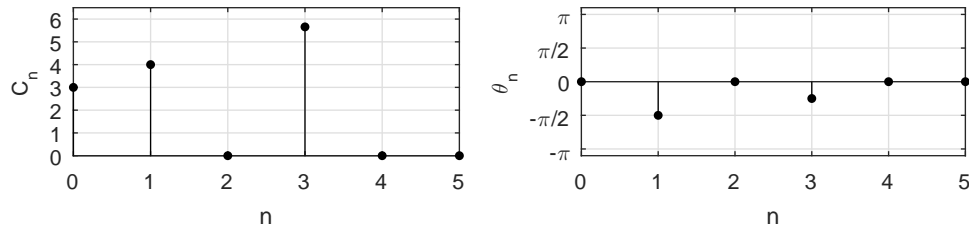
Figure S6.3-5b

(c) Since, the trigonometric series in part (b) is obtained from the exponential series in part (a), the two series are equivalent.

(d) The lowest frequency in the spectrum is 0 and the highest frequency is 3 rad/s. Therefore, the bandwidth is 3 rad/s or $\frac{3}{2\pi}$ Hz.

Solution 6.3-6

(a) By inspection of the spectra given in Fig. P6.3-6, the trigonometric Fourier series is

$$x(t) = 2 + 2\cos(2t - \pi) + \cos(3t - \frac{\pi}{2})$$
$$= 2 - 2\cos 2t + \sin 3t$$

(b) The exponential spectra are shown in Fig. S6.3-6b.

(c) By inspection of Fig. S6.3-6b, the exponential Fourier series is

$$x(t) = 2 + \left[e^{(2t-\pi)} + e^{-j(2t-\pi)}\right] + \frac{1}{2}\left[e^{j(3t-\frac{\pi}{2})} + e^{-j(3t-\frac{\pi}{2})}\right]$$
$$= 2 + 2\cos(2t - \pi) + \cos\left(3t - \frac{\pi}{2}\right).$$

(d) Parts (a) and (c) demonstrate that the trigonometric and exponential Fourier series are equivalent.
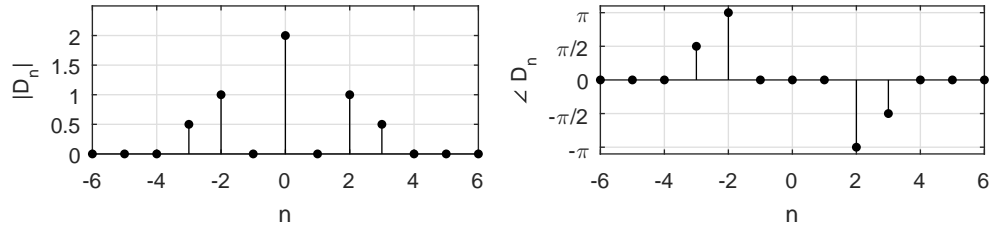
Figure S6.3-6b

Solution 6.3-7

(a) By inspection of the spectra given in Fig. P6.3-7, the exponential Fourier series is

$$x(t) = 2 + 2e^{j(t+\frac{2\pi}{3})} + 2e^{-j(t+\frac{2\pi}{3})} + e^{j(2t+\frac{\pi}{3})} + e^{-j(2t+\frac{\pi}{3})}.$$

(b) Using Table 6.1, we convert the exponential Fourier series coefficients to the trigonometric Fourier series coefficients. The corresponding spectrum is shown in Fig. S6.3-7b.
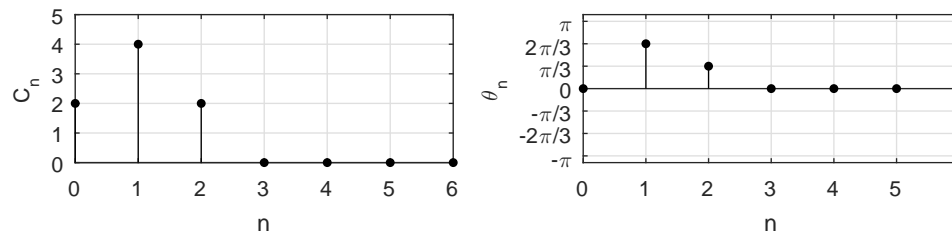
Figure S6.3-7b

(c) By inspection of Fig. S6.3-7b, the trigonometric Fourier series is

$$x(t) = 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right).$$

(d) To show that the trigonometric and exponential Fourier series are equivalent, we simply use Euler's formula to express the trigonometric series as

$$x(t) = 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right)$$

$$= 2 + 2e^{j(t+\frac{2\pi}{3})} + 2e^{-j(t+\frac{2\pi}{3})} + e^{j(2t+\frac{\pi}{3})} + e^{-j(2t+\frac{\pi}{3})}.$$

Clearly the two forms (trigonometric and exponential) are equivalent.

Solution 6.3-8

In this problem, periodic signal $x(t)$ has exponential Fourier series spectrum

$$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} \, dt.$$

(a) If $x(t)$ has even symmetry, we know that $x(t) = x(-t)$. Thus,

$$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt$$

$$= \frac{1}{T_0} \int_0^{T_0} x(-t) \, e^{-jn\omega_0 t} \, dt.$$

Letting $t' = -t$, we obtain

$$D_n = \frac{1}{T_0} \int_0^{-T_0} x(t') \, e^{-jn\omega_0(-t')} \, (-dt')$$

$$= \frac{1}{T_0} \int_{-T_0}^{0} x(t') \, e^{-j(-n)\omega_0 t'} \, dt'$$

$$= D_{-n}.$$

This proves that if $x(t)$ has even symmetry, then D_n also has even symmetry.

(b) If $x(t)$ has odd symmetry, we know that $x(t) = -x(-t)$. Thus,

$$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt$$

$$= -\frac{1}{T_0} \int_0^{T_0} x(-t) \, e^{-jn\omega_0 t} \, dt.$$

Letting $t' = -t$, we obtain

$$D_n = -\frac{1}{T_0} \int_0^{-T_0} x(t') \, e^{-jn\omega_0(-t')} \, (-dt')$$

$$= -\frac{1}{T_0} \int_{-T_0}^{0} x(t') \, e^{-j(-n)\omega_0 t'} \, dt'$$

$$= -D_{-n}.$$

This proves that if $x(t)$ has odd symmetry, then D_n also has odd symmetry.

(c) To begin, we note that

$$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt.$$

Taking the conjugate reflection of D_n, we obtain

$$D_{-n}^* = \frac{1}{T_0} \int_0^{T_0} x^*(t) \, e^{j(-n)\omega_0 t} \, dt.$$

Since $x(t)$ is real, we know that $x(t) = x^*(t)$. Hence

$$D_{-n}^* = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt$$

$$= D_n.$$

This proves that if $x(t)$ is real, then D_n is conjugate symmetric ($D_n = D_{-n}^*$).

(d) To begin, we note that

$$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt.$$

Conjugating, negating, and reflecting D_n, we obtain

$$-D_{-n}^* = -\frac{1}{T_0} \int_0^{T_0} x^*(t) \, e^{j(-n)\omega_0 t} \, dt.$$

Since $x(t)$ is imaginary, we know that $x(t) = -x^*(t)$. Hence

$$-D_{-n}^* = \frac{1}{T_0} \int_0^{T_0} x(t) \, e^{-jn\omega_0 t} \, dt$$

$$= D_n.$$

This proves that if $x(t)$ is imaginary, then D_n is conjugate antisymmetric $(D_n = -D_{-n}^*)$.

Solution 6.3-9

(a) By inspection, $T_0 = 8$, $\omega_0 = \pi/4$, and $D_0 = 0$. For $n \neq 0$,

$$D_n = \frac{1}{8} \left[\int_{-4}^0 \left(\frac{t}{2} + 1 \right) e^{-jn(\pi/4)t} \, dt + \int_0^4 \left(-\frac{t}{2} + 1 \right) e^{-jn(\pi/4)t} \, dt \right]$$

$$= \frac{1}{8} \int_0^4 \left(-\frac{t}{2} + 1 \right) \left(e^{jn(\pi/4)t} + e^{-jn(\pi/4)t} \right) dt$$

$$= \frac{1}{8} \int_0^4 2 \left(-\frac{t}{2} + 1 \right) \cos(n\pi t/4) \, dt$$

$$= -\frac{1}{8} \int_0^4 t \cos(n\pi t/4) \, dt$$

$$= -\frac{1}{8(n\pi/4)^2} \left(\cos(n\pi t/4) + \frac{n\pi t}{4} \sin(n\pi t/4) \Big|_{t=0}^4 \right)$$

$$= -frac{2}{\pi^2 n^2} \left(\cos(n\pi) - 1 \right)$$

$$= \frac{2}{\pi^2 n^2} \left(1 - \cos(n\pi) \right).$$

Therefore, the exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{2}{\pi^2 n^2} \left(1 - \cos(n\pi) \right) e^{jn\frac{\pi}{4}t}.$$

Notice that since $D_n = 0$ for all even-values of n, this summation can be simplified to

$$x(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi^2 n^2} e^{jn\frac{\pi}{4}t}.$$

(b) Observe that $\hat{x}(t)$ is the same as $x(t)$ in Fig. P6.3-9a delayed by 2 seconds. Therefore,

$$\hat{x}(t) = x(t-2) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi^2 n^2} \left(1 - \cos(n\pi)\right) e^{jn\frac{\pi}{4}(t-2)}$$

$$= \sum_{n=-\infty}^{\infty} \frac{2}{\pi^2 n^2} \left(1 - \cos(n\pi)\right) e^{-jn\frac{\pi}{2}} e^{jn\frac{\pi}{4}t}.$$

As expected, the Fourier series coefficients $\hat{D}_n$ for $\hat{x}(t)$ are related to the Fourier series coefficients D_n for $x(t)$ through a simple (complex exponential) multiplicative factor. That is,

$$\hat{D}_n = D_n e^{-jn\frac{\pi}{2}}.$$

(c) Observe that $\tilde{x}(t)$ is the same as $x(t)$ in Fig. P6.3-9a time-compressed by a factor 2. Therefore,

$$\tilde{x}(t) = x(2t) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi^2 n^2} \left(1 - \cos(n\pi)\right) e^{jn\frac{\pi}{4}2t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{2}{\pi^2 n^2} \left(1 - \cos(n\pi)\right) e^{jn\frac{\pi}{2}t}.$$

As expected, the Fourier series coefficients $\tilde{D}_n$ for $\tilde{x}(t)$ exactly equal the Fourier series coefficients D_n for $x(t)$. That is,

$$\tilde{D}_n = D_n.$$

Notice, however, that the fundamental frequency $\tilde{\omega}_0 = \frac{\pi}{2}$ of $\tilde{x}(t)$ is double the fundamental frequency $\omega_0 = \frac{\pi}{4}$ of $x(t)$.

Solution 6.3-10

Periodic signal $x(t)$ is expressed as an exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

(a) Now, the exponential Fourier series of $x(t-T)$ is given as

$$\hat{x}(t) = x(t-T) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0(t-T)} = \sum_{n=-\infty}^{\infty} \left(D_n e^{-jn\omega_0 T}\right) e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \hat{D}_n e^{jn\omega_0 t}.$$

Clearly,

$$\hat{D}_n = D_n e^{-jn\omega_0 t} \quad \text{so that} \quad |\hat{D}_n| = |D_n|, \quad \text{and} \quad \angle \hat{D}_n = \angle D_n - jn\omega_0 T.$$

Thus, time-shifting of a periodic signal by T seconds merely changes the phase spectrum by $n\omega_0 T$. The amplitude spectrum is unchanged.

(b) Next, the exponential Fourier series of $x(at)$ is given as

$$\hat{x}(t) = x(at) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0(at)} = \sum_{n=-\infty}^{\infty} D_n e^{jn(a\omega_0)t}$$

Clear, the Fourier series of $x(at)$ is identical to the Fourier series of $x(t)$ except that frequency is scaled by a factor a. If $x(t)$ is compressed ($|a| > 1$), the Fourier spectra expands by the same factor a. If $x(t)$ is expanded ($|a| < 1$), then the Fourier spectra is compressed by the same factor a. This makes sense. Time compression makes a signal change faster (have higher frequency), while time expansion makes a signal change slower (have lower frequency).

Solution 6.3-11

(a) From Drill 6.1a,

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t, \qquad -1 \leq t \leq 1.$$

The power of $x(t)$ is

$$P_x = \frac{1}{2} \int_{-1}^{1} t^4 \, dt = \frac{1}{5}$$

Moreover, Parseval's theorem [Eq. (6.26)] states

$$P_x = C_0^2 + \sum_{1}^{\infty} \frac{C_n^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{\pi^2 n^2}\right)^2 = \frac{1}{9} + \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{9} + \frac{8}{90} = \frac{1}{5}.$$

Clearly Parseval's theorem holds.

(b) If the N-term Fourier series is denoted by $w(t)$, then

$$w(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{N-1} \frac{(-1)^n}{n^2} \cos n\pi t, \qquad -1 \leq t \leq 1.$$

To make the power of the error signal less that 1% of P_x, the power P_w is required to be at least $0.99 P_x = 0.198$. Therefore,

$$P_w = \frac{1}{9} + \frac{8}{\pi^4} \sum_{n=1}^{N-1} \frac{1}{n^4} \geq 0.198.$$

For $N = 1$, $P_w = 0.1111$, and for $N = 2$, $P_w = 0.19323$. For $N = 3$, $P_w = 0.19837$, which is greater than 0.198. Thus, $N = 3$.

Solution 6.3-12

(a) From Drill 6.1b

$$x(t) = \frac{2A}{\pi} (-1)^{n+1} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi t, \qquad -\pi \leq t \leq \pi.$$

The power of $x(t)$ is

$$P_x = \frac{1}{2} \int_{-1}^{1} (At)^2 \, dt = \frac{A^2}{3}.$$

Moreover, Parseval's theorem [Eq. (6.26)] states

$$P_x = C_0^2 + \sum_{1}^{\infty} \frac{C_n^2}{2} = \frac{1}{2} \sum_{1}^{\infty} \frac{4A^2}{\pi^2 n^2} = \frac{2A^2}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} = \frac{A^2}{3}.$$

Clearly Parseval's theorem holds.

(b) If the N-term Fourier series is denoted by $w(t)$, then

$$w(t) = \frac{2A}{\pi} (-1)^{n+1} \sum_{n=1}^{N} \frac{1}{n} \sin n\pi t, \qquad -\pi \leq t \leq \pi.$$

The power P_w is required to be no less than $0.90\frac{A^2}{3} = 0.3A^2$. Therefore,

$$P_w = \frac{1}{2}\sum_1^N \frac{4A^2}{\pi^2 n^2} \geq 0.3A^2.$$

For $N = 1$, $P_w = 0.2026A^2$, and for $N = 2$, $P_w = 0.2533A^2$. Continuing to $N = 5$, $P_w = 0.29658A^2$. For $N = 6$, $P_w = 0.30222A^2$, which is greater than $0.3A^2$. Thus, $N = 6$.

Solution 6.3-13

The power of a rectified sine wave is the same as that of a sine wave, that is, $1/2$. Thus, $P_x = 0.5$. Let the $2N + 1$ term truncated Fourier series be denoted as $\hat{x}(t)$. The power $P_{\hat{x}}$ is required to be no less than $0.9975P_x = 0.49875$. Using the Fourier series coefficients in Drill 6.5, we have

$$P_{\hat{x}} = \sum_{n=-N}^{N} |D_n|^2 = \frac{4}{\pi^2}\sum_{n=-N}^{N} \frac{1}{(1-4n^2)^2} \geq 0.49875.$$

Direct calculations using the above equation gives $P_{\hat{x}} = 4/\pi^2 = 0.4053$ for $N = 0$ (only dc), $P_{\hat{x}} = 0.49535$ for $N = 1$ (3 terms), and $P_{\hat{x}} = 0.49895$ for $N = 2$ (5 terms). Thus, a 5-term Fourier series yields a signal whose power is 99.79% of the power of the rectified sine wave. The power of the error in the approximation of $x(t)$ by $\hat{x}(t)$ is only 0.21% of the signal power P_x.

Solution 6.3-14

(a) An $\omega_0 = 2$ rad/s periodic signal $x_1(t)$ has Fourier series specturm $X_1[n]$. Using Fourier series properties,

$$x_1(t) \Longleftrightarrow X_1[n]$$
$$x_1(t-5) \Longleftrightarrow e^{-jn\omega_0 5}X_1[n]$$
$$x_1(-t-5) \Longleftrightarrow e^{jn\omega_0 5}X_1[-n]$$
$$\frac{1}{3}x_1(-t-5) \Longleftrightarrow \frac{1}{3}e^{jn\omega_0 5}X_1[-n]$$

Since $x_2(t) = \frac{1}{3}x_1(-t-5)$ and $\omega_0 = 2$, we therefore see that

$$X_2[n] = \frac{1}{3}e^{j10n}X_1[-n].$$

(b) An $\omega_0 = 2$ rad/s periodic signal $x_1(t)$ has Fourier series specturm $X_1[n]$. Using Fourier series properties,

$$x_1(t) \Longleftrightarrow X_1[n]$$
$$\frac{1}{2}e^{j10t}x_1(t) = \frac{1}{2}e^{j5\omega_0 t}x(t) \Longleftrightarrow \frac{1}{2}X_1[n-5]$$
$$\frac{1}{2}e^{-j10t}x_1(t) = \frac{1}{2}e^{-j5\omega_0 t}x(t) \Longleftrightarrow \frac{1}{2}X_1[n+5]$$
$$\frac{1}{2}\left(e^{j10t} + e^{-j10t}\right)x_1(t) = \cos(10t)x_1(t) \Longleftrightarrow \frac{1}{2}\left(X_1[n-5] + X_1[n+5]\right)$$

Since $x_2(t) = \cos(10t)x_1(t)$, we therefore see that

$$X_2[n] = \frac{1}{2}\left(X_1[n-5] + X_1[n+5]\right).$$

(c) An $\omega_0 = 3$ rad/s periodic signal $x_1(t)$ has Fourier series specturm $X_1[n]$. Using Fourier series properties,

$$x_1(t) \Longleftrightarrow X_1[n]$$
$$x_1(-t) \Longleftrightarrow X_1[-n]$$
$$x_1(t+2) \Longleftrightarrow e^{jn2\omega_0} X_1[n]$$
$$x_1(-t) - 3x_1(t+2) \Longleftrightarrow X_1[-n] - 3e^{jn2\omega_0} X_1[n]$$

Since $x_2(t) = x_1(-t) - 3x_1(t+2)$ and $\omega_0 = 3$, we therefore see that

$$X_2[n] = X_1[-n] - 3e^{j6n} X_1[n].$$

Solution 6.3-15

This problem defines a 2-periodic signal $x(t)$ as

$$x(t) = \begin{cases} -t^2 - t + 0.25 & -1 \le t < 0 \\ t^2 - t + 0.25 & 0 \le t < 1 \\ x(t+2) & \forall t \end{cases}$$

Since $x(t)$ is 2-periodic, we know that $\omega_0 = \pi$.

(a) We use MATLAB to plot $x(t)$ over $-2 \le t \le 2$.

```
>>  xT = @(t) (-t.^2-t+0.25).*((-1<=t)&(t<0))+(t.^2-t+0.25).*((0<=t)&(t<1));
>>  x = @(t) xT(mod(t+1,2)-1);
>>  t = -2:.001:2; plot(t,x(t),'k'); xlabel('t'); ylabel('x(t)');
>>  axis([-2 2 -.05 .55]); set(gca,'xtick',-2:2,'ytick',-.5:.25:.5); grid on
```

As shown in Fig. S6.3-15a, $x(t)$ looks quite similar to $\frac{1}{4} - \frac{1}{4}\sin(\pi t)$.
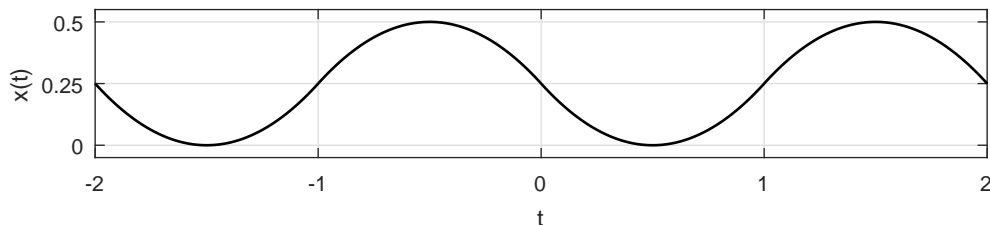
Figure S6.3-15a

(b) Next, we determine the dc content, D_0.

$$D_0 = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t}\, dt \bigg|_{n=0} = \frac{1}{2}\left[\int_{-1}^{0} \left(-t^2 - t + \frac{1}{4}\right) dt + \int_{0}^{1} \left(t^2 - t + \frac{1}{4}\right) dt \right]$$
$$= \frac{1}{2}\left[-\frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4}\bigg|_{-1}^{0} + \frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4}\bigg|_{0}^{1} \right] = \frac{1}{2}\left[-\frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right].$$

Thus,

$$D_0 = \frac{1}{4}.$$

(c) Over the period $-1 \leq t < 1$, we see that

$$\frac{d}{dt}\{x(t)\} = \begin{cases} -2t - 1 & -1 \leq t < 0 \\ 2t - 1 & 0 \leq t < 1 \end{cases}.$$

Differentiating again, we obtain

$$\frac{d^2}{dt^2}\{x(t)\} = \begin{cases} -2 & -1 \leq t < 0 \\ 2 & 0 \leq t < 1 \end{cases}.$$

Differentiating a third time, we obtain

$$\frac{d^3}{dt^3}\{x(t)\} = -4\delta(t+1) + 4\delta(t), \qquad -1 \leq t < 1.$$

From Ex. 6.9, the Fourier series coefficients of a 2-periodic impulse train is known to be $\frac{1}{T_0} = \frac{1}{2}$. Using this fact and appropriate Fourier series properties, we transform the expression for $\frac{d^3}{dt^3}\{x(t)\}$ to the frequency domain as

$$(jn\pi)^3 D_n = \frac{1}{2}\left(-4e^{jn\pi} + 4\right).$$

Thus,

$$D_n = \frac{2 - 2e^{jn\pi}}{(jn\pi)^3}, \qquad n \neq 0.$$

(d) We use MATLAB to plot D_n over $-10 \leq n \leq 10$.

```
>>  n = -10:10; Dn = (2-2*exp(j*n*pi))./((j*n*pi).^3); Dn(n==0) = 1/4;
>>  stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|'); grid on
>>  axis([-10 10 0 0.3]); set(gca,'xtick',-10:2:10,'ytick',[0:1/8:.3]);
```
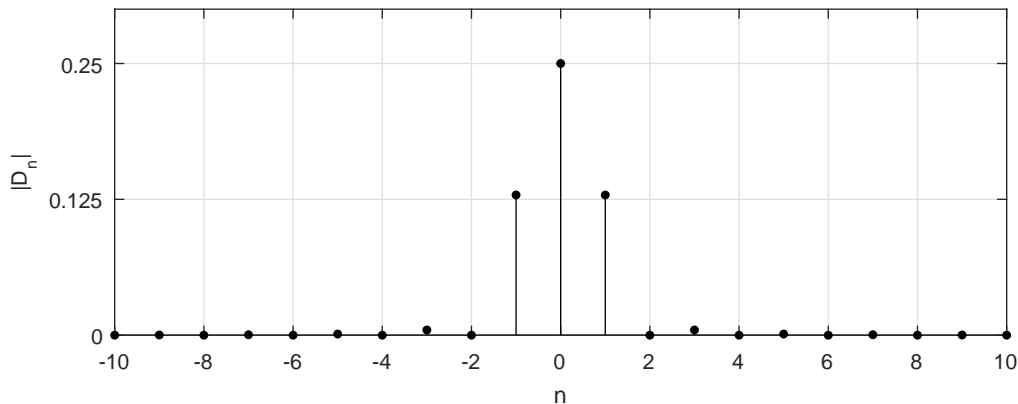

Figure S6.3-15c

(e) As shown in Fig. S6.3-15c, D_n is dominated by dc and first harmonic content. This is consistent with the observation of part (a) that $x(t)$ looks quite similar to $\frac{1}{4} - \frac{1}{4}\sin(\pi t)$. Thus, we expect D_n to have D_0 close to $\frac{1}{4}$ (it does exactly) and $|D_{\pm 1}|$ close to $\frac{1}{8}$ ($|D_{\pm 1}|$ is slightly greater than $\frac{1}{8}$). The 2-periodic signal $y(t) = \cos(\pi t)$ also has spectral content at the first harmonic ($\frac{1}{2}$ at $n = \pm 1$) but is zero everywhere else. The magnitude spectrum of $4x(t) - 1 \approx \sin(\pi t)$ would be nearly identical to the magnitude spectrum of $y(t) = \cos(\pi t)$.

Solution 6.3-16

This problem defines a 3-periodic signal $x(t)$ as

$$x(t) = \begin{cases} |t| & -1 \le t \le 1 \\ 0 & 1 < |t| \le 1.5 \\ x(t+3) & \forall t \end{cases}$$

Since $x(t)$ is 3-periodic, we know that $\omega_0 = \frac{2\pi}{3}$.

(a) We use MATLAB to plot $x(t)$ over $-3 \le t \le 3$ (see Fig. S6.3-16a).

```
>>  xT = @(t) abs(t).*((-1<=t)&(t<1)); x = @(t) xT(mod(t+1.5,3)-1.5);
>>  t = -3:.001:3; plot(t,x(t),'k'); xlabel('t'); ylabel('x(t)');
>>  axis([-3 3 -.1 1.1]); set(gca,'xtick',-3:3,'ytick',0:.5:1); grid on
```


Figure S6.3-16a

(b) Next, we determine the dc content, D_0.

$$D_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt = \frac{1}{3} \int_{-1}^{1} |t|\, dt = \frac{2}{3} \int_{0}^{1} t\, dt = \frac{2t^2}{6} \Big|_{0}^{1} = \frac{1}{3}.$$

(c) Let $y(t) = \frac{d}{dt}\{x(t)\}$. We obtain $y(t)$, shown left in Fig. S6.3-16b, by graphically differentiating Fig. S6.3-16a. Over $-1.5 \le t < 1.5$, we see that

$$y(t) = \delta(t+1) - \delta(t-1) + \underbrace{-u(t+1) + 2u(t) - u(t-1)}_{y_{\text{non}-\delta}(t)}.$$

Next, let

$$z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\} = -\delta(t+1) + 2\delta(t) - \delta(t-1).$$

Signal $z(t)$ is shown right in Fig. S6.3-16b.
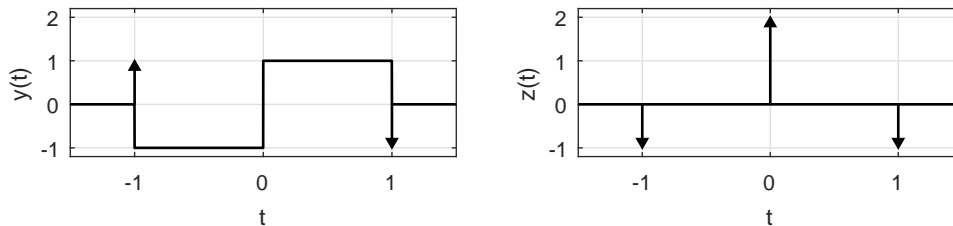
Figure S6.3-16b

From Ex. 6.9, the Fourier series coefficients of a 3-periodic impulse train is known to be $\frac{1}{T_0} = \frac{1}{3}$. Using this fact and appropriate Fourier series properties, we transform the expression for $z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\}$ to the frequency domain as

$$Z[n] = jn\omega_0 Y_{\text{non}-\delta}[n] = \frac{1}{3}\left(-e^{j\omega_0 n} + 2 - e^{-j\omega_0 n}\right).$$

Thus,

$$Y_{\text{non}-\delta}[n] = \frac{1}{3jn\omega_0}\left(-e^{j\omega_0 n} + 2 - e^{-j\omega_0 n}\right).$$

Similar transformation of $y(t) = \delta(t+1) - \delta(t-1) + y_{\text{non}-\delta}(t)$ yields

$$Y[n] = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{3} + \frac{1}{3jn\omega_0}\left(-e^{j\omega_0 n} + 2 - e^{-j\omega_0 n}\right).$$

Since $y(t) = \frac{d}{dt}\{x(t)\}$, it follows that the Fourier coefficients D_n of $x(t)$ are $D_n = \frac{1}{j\omega_0 n}Y[n]$. Thus, for $n \neq 0$,

$$D_n = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{3j\omega_0 n} + \frac{-e^{j\omega_0 n} + 2 - e^{-j\omega_0 n}}{-3n^2\omega_0^2}, \qquad \omega_0 = \frac{2\pi}{3}.$$

(d) We use MATLAB to plot D_n over $-10 \leq n \leq 10$.

```
>>  n = -10:10; omega0 = 2*pi/3;
>>  Dn = (exp(1j*omega0*n)-exp(-1j*omega0*n))./(3j*omega0*n)+...
>>      (-exp(1j*omega0*n)+2-exp(-j*omega0*n))./(-3*n.^2*omega0^2);
>>  Dn(n==0) = 1/3;
>>  stem(n,abs(Dn),'k.'); xlabel('n'); ylabel('|D_n|'); grid on
>>  axis([-10 10 0 0.35]); set(gca,'xtick',-10:2:10,'ytick',[0:.05:.3]);
```
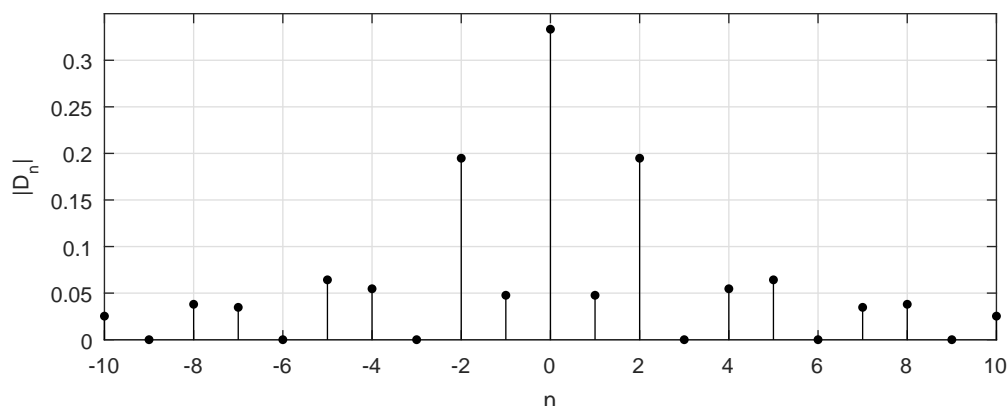

Figure S6.3-16d

From Fig. S6.3-16d, we see that the dc component ($n = 0$) is the most dominant component of this signal followed by the second harmonic ($n = \pm 2$).

Solution 6.4-1

The input shown in Fig. 6.2a has period $T_0 = \pi$ and $\omega_0 = 2$. From Drill 6.6, this signal has Fourier series coefficients

$$D_n = \frac{0.504}{1 + j4n}.$$

The system frequency response is

$$H(j\omega) = \frac{j\omega}{(-\omega^2 + 3) + j2\omega}.$$

Therefore,

$$y(t) = \sum_{n=-\infty}^{\infty} D_n H(jn\omega_0)e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{j1.08n}{(1 + j4n)(-4n^2 + 3 + j4n)}e^{j2nt}.$$

Solution 6.4-2

This problem considers the periodic signal $x(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t)$.

(a) To be periodic, $x(t)$ must equal $x(t+T_0)$ for all t. Since sinusoids are 2π-periodic, this requires that

$$5\pi T_0 = 2\pi k_1 \quad \text{and} \quad 14\pi T_0 = 2\pi k_2.$$

This requires that $T_0 = \frac{2k_1}{5} = \frac{k_2}{7}$ or that $14k_1 = 5k_2$, which is satisfied (smallest T_0) by choosing $k_1 = 5$ and $k_2 = 14$. Thus, $T = \frac{2(5)}{5} = 2$ and

$$\omega_0 = \frac{2\pi}{T_0} = \pi \quad \text{and} \quad f_0 = \frac{1}{T_0} = \frac{1}{2}.$$

(b) Using Euler's formula and $\omega_0 = \pi$, we write $x(t)$ as

$$x(t) = 1e^{j0\omega_0 t} + e^{j5\omega_0 t} + e^{-j5\omega_0 t} + \frac{3}{2j}e^{j14\omega_0 t} - \frac{3}{2j}e^{-j14\omega_0 t}.$$

By inspection, the exponential Fourier series coefficients D_n are

$$D_n = \begin{cases} 1 & n = 0, 5, -5 \\ \frac{3}{2j} & n = 14 \\ -\frac{3}{2j} & n = -14 \end{cases}.$$

In terms of Hertzian frequencies, we see that $x(t)$ has content at 0, ± 2.5, and ± 7 Hz.

(c) If $x(t)$ is applied to an ideal lowpass filter with cutoff frequency $f_c = 2$ Hz, the dc content is passed and the ± 2.5 and ± 7 Hz components are rejected. The output in this case is thus

$$y(t) = 1.$$

(d) If $x(t)$ is applied to an ideal highpass filter with cutoff frequency $f_c = 2$ Hz, the dc content is rejected and the ± 2.5 and ± 7 Hz components are passed. The output in this case is thus

$$y(t) = 2\cos(5\pi t) + 3\sin(14\pi t).$$

(e) An ideal bandpass filter with a 4 Hz passband centered at 4 Hz has passband from 2 to 6 Hz. If $x(t)$ is applied to this system, the ± 2.5 Hz is passed and the dc and ± 7 Hz components are rejected. Thus, the output is

$$y(t) = 2\cos(5\pi t).$$

(f) An ideal bandstop filter with a 5 Hz stopband centered at 10 Hz has stopband from 7.5 to 12.5 Hz. Since no components of $x(t)$ fall within this stopband, the system will not change $x(t)$ at all. That is, the output is

$$y(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t).$$

(g) There are infinitely many filters whose frequency response will produce the output $y(t) = 4\cos(5\pi t) - 9\sin(14\pi t)$ in response to input $x(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t)$. All such systems share the following characteristics.

System must: reject dc; pass ± 2.5 Hz with gain 2; pass ± 7 Hz with gain -3.

One way to achieve this behavior is to subtract the output of gain-3 bandpass filter with 1 Hz bandwidth centered at 7 Hz from the output of a gain-2 bandpass filter with 1 Hz bandwidth centered at 2.5 Hz. The frequency response of this (example) filter is shown in Fig. S6.4-2g.
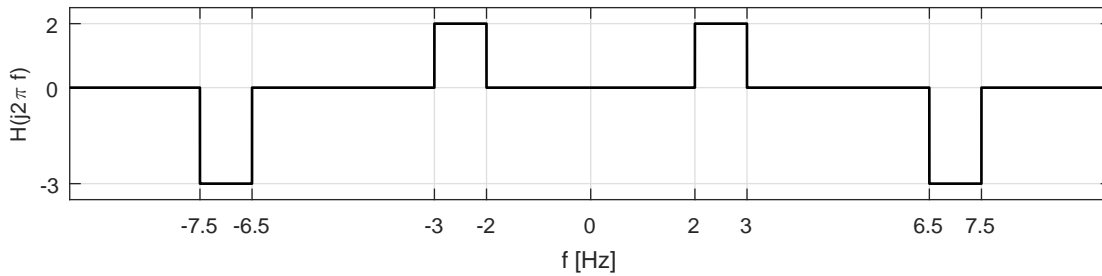
Figure S6.4-2g

Solution 6.4-3

This problem considers a $T_0 = 1$ periodic signal $x(t)$ defined as

$$x(t) = \begin{cases} 1 - t^2 & 0 < t \leq 1 \\ x(t+1) & \forall t \end{cases}$$

(a) MATLAB is well suited to plot $x(t)$ over $-2 \leq t \leq 2$ (see Fig. S6.4-3a).

```
>>  xT = @(t) (1-t.^2).*((t>=0)&(t<1)); x = @(t) xT(mod(t,1));
>>  t = -2:.001:2; plot(t,x(t),'k'); xlabel('t'); ylabel('x(t)');
>>  axis([-2 2 -.1 1.1]); set(gca,'xtick',-2:2,'ytick',0:.5:1); grid on
```
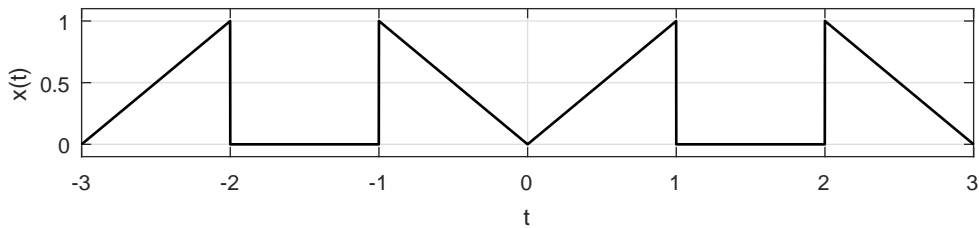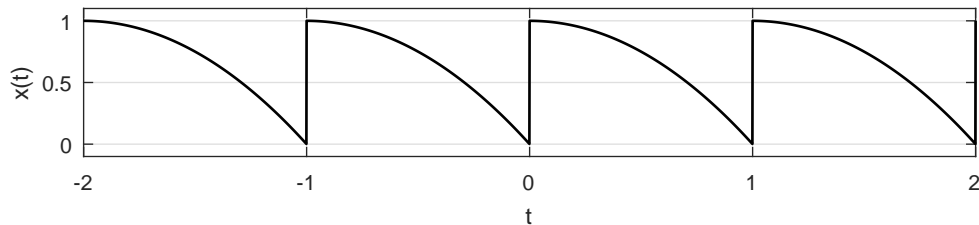

Figure S6.4-3a

(b) Let $y(t) = \frac{d}{dt}\{x(t)\}$. We obtain $y(t)$, shown left in Fig. S6.4-3b, by graphically differentiating Fig. S6.4-3a. Over $0 \leq t < 1$, we see that

$$y(t) = \delta(t) + \underbrace{-2t(u(t) - u(t-1))}_{y_{\text{non}-\delta}(t)}.$$

Letting $z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\}$, we see over $0 \leq t < 1$ that

$$z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\} = 2\delta(t) - 2.$$

Signal $z(t)$ is shown right in Fig. S6.4-3b.

From Ex. 6.9, the Fourier series coefficients of a 1-periodic impulse train is known to be $\frac{1}{T_0} = 1$. Using this fact and appropriate Fourier series properties, we transform the expression for $z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\}$ to the frequency domain as

$$Z[n] = jn\omega_0 Y_{\text{non}-\delta}[n] = 2 - 2\delta[n].$$

The $2\delta[n]$ term in unimportant since dc is computed separately, and it will be therefore ignored. Thus,

$$Y_{\text{non}-\delta}[n] = \frac{2}{jn\omega_0}.$$
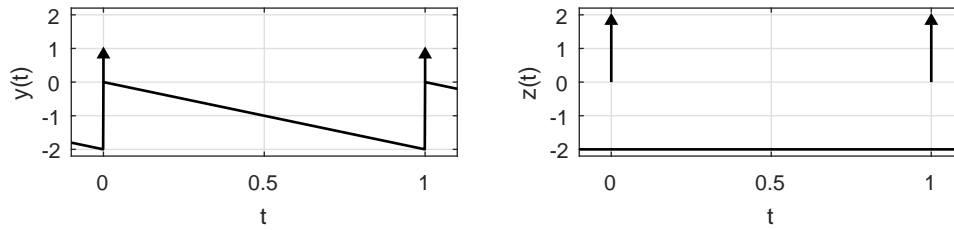
Figure S6.4-3b

Similar transformation of $y(t) = \delta(t) + y_{\text{non}-\delta}(t)$ yields

$$Y[n] = 1 + \frac{2}{jn\omega_0}.$$

Since $y(t) = \frac{d}{dt}\{x(t)\}$, it follows that the Fourier coefficients D_n of $x(t)$ are $D_n = \frac{1}{j\omega_0 n}Y[n]$. Thus, for $n \neq 0$,

$$D_n = \frac{1}{jn\omega_0} - \frac{2}{n^2\omega_0^2}, \qquad \omega_0 = 2\pi.$$

Next, we compute the dc portion.

$$D_0 = \int_0^1 (1 - t^2)\, dt = t - \frac{t^3}{3}\bigg|_0^1 = \frac{2}{3}.$$

Putting everything together, we obtain

$$D_n = \begin{cases} \frac{2}{3} & n = 0 \\ \frac{1}{j2\pi n} - \frac{1}{2n^2\pi^2} & n \neq 0 \end{cases}.$$

(c) Since $x(t)$ is 1-periodic, it has content at dc, 1 Hz, 2 Hz, 3 Hz, and so forth. An ideal bandpass filter with 1 Hz passband centered at 3 Hz will only let the 3 Hz components ($n = \pm 3$) through. Thus, the ideal bandpass filter output is

$$y(t) = H(j2\pi 3)D_3 e^{j2\pi 3t} + H(-j2\pi 3)D_{-3} e^{-j2\pi 3t}$$

$$= 1\left(\frac{1}{6\pi j} - \frac{1}{18\pi^2}\right)e^{j6\pi t} + 1\left(\frac{1}{6\pi j} - \frac{1}{18\pi^2}\right)e^{j6\pi t}$$

Simplifying, the final result is

$$y(t) = \frac{1}{3\pi}\sin(6\pi t) - \frac{1}{9\pi^2}\cos(6\pi t).$$

Solution 6.4-4

(a) Using trigonometric properties, we obtain

$$\cos 5t \sin 3t = \frac{1}{2}\left[\sin 8t - \sin 2t\right]$$

$$= \frac{1}{4j}\left[e^{j8t} - e^{-j8t} - e^{j2t} + e^{-j2t}\right]$$

$$= \frac{1}{4}\left[e^{j\pi/2}e^{-j8t} + e^{-j\pi/2}e^{-j2t} + e^{j\pi/2}e^{j2t} + e^{-j\pi/2}e^{j8t}\right]$$

$$= \frac{e^{j\pi/2}}{4}e^{-j4\omega_0 t} + \frac{e^{-j\pi/2}}{4}e^{-j\omega_0 t} + \frac{e^{j\pi/2}}{4}e^{j\omega_0 t} + \frac{e^{-j\pi/2}}{4}e^{j4\omega_0 t}.$$

This is the desired exponential Fourier series, where $\omega_0 = 2$.

(b) There are four non-zero spectral components, located at $n = \pm 1$ and ± 4 ($\omega = \pm 2$ and ± 8). The phases are either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, as shown in the spectrum in Fig. S6.4-4b.
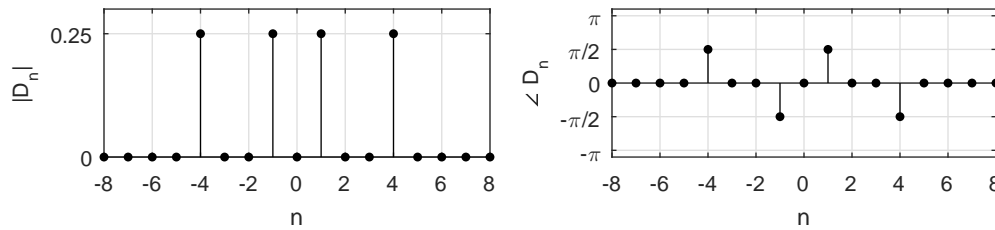
Figure S6.4-4b

(c) Since none of the spectral components of $x(t)$ appear in the pass-band of the filter, the output is $y(t) = 0$.

Solution 6.4-5

(a) For $x(t)$ in this problem, $\omega_0 = 2\pi$ and

$$D_n = \int_0^1 e^{-t}e^{-jn\omega_0 t}dt == \frac{e^{-1-jn\omega_0} - 1}{-1 - jn\omega_0} = \frac{1 - e^{-1}}{1 + j2\pi n}.$$

Hence, the exponential Fourier series of $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1 - e^{-1}}{1 + j2\pi n}e^{j2\pi nt}.$$

(b) The transfer function of the RC circuit is

$$H(j\omega) = \frac{1}{1 + (\frac{1}{j\omega})} = \frac{j\omega}{j\omega + 1}$$

Using $H(j\omega)$ and the exponential Fourier series of $x(t)$, the output $y(t)$ is given by

$$y(t) = \sum_{n=-\infty}^{\infty} D_n H(j2\pi n)e^{j2\pi nt}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1 - e^{-1}}{1 + j2\pi n}\right)\left(\frac{j2\pi n}{j2\pi n + 1}\right)e^{j2\pi nt}$$

$$= \sum_{n=-\infty}^{\infty} \frac{j2\pi n(1 - e^{-1})}{(1 + j2\pi n)^2}e^{j2\pi nt}.$$

Solution 6.4-6

In this problem, a T-periodic τ/T duty-cycle square wave $p(t)$ is defined as

$$p(t) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & \frac{\tau}{2} < |t| < \frac{T}{2} \\ p(t + T) & \forall t \end{cases},$$

where $0 < \tau < T$. Of interest is the frequency response $H(\omega)$ of a low-pass communications channel with 10 rad/s bandwidth (e.g., $|H(\omega)| \approx 0$ for $\omega > 10$. If T and τ are properly chosen, we can estimate $H(\omega)$ at points $\omega = n\omega_0$ as $\hat{H}(n\omega_0) = \frac{1}{P_0}Y_n$, where Y_n is the exponential FS spectrum of the channel output $y(t)$ in response to input $p(t)$ and P_0 is the dc component of $p(t)$.

(a) Here, we use direct integration to determine the exponential Fourier series coefficients P_n of signal $p(t)$. For $k = 0$ (the dc component),

$$P_0 = \frac{1}{T} \int_T p(t)\, dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} dt = \frac{\tau}{T}.$$

For $k \neq 0$,

$$
\begin{aligned}
P_k &= \frac{1}{T} \int_T p(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-jk\omega_0 t}\, dt \\
&= \frac{e^{-jk\omega_0 t}}{-jk\omega_0 T}\bigg|_{-\tau/2}^{\tau/2} = \frac{e^{-jk\omega_0\tau/2} - e^{jk\omega_0\tau/2}}{\frac{T}{\tau}k\omega_0\frac{\tau}{2}(-2j)} \\
&= \frac{\tau}{T}\left(\frac{sin(k\omega_0\tau/2)}{k\omega_0\tau/2}\right) = \frac{\tau}{T}\text{sinc}(k\omega_0\tau/2).
\end{aligned}
$$

Since $\omega_0 = \frac{2\pi}{T}$, we see that

$$
P_k = \begin{cases} \frac{\tau}{T}\text{sinc}\left(\frac{k\pi\tau}{T}\right) & k \neq 0 \\ \frac{\tau}{T} & k = 0 \end{cases}.
$$

(b) Next, we determine a value T so that $p(t)$ applied to system $H(\omega)$ has 21 component frequencies over the system bandwidth $0 \leq \omega \leq 10$. This requires a spacing of $\omega_0 = \frac{1}{2}$. Since $T = \frac{2\pi}{\omega_0}$, we see that a suitable value of T is

$$T = 4\pi.$$

(c) Assuming T is properly chosen, we next determine a suitable duty cycle τ/T so that $H(n\omega_0) \approx Y_n$. To this end, we desire P_k, whose shape follow $\text{sinc}(k\pi\tau/T)$, to be approximately flat over $0 \leq \omega \leq 10$ or, using the discussion in part (b), $0 \leq k \leq 20$. Given the nature of the sinc function, this requires the first zero crossing of P_k occur at approximately a $10\times$ higher frequency, or $k = 200$. The first zero crossing of P_k occurs at $k\pi\tau/T = \pi$. Substituting $k = 200$, we thus see that

a ratio $\frac{\tau}{T} = \frac{1}{200}$ should work nicely to ensure $H(n\omega_0) \approx Y_n$.

In fact, this ratio gives a maximum relative error of about 1.8%.

(d) From the perspective of using $p(t)$ to help measure the system frequency response $H(\omega)$, two primary things happen if T is properly chosen but τ/T is chosen too small.

- Pro: small $\frac{\tau}{T}$ flattens the sinc function for low frequencies, which ideally improves our ability to estimate the channel response over these frequencies.

- Con: small $\frac{\tau}{T}$ reduces the power of $p(t)$, which makes channel response estimates more susceptible to measurement noise.

(e) From the perspective of using $p(t)$ to help measure the system frequency response $H(\omega)$, two primary things happen if T is properly chosen but τ/T is chosen too large:

- Pro: large $\frac{\tau}{T}$ increases the power of $p(t)$, which makes channel response estimates less susceptible to measurement noise.

- Con: large $\frac{\tau}{T}$ makes the spectrum P_k roll off faster, which reduces the accuracy of channel response estimates, especially at higher frequencies.

Solution 6.5-1

We can find the minimum of $|\mathbf{e}|^2 = |\mathbf{x}|^2 + c^2|\mathbf{y}|^2 - 2c\mathbf{x}\cdot\mathbf{y}$ by differentiating with respect to c and setting the result equal to zero:

$$2c|\mathbf{y}|^2 = 2\mathbf{x}\cdot\mathbf{y}.$$

Solving for c, we obtain the desired result of

$$c = \frac{1}{|\mathbf{y}|^2}\mathbf{x}\cdot\mathbf{y}.$$

Solution 6.5-2

(a) Here,

$$e(t) = x(t) - cy(t).$$

Next, we consider the inner product between $y(t)$ and $e(t)$,

$$\int_{t_1}^{t_2} y(t)[x(t) - cy(t)]\, dt = \int_{t_1}^{t_2} y(t)x(t)\, dt - c\int_{t_1}^{t_2} y^2(t)\, dt.$$

But since c is chosen to minimize the error energy, we know from Eq. (6.32) that $c = \frac{\int_{t_1}^{t_2} x(t)y(t)\, dt}{\int_{t_1}^{t_2} y^2(t)\, dt}$. Substituting this expression for c into the expression of the inner product between $y(t)$ and $e(t)$ yields

$$\int_{t_1}^{t_2} y(t)[x(t) - cy(t)]\, dt = \int_{t_1}^{t_2} y(t)x(t)\, dt - c\int_{t_1}^{t_2} y^2(t)\, dt$$

$$= \int_{t_1}^{t_2} y(t)x(t)\, dt - \left(\frac{\int_{t_1}^{t_2} x(t)y(t)\, dt}{\int_{t_1}^{t_2} y^2(t)\, dt}\right)\int_{t_1}^{t_2} y^2(t)\, dt$$

$$= \int_{t_1}^{t_2} y(t)x(t)\, dt - \int_{t_1}^{t_2} y(t)x(t)\, dt = 0$$

Since $\int_{t_1}^{t_2} y(t)e(t)\, dt = 0$, $y(t)$ and $e(t) = x(t) - cy(t)$ are orthogonal.

(b) We can readily see the result in a signal-vector analogy using Fig. 6.21. The error vector $\mathbf{e}$ is orthogonal (at right angles) to vector $\mathbf{y}$.

(c) In this case,

$$e(t) = \begin{cases} 1 - \frac{4}{\pi}\sin t & 0 \le t \le \pi \\ -1 - \frac{4}{\pi}\sin t & \pi \le t \le 2\pi \end{cases}.$$

To show this error signal is orthogonal to $\sin(t)$, we compute the inner product between $e(t)$ and $c\sin(t)$ as

$$\int_0^{2\pi} e(t)c\sin(t)\, dt = c\int_0^\pi \left(1 - \frac{4}{\pi}\sin t\right)\sin(t)\, dt + c\int_\pi^{2\pi}\left(-1 - \frac{4}{\pi}\sin t\right)\sin(t)\, dt$$

$$= c\int_0^\pi \sin(t)\, dt - c\int_\pi^{2\pi}\sin(t)\, dt - \frac{4c}{\pi}\int_0^{2\pi}\sin^2(t)\, dt$$

$$= -c\cos(t)\Big|_0^\pi + c\cos(t)\Big|_\pi^{2\pi} - \frac{4c}{\pi}\left[\frac{t}{2} - \frac{\sin(2t)}{4}\right]\Big|_0^{2\pi}.$$

$$= c\left[-(-1-1)+1-(-1) - \frac{4}{\pi}(\pi - 0 - (0-0))\right] = 4c - 4c = 0.$$

Since the inner product is 0, $e(t)$ and $cy(t)$ are orthogonal.

Solution 6.5-3

For real constants c_1 and c_2, the energy of $c_1 x(t) \pm c_2 y(t)$ is

$$
E = \int_{-\infty}^{\infty} |c_1 x(t) \pm c_2 y(t)|^2 \, dt
$$

$$
= c_1^2 \int_{-\infty}^{\infty} |x(t)|^2 \, dt + c_2^2 \int_{-\infty}^{\infty} |y(t)|^2 \, dt \pm \int_{-\infty}^{\infty} c_1 x(t) c_2 y^*(t) \, dt \pm \int_{-\infty}^{\infty} c_2 y(t) c_1 x^*(t) \, dt
$$

$$
= c_1^2 E_x + c_2^2 E_y.
$$

The last result follows from the fact that because of orthogonality, the two integrals of the cross products $x(t) y^*(t)$ and $x^*(t) y(t)$ are zero. Clearly, the energies of $c_1 x(t) + c_2 y(t)$ and $c_1 x(t) - c_2 y(t)$ both equal $c_1^2 E_x + c_2^2 E_y$.

Solution 6.5-4

(a) In this case $E_y = \int_0^1 dt = 1$, and

$$
c = \frac{1}{E_y} \int_0^1 x(t) y(t) \, dt = \frac{1}{1} \int_0^1 t \, dt = 0.5.
$$

Thus, $x(t) \approx 0.5 y(t)$.

(b) For $x(t) \approx 0.5 y(t)$, the error is $e(t) = t - 0.5$ over $(0 \leq t \leq 1)$, and zero outside this interval. Also E_x and E_e (the energy of the error) are

$$
E_x = \int_0^1 x^2(t) \, dt = \int_0^1 t^2 \, dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 (t - 0.5)^2 \, dt = 1/12.
$$

The error $(t - 0.5)$ is orthogonal to $y(t)$ because

$$
\int_0^1 (t - 0.5)(1) \, dt = 0.
$$

Using all our previous calculations, we see that

$$
c^2 E_y + E_e = \frac{1}{4} + \frac{1}{12} = \frac{1}{3} = E_x.
$$

To explain these results in terms of vector concepts we observe from Fig. 6.21 that the error vector **e** is orthogonal to the component $c\mathbf{y}$. Because of this orthogonality, the length-square of **x** [energy of $x(t)$] is equal to the sum of the square of the lengths of $c\mathbf{y}$ and **e** [sum of the energies of $cy(t)$ and $e(t)$].

Solution 6.5-5

In this case $E_x = \int_0^1 x^2(t) \, dt = \int_0^1 t^2 \, dt = 1/3$, and

$$
c = \frac{1}{E_x} \int_0^1 y(t) x(t) \, dt = 3 \int_0^1 t \, dt = 1.5.
$$

Thus, $y(t) \approx 1.5 x(t)$. The error is $e(t) = y(t) - 1.5 x(t) = 1 - 1.5t$ over $(0 \leq t \leq 1)$ and is zero outside this interval. The energy of the error is $E_e = \int_0^1 (1 - 1.5t)^2 \, dt = 1/4$. As expected,

$$
c^2 E_x + E_e = \frac{9}{4} \left(\frac{1}{3} \right) + \frac{1}{4} = \frac{4}{4} = E_y.
$$

Solution 6.5-6

The trigonometric Fourier series of $x(t)$ (over 0 to 1 with $\omega_0 = 2\pi$) is

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nt + b_n \sin 2\pi nt \qquad \left(\omega_0 = \frac{2\pi}{1} \right),$$

where

$$a_0 = 1 \int_0^1 x(t)\, dt = \int_0^1 t\, dt = \frac{1}{2},$$

$$a_n = 2 \int_0^1 t \cos 2\pi nt\, dt = 0,$$

and

$$b_n = 2 \int_0^1 t \sin 2\pi nt\, dt = \frac{-1}{\pi n}.$$

Hence,

$$x(t) = \frac{1}{2} - \frac{1}{\pi} \left(\sin 2\pi t + \frac{1}{2} \sin 4\pi t + \frac{1}{3} \sin 6\pi t + \cdots \right) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi nt.$$

From Eq. (6.44)

$$E_e = \int_0^1 x^2(t)\, dt - \left[\left(\frac{1}{2} \right)^2 + \frac{1}{2} \left[\left(\frac{1}{\pi} \right)^2 + \left(\frac{1}{2\pi} \right)^2 + \cdots + \left(\frac{1}{(k-1)\pi} \right)^2 \right] \right]$$

Thus,

$$E_e = \int_0^1 t^2\, dt - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = 0.08333, \qquad N = 1$$

$$E_e = \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} = 0.03267, \qquad N = 2$$

$$E_e = \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} - \frac{1}{8\pi^2} = 0.02, \qquad N = 3$$

$$E_e = \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} - \frac{1}{8\pi^2} - \frac{1}{18\pi^2} = 0.014378, \qquad N = 4.$$

Solution 6.5-7

(a) Figure S6.5-7a shows $x_a(t)$ that is a periodic extension of $x(t)$ to yield a series with $\omega_0 = 2\pi$ and only sine terms. This requires $T_0 = 2\pi/2\omega = 1$ and odd symmetry. By inspection, the dc component is 0.5. If we subtract dc (0.5) from $x_a(t)$, the remaining signal $x_a(t) - 0.5$ has odd symmetry (only sine terms). Therefore,

$$x_a(t) = 0.5 + \sum_{n=1}^{\infty} b_n \sin 2\pi nt,$$

where

$$b_n = 2 \int_0^1 t \sin 2\pi nt\, dt = -\frac{1}{\pi n}.$$

That is,

$$x_a(t) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi nt.$$
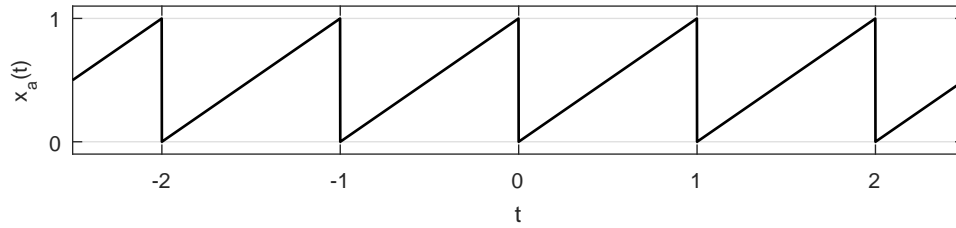
Figure S6.5-7a

(b) In this case, $\omega_0 = \pi$ and $T_0 = 2\pi/\pi = 2$. For sine terms only, we need odd symmetry. Figure S6.5-7b shows a suitable function $x_b(t)$. Since the dc component is zero, we see that

$$x_b(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t,$$

where

$$b_n = \frac{4}{2} \int_0^1 t \sin n\pi t \, dt = (-1)^{n+1} \frac{2}{n\pi}.$$
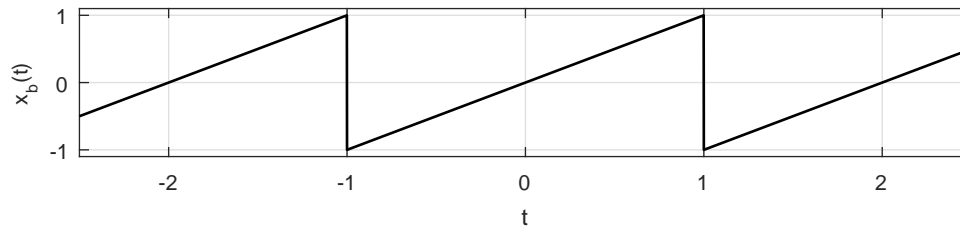
Figure S6.5-7b

(c) Here, $\omega_0 = \pi$ and $T_0 = 2\pi/\omega_0 = 2$. For cosine terms only, we need an even function. Figure S6.5-7c shows a suitable function $x_c(t)$. By inspection, the dc component is 0.5. Therefore,

$$x_c(t) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t,$$

where

$$a_n = \frac{4}{2} \int_0^1 t \cos n\pi t \, dt = -\frac{4}{\pi^2 n^2}, \quad n = 1,\ 3,\ 5,\ \ldots.$$
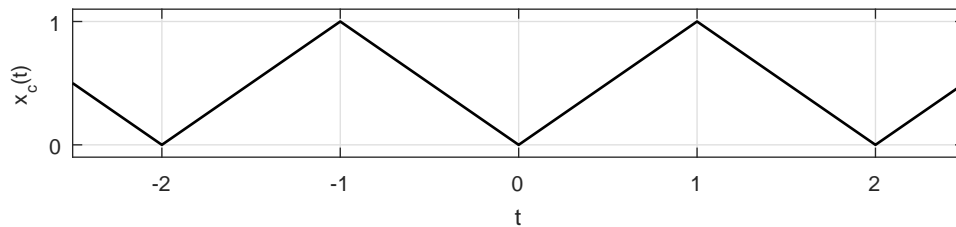
Figure S6.5-7c

Solution 6.5-8

(a) The signal $g(t)$ is the same as the signal $x(t)$ in Ex. 6.15 time-expanded by a factor π. Substituting t/π for t in the results of that example yields

$$g(t) = x\left(\frac{t}{\pi}\right) = -\frac{3}{2}\left(\frac{t}{\pi}\right) + \frac{7}{8}\left[\frac{5}{2}\left(\frac{t}{\pi}\right)^3 - \frac{3}{2}\left(\frac{t}{\pi}\right)\right] + \cdots$$

(b) Returning to the signal $x(t)$ in Ex. 6.15, we see that

$$\int_{-1}^{1} x^2(t)\,dt = 2.$$

From Eq. (6.44), we see that

$$E_e = \int x^2(t)\,dt - \frac{2}{3}c_1^2 = 2 - \frac{3}{2} = 0.5, \qquad N = 1,$$

and

$$E_e = \int x^2(t)\,dt - \frac{2}{3}c_1^2 - \frac{2}{7}c_3^2 = 0.28125, \qquad N = 2.$$

Since $g(t)$ is the same as $x(t)$ time-expanded by a factor π, all energies are increased by the same factor (π). Therefore,

$$E_e = 0.5\pi \text{ for } N = 1 \qquad \text{and} \qquad E_e = 0.28125\pi \text{ for } N = 2.$$

Solution 6.5-9

The 8-term Walsh Fourier series is given as

$$x(t) = c_0 x_0(t) + c_1 x_1(t) + \cdots + c_7 x_7(t).$$

The energy E_n of $x_n(t)$, for all $n = 1, 2, 3, \ldots, 8$, is given by

$$E_n = \int_0^1 x_n^2(t)\,dt = 1.$$

Hence

$$c_0 = \int_0^1 x(t)x_0(t)\,dt = \frac{1}{2}$$

$$c_1 = \int_0^1 x(t)x_1(t)\,dt = -\frac{1}{4}$$

$$c_2 = c_4 = c_5 = c_6 = 0$$

$$c_3 = \int_0^1 x(t)x_3(t)\,dt = -\frac{1}{8}$$

$$c_7 = \int_0^1 x(t)x_7(t)\,dt = -\frac{1}{16}$$

Hence

$$x(t) \simeq \frac{1}{2}x_0(t) - \frac{1}{4}x_1(t) - \frac{1}{8}x_3(t) - \frac{1}{16}x_7(t).$$

Also,

$$\int_0^1 x^2(t)\,dt = \frac{1}{3} \qquad \text{and} \qquad E_n = 1.$$

If $E_e(N)$ is the energy of the error signal in the approximation using first N terms, then from Eq. (6.44)

$$E_e(1) = \frac{1}{3} - c_0^2 = \frac{1}{12} = 0.0833$$

$$E_e(2) = \frac{1}{3} - c_0^2 - c_1^2 = \frac{1}{48} = 0.0204$$

$$E_e(3) = \frac{1}{3} - c_0^2 - c_1^2 - c_3^2 = \frac{1}{192} = 0.0052$$

$$E_e(4) = \frac{1}{3} - c_0^2 - c_1^2 - c_3^2 - c_7^2 = \frac{1}{768} = 0.001302$$

The error energies of the corresponding trigonometric Fourier series found in Prob. 6.5-6 are 0.0833, 0.03267, 0.02, 0.014378. Clearly, the Walsh Fourier series gives smaller error than the corresponding trigonometric Fourier series for the same number of terms in the approximation.

Solution 6.5-10

For the four-dimensional real space $\mathcal{R}^4$, the Walsh basis is given by: $\phi_1 = [1, 1, 1, 1]$, $\phi_2 = [1, 1, -1, -1]$, $\phi_3 = [1, -1, -1, 1]$, and $\phi_4 = [1, -1, 1, -1]$. For vectors x and y, the problem defines orthogonality as $\sum_{k=1}^{4} x_k y_k^* = 0$. Lastly, the problem defines vector $z = [-4, 0, 1, -7]$.

(a) Here, we show that the four Walsh basis functions are mutually orthogonal.

$$\sum_{k=1}^{4} \phi_1[k]\phi_2[k]^* = 1(1) + 1(1) + 1(-1) + 1(-1) = 0$$

$$\sum_{k=1}^{4} \phi_1[k]\phi_3[k]^* = 1(1) + 1(-1) + 1(-1) + 1(1) = 0$$

$$\sum_{k=1}^{4} \phi_1[k]\phi_4[k]^* = 1(1) + 1(-1) + 1(1) + 1(-1) = 0$$

$$\sum_{k=1}^{4} \phi_2[k]\phi_3[k]^* = 1(1) + 1(-1) - 1(-1) - 1(1) = 0$$

$$\sum_{k=1}^{4} \phi_2[k]\phi_4[k]^* = 1(1) + 1(-1) - 1(1) - 1(-1) = 0$$

$$\sum_{k=1}^{4} \phi_3[k]\phi_4[k]^* = 1(1) - 1(-1) - 1(1) + 1(-1) = 0$$

Since the inner products of ϕ_i and ϕ_j are zero for all $i \neq j$, the Walsh basis functions are mutually orthogonal.

(b) To see if the Walsh basis functions are normal, notice that

$$\sum_{k=1}^{4} \phi_i[k]\phi_i[k]^* = 1 + 1 + 1 + 1 = 4 \neq 1.$$

Since the inner product of any Walsh basis function with itself does not equal one, the Walsh basis functions are not normal.

(c) Here, we determine the coefficients $[c_1, c_2, c_3, c_4]$ to represent z using Walsh basis functions as $\hat{z} = \sum_{k=1}^{4} c_k \phi_k$. Since the basis functions ϕ_k are mutually orthogonal, the c_k coefficients are easy to compute.

$$c_1 = \frac{\sum_{k=1}^{4} x \phi_1[k]^*}{\sum_{k=1}^{4} \phi_1[k]\phi_1[k]^*} = \frac{-4 + 0 + 1 - 7}{4} = -2.5$$

$$c_2 = \frac{\sum_{k=1}^{4} x \phi_2[k]^*}{\sum_{k=1}^{4} \phi_2[k]\phi_2[k]^*} = \frac{-4 + 0 - 1 + 7}{4} = 0.5$$

$$c_3 = \frac{\sum_{k=1}^{4} x \phi_3[k]^*}{\sum_{k=1}^{4} \phi_3[k]\phi_3[k]^*} = \frac{-4 + 0 - 1 - 7}{4} = -3$$

$$c_4 = \frac{\sum_{k=1}^{4} x \phi_4[k]^*}{\sum_{k=1}^{4} \phi_4[k]\phi_4[k]^*} = \frac{-4 + 0 + 1 + 7}{4} = 1$$

Since $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ span $\mathcal{R}^4$, we know that $\hat{z} = z = [-4, 0, 1, -7]$ (no error).

(d) The best three-dimensional approximation $\hat{z}_{3D}$ to z is found by using the 3 largest coefficients from part (c). Thus,

$$\hat{z}_{3D} = -2.5\phi_1 - 3\phi_3 + \phi_4 = [-4.5, -0.5, 1.5, -6.5].$$

This estimate has the smallest (mean-square) error of any three-term Walsh approximation to z.

Solution 6.5-11

This problem considers the Laguerre expansion of signal $x(t)$, which is given by $x(t) = \sum_{k=0}^{\infty} c_k L_k(t)$, where $L_0(t) = 1$, $L_1(t) = (1 - t)$, ..., $L_k(t) = e^t \frac{d^k}{dt^k}\left(t^k e^{-t}\right)$.

(a) For $L_0(t) = 1$, we see that

$$\int_0^\infty e^{-t} L_0(t) L_0^*(t) dt = \int_0^\infty e^{-t} dt = -e^{-t}\big|_0^\infty = 0 - (-e^0) = 1.$$

Since $\int_0^\infty e^{-t} L_0(t) L_0^*(t) dt = 1$, we see that $L_0(t)$ is normal (unit size).

(b) For $L_1(t) = 1 - t$, we see that

$$\int_0^\infty e^{-t} L_1(t) L_1^*(t) dt = \int_0^\infty (1 - t)^2 e^{-t} dt = \int_0^\infty (1 - 2t + t^2) e^{-t} dt$$

$$= \int_0^\infty e^{-t} dt - 2\int_0^\infty t e^{-t} dt + \int_0^\infty t^2 e^{-t} dt.$$

From part (a), we know the first integral is 1. Using integration by parts, the second integral is

$$-2\int_0^\infty t e^{-t} dt = 2t e^{-t}\big|_0^\infty + 2\int_0^\infty e^{-t} dt = 0 - 0 + 2(0 - (-1)) = 2.$$

Using integration by parts as well as previous results, the third integral is

$$\int_0^\infty t^2 e^{-t} dt = -t^2 e^{-t}\big|_0^\infty + 2\int_0^\infty t e^{-t} dt = 0 - 0 - 2 = -2.$$

Combining these results, we see that

$$\int_0^\infty e^{-t} L_1(t) L_1^*(t) dt = \int_0^\infty e^{-t} dt - 2\int_0^\infty t e^{-t} dt + \int_0^\infty t^2 e^{-t} dt = 1 + 2 - 2 = 1.$$

Since $\int_0^\infty e^{-t} L_1(t) L_1^*(t) dt = 1$, we see that $L_1(t)$ is normal (unit size).

(c) Here,

$$\int_0^\infty e^{-t} L_0(t) L_1^*(t) dt = \int_0^\infty (1 - t) e^{-t} dt = \int_0^\infty e^{-t} dt - \int_0^\infty t e^{-t} dt.$$

Using the results from part (b), these integrals evaluate to

$$\int_0^\infty e^{-t} L_0(t) L_1^*(t) dt = 1 - \frac{1}{2}(2) = 0.$$

Since $\int_0^\infty e^{-t} L_0(t) L_1^*(t) dt = 0$, $L_0(t)$ is orthogonal to $L_1(t)$.

(d) Since the basis functions are orthonormal, we can compute the coefficient c_0 as the inner product between $x(t)$ and $L_0(t)$. That is,

$$c_0 = \int_0^\infty e^{-t} x(t) L_0(t) \, dt = \int_0^\infty e^{-t} e^{-t} \, dt = \left. \frac{-e^{-2t}}{-2} \right|_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.$$

Thus,

$$\hat{x}_0(t) = c_0 L_0(t) = \tfrac{1}{2} \text{ is the one-term Laguerre approximation of } x(t) = e^{-t} u(t).$$

(e) Because Laguerre basis functions $L_k(t)$ are mutually orthogonal, the coefficients have the *finality property*, which is to say that Laguerre expansion coefficients are each computed independently of one another. For the best estimate $\hat{x}_1(t) = c_0 L_0(t) + c_1 L_1(t)$ of the function $x(t) = e^{-t} u(t)$, coefficients c_0 and c_1 are determined independently. From part (d), we know that $c_0 = \frac{1}{2}$. To verify $c_1 = \frac{1}{4}$, we begin with

$$c_1 = \int_0^\infty e^{-t} x(t) L_1(t) \, dt = \int_0^\infty e^{-t} e^{-t} (1 - t) \, dt = \int_0^\infty e^{-2t} dt - \int_0^\infty t e^{-2t} dt.$$

From part (d), we know the first integral is $\frac{1}{2}$. Using integration by parts and previous integration results, the second integral is

$$-\int_0^\infty t e^{-2t} dt = \left. \frac{t e^{-2t}}{2} \right|_0^\infty - \int_0^\infty \frac{e^{-2t}}{2} dt = 0 - 0 - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{4}.$$

Combining these results, we see that

$$c_1 = \int_0^\infty e^{-2t} dt - \int_0^\infty t e^{-2t} dt = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

This is the result we sought to prove.

Solution 6.7-1

A periodic signal $x(t)$ has $\omega_0 = \frac{2}{3}\pi$ and exponential Fourier series spectrum $D_n = j\cos(\pi n/10)\,(u[n+10] - u[n-11])$.

(a) Since $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3}$, we see that

$$T_0 = 3.$$

(b) By inspection, we see that $D_n = -D_{-n}^*$, which is to say that the spectrum D_n is conjugate antisymmetric (skew-Hermitian). Using properties to express this relationship in the time domain, we see that

$$x(t) = -x^*(t), \text{ meaning that signal } x(t) \text{ is imaginary.}$$

(c) By inspection, we see that $D_n = D_{-n}$, which is to say that the spectrum D_n is even. Using properties to express this relationship in the time domain, we see that

$$x(t) = x(-t), \text{ meaning that signal } x(t) \text{ is even.}$$

(d) Since D_n is zero outside the finite interval $-10 \le n \le 10$, it is straightforward to reconstruct $x(t)$ exactly; no approximation is required. The resulting plot, shown in Fig. S6.7-1d, helps confirm that $x(t)$ is 3-periodic, imaginary, and even.

```
>>   T0 = 3; omega0 = 2*pi/T0; t = -T0:2*T0/2001:T0; x = zeros(size(t));
>>   D = @(n) 1j*cos(pi*n/10).*((n>=-10)&(n<=10));
>>   for n = -10:10, x = x+D(n)*exp(1j*omega0*n*t); end
>>   plot(t,imag(x),'k'); xlabel('t'); ylabel('Im\{x(t)\}'); grid on
>>   axis([-T0,T0,-4,12]); set(gca,'ytick',-4:4:12,'xtick',-T0:T0/2:T0);
```
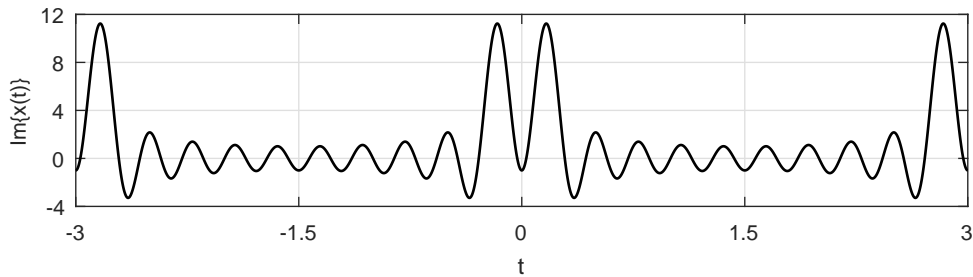

Figure S6.7-1d

Solution 6.7-2

A periodic signal $x(t)$ has $\omega_0 = \frac{3}{2}\pi$ and exponential Fourier series spectrum $D_n = 2\sin(\pi n/10)(u[n+10] - u[n-11])$.

(a) Since $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3\pi/2}$, we see that

$$T_0 = \frac{4}{3}.$$

(b) By inspection, we see that $D_n = -D_{-n}^*$, which is to say that the spectrum D_n is conjugate antisymmetric (skew-Hermitian). Using properties to express this relationship in the time domain, we see that

$$x(t) = -x^*(t), \text{ meaning that signal } x(t) \text{ is imaginary.}$$

(c) By inspection, we see that $D_n = -D_{-n}$, which is to say that the spectrum D_n is odd. Using properties to express this relationship in the time domain, we see that

$$x(t) = -x(-t), \text{ meaning that signal } x(t) \text{ is odd.}$$

(d) Since D_n is zero outside the finite interval $-10 \le n \le 10$, it is straightforward to reconstruct $x(t)$ exactly; no approximation is required. The resulting plot, shown in Fig. S6.7-2d, helps confirm that $x(t)$ is $\frac{4}{3}$-periodic, imaginary, and odd.

```
>>   T0 = 4/3; omega0 = 2*pi/T0; t = -T0:2*T0/2001:T0; x = zeros(size(t));
>>   D = @(n) 2*sin(pi*n/10).*((n>=-10)&(n<=10));
>>   for n = -10:10, x = x+D(n)*exp(1j*omega0*n*t); end
>>   plot(t,imag(x),'k'); xlabel('t'); ylabel('Im\{x(t)\}'); grid on
>>   axis([-T0,T0,-22,22]); set(gca,'ytick',-20:5:20,'xtick',-T0:T0/4:T0);
```
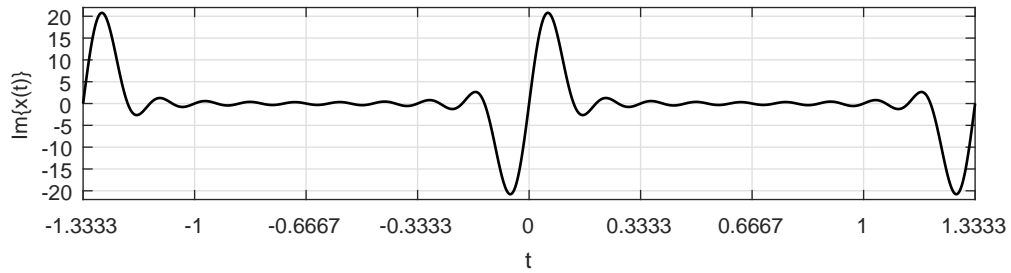
Figure S6.7-2d

Solution 6.7-3

This problem considers a $T_0 = 1$ periodic signal $x(t)$ defined as

$$x(t) = \begin{cases} 2t - t^2 & 0 < t \leq 1 \\ x(t+1) & \forall t \end{cases}.$$

(a) MATLAB is well suited to plot $x(t)$ over $-2 \leq t \leq 2$ (see Fig. S6.7-3a).

```
>>  xT = @(t) (2*t-t.^2).*((t>=0)&(t<1)); x = @(t) xT(mod(t,1));
>>  t = -2:.001:2; plot(t,x(t),'k'); xlabel('t'); ylabel('x(t)');
>>  axis([-2 2 -.1 1.1]); set(gca,'xtick',-2:2,'ytick',0:.5:1); grid on
```
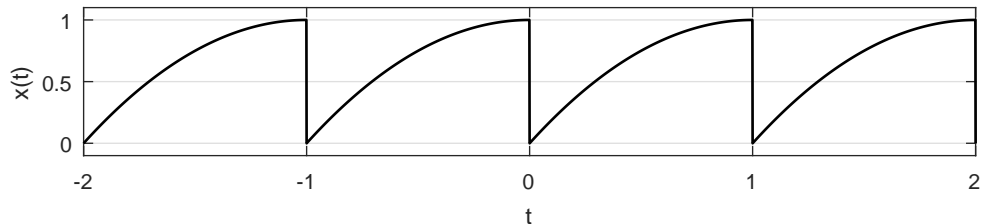

Figure S6.7-3a

(b) Let $y(t) = \frac{d}{dt}\{x(t)\}$. We obtain $y(t)$, shown left in Fig. S6.7-3b, by graphically differentiating Fig. S6.7-3a. Over $0 \leq t < 1$, we see that

$$y(t) = -\delta(t) + \underbrace{2 - 2t(u(t) - u(t-1))}_{y_{\text{non}-\delta}(t)}.$$

Letting $z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\}$, we see over $0 \leq t < 1$ that

$$z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\} = 2\delta(t) - 2.$$

Signal $z(t)$ is shown right in Fig. S6.7-3b.

From Ex. 6.9, the Fourier series coefficients of a 1-periodic impulse train is known to be $\frac{1}{T_0} = 1$. Using this fact and appropriate Fourier series properties, we transform the expression for $z(t) = \frac{d}{dt}\{y_{\text{non}-\delta}(t)\}$ to the frequency domain as

$$Z[n] = jn\omega_0 Y_{\text{non}-\delta}[n] = 2 - 2\delta[n].$$

The $2\delta[n]$ term in unimportant since dc is computed separately, and it will be therefore ignored. Thus,

$$Y_{\text{non}-\delta}[n] = \frac{2}{jn\omega_0}.$$
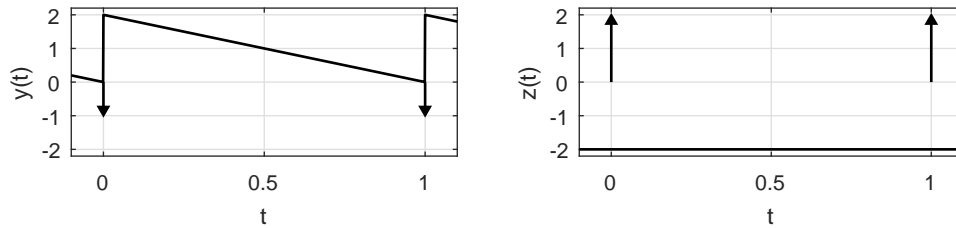
Figure S6.7-3b

Similar transformation of $y(t) = -\delta(t) + y_{\text{non}-\delta}(t)$ yields

$$Y[n] = -1 + \frac{2}{jn\omega_0}.$$

Since $y(t) = \frac{d}{dt}\{x(t)\}$, it follows that the Fourier coefficients D_n of $x(t)$ are $D_n = \frac{1}{j\omega_0 n}Y[n]$. Thus, for $n \neq 0$,

$$D_n = -\frac{1}{jn\omega_0} - \frac{2}{n^2\omega_0^2}, \qquad \omega_0 = 2\pi.$$

Next, we compute the dc portion.

$$D_0 = \int_0^1 (2t - t^2)\, dt = t^2 - \left.\frac{t^3}{3}\right|_0^1 = \frac{2}{3}.$$

Putting everything together, we obtain

$$D_n = \begin{cases} \frac{2}{3} & n = 0 \\ -\frac{1}{j2\pi n} - \frac{1}{2n^2\pi^2} & n \neq 0 \end{cases}.$$

(c) Next, we use MATLAB to synthesize $x(t)$ as $x_{10}(t)$ from the Fourier series coefficients D_n over $-10 \leq n \leq 10$. As shown in Fig. S6.7-3c, the synthesized signal $x_{10}(t)$ (solid line) is a close approximation to the true signal $x(t)$ (dashed line), thereby confirming the correctness of the Fourier series coefficients D_n derived in part (b).

```
>>  LSS3eSMMATLABFigFormat(6,1.25,10);
>>  xT = @(t) (2*t-t.^2).*((t>=0)&(t<1)); x = @(t) xT(mod(t,1));
>>  t = -2:.001:2; xhat = zeros(size(t)); T0 = 1; omega0 = 2*pi/T0; n = -10:10;
>>  Dn = -1./(2j*pi*n)-1./(2*n.^2*pi^2); Dn(n==0) = 2/3;
>>  for nval = n,
>>      xhat = xhat+Dn(nval==n)*exp(j*nval*omega0*t);
>>  end
>>  plot(t,real(xhat),'k',t,x(t),'k--'); xlabel('t'); ylabel('x_{10}(t)');
>>  axis([-2 2 -.1 1.1]); set(gca,'xtick',-2:2,'ytick',0:.5:1); grid on
```
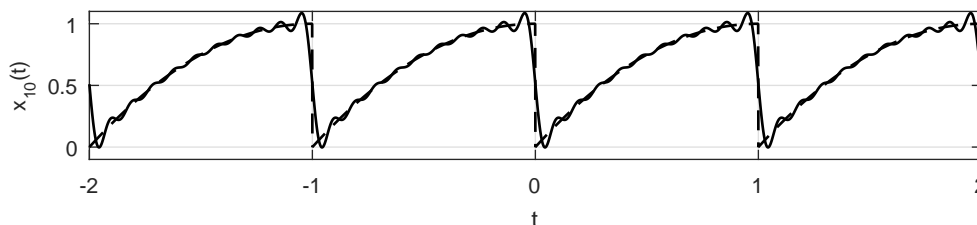

Figure S6.7-3c

(d) Since $x(t)$ is 1-periodic, it has content at dc, 1 Hz, 2 Hz, 3 Hz, and so forth. An ideal bandpass filter with 1 Hz passband centered at 3 Hz will only let the 3 Hz components $(n = \pm 3)$ through. Thus, the ideal bandpass filter output is

$$y(t) = H(j2\pi3)D_3 e^{j2\pi3t} + H(-j2\pi3)D_{-3} e^{-j2\pi3t}$$

$$= 1\left(-\frac{1}{6\pi j} - \frac{1}{18\pi^2}\right) e^{j6\pi t} + 1\left(-\frac{1}{6\pi j} - \frac{1}{18\pi^2}\right) e^{j6\pi t}$$

Simplifying, the final result is

$$y(t) = -\frac{1}{3\pi}\sin(6\pi t) - \frac{1}{9\pi^2}\cos(6\pi t).$$

Solution 6.7-4

(a) To determine a suitable set of $N = 10$ frequencies ω_n, we first determine ten points logarithmically spaced from 1 to 100.

```
>> N = 10; f = logspace(0,2,N)
   f =    1.0000    1.6681    2.7826    4.6416    7.7426
         12.9155   21.5443   35.9381   59.9484  100.0000
```

The problem with these points is that they are not all rational, and the resulting signal $m(t)$ is thus aperiodic. Truncating to the four decimal places shown makes the frequencies rational, but the resulting period T_0 is excessively long. An approximately logarithmic sequence that results in smaller T_0 is generated by rounding the logarithmic frequencies to the nearest tenths of a hertz.

```
>> f = round(10*logspace(0,2,N))/10
   f =    1.0000    1.7000    2.8000    4.6000    7.7000
         12.9000   21.5000   35.9000   59.9000  100.0000
```

With these frequencies, the signal $m(t) = \sum_{n=1}^{N} \cos(\omega_n t + \theta_n)$ has period $T_0 = 10$. Thus, one reasonable choice of frequencies is

$$\omega_n = 2\pi[1, 1.7, 2.8, 4.6, 7.7, 12.9, 21.5, 35.9, 59.9, 100] \text{ for which } m(t) \text{ has period } T_0 = 10.$$

MATLAB is used to plot $m(t)$ when all θ_n are set to zero.

```
>>  m = @(theta,t,omega) sum(cos(omega*t+theta*ones(size(t))));
>>  omega = 2*pi*f'; theta = zeros(size(omega));
>>  t = (-5:.01:5); plot(t,m(theta,t,omega),'k');
>>  xlabel('t [sec]'); ylabel('m(t) [volts]');
```

As expected, this worst-case version of $m(t)$ has a maximum amplitude of 10, which is also the number of sinusoids comprising the signal.

(b) MATLAB is used to try and find an optimal set of phases θ_n that minimizes the maximum amplitude of $m(t)$. The procedure followed is the same as that presented in Sec. 6.7. To proceed, the code from part ?? needs to be first executed.

```
>>  maxmagm = @(theta,t,omega) max(abs(sum(cos(omega*t+theta*ones(size(t))))));
>>  t = [-5:.001 :5]; rng(0); theta_init = 2*pi*rand(N,1);
>>  theta_opt = fminsearch(maxmagm,theta_init,[],t,omega);
>>  mmag = max(abs(m(theta_opt,t,omega)))
    mmag = 6.8734
```
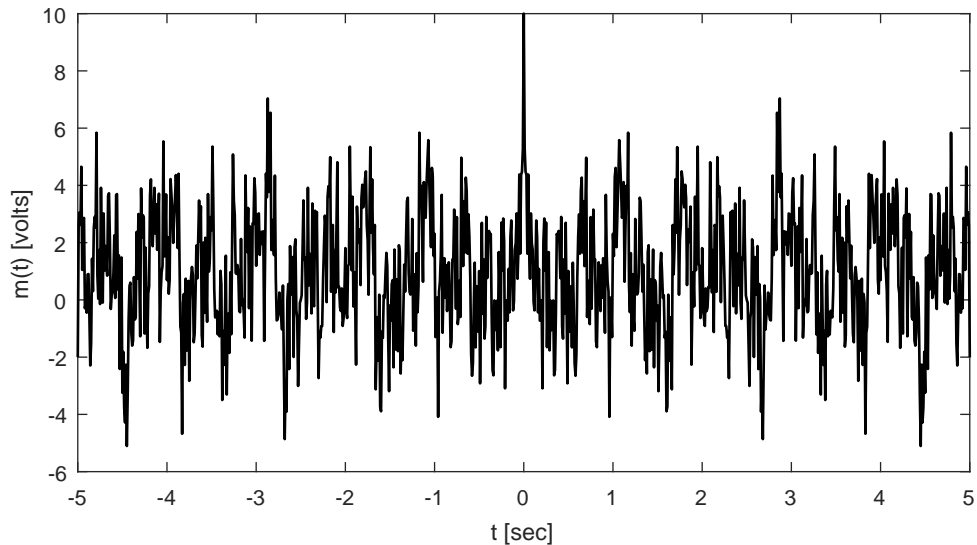
Figure S6.7-4a

The result of 6.8734 shows a reasonable reduction in maximum amplitude from the worst-case value of 10. Notice, a finely-spaced time vector `t` is required for the function `fminsearch` to determine a reliable result.

To make sure the result is good and not just a local minimum, the sequence is run again with a different initial guess for the phases.

```
>>  theta_init = 2*pi*rand(N,1);
>>  theta_opt = fminsearch(maxmagm,theta_init,[],t,omega);
>>  mmag = max(abs(m(theta_opt,t,omega)))
    mmag = 6.8049
```

Although the second result coincides well with the first, it is not exactly the same. To be safe, the sequence is therefore run several times, and the best solution is preserved.

```
>>  mmag_opt = mmag; mmag = [mmag,zeros(1,9)];
>>  for trial = 2:10;
>>      theta_init = 2*pi*rand(N,1);
>>      theta = fminsearch(maxmagm,theta_init,[],t,omega);
>>      mmag(trial) = max(abs(m(theta,t,omega)));
>>      if (mmag(trial)<mmag_opt),
>>          theta_opt = theta; mmag_opt = mmag(trial);
>>      end
>>  end
>>  mmag, theta_opt'
    mmag =      6.8049  6.6543  6.5756  6.4074  6.4805
                6.5585  6.7437  6.3758  6.5909  6.6296
    theta_opt = 2.0243  5.5845  4.0227  3.3761  5.5536
                1.7900  4.5281  4.9452  2.1278  3.8349
```

Thus, a good (but unlikely globally best) choice of phases is

$$\theta_n = [2.0243, 5.5845, 4.0227, 3.3761, 5.5536, 1.7900, 4.5281, 4.9452, 2.1278, 3.8349].$$

In this case, the maximum value of $m(t)$ is 6.3758, as shown in Fig. S6.7-4b.
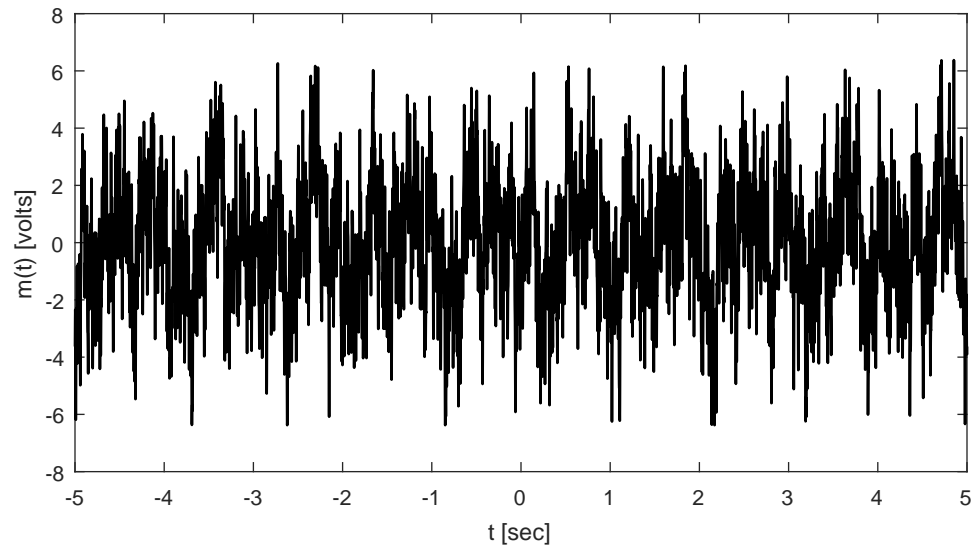
Figure S6.7-4b

```
>>  plot(t,m(theta_opt,t,omega),'k');
>>  xlabel('t [sec]'); ylabel('m(t) [volts]');
```

(c) For environments with $1/f$ noise, it is appropriate to have lower frequency components have greater strength than higher frequency components. One simple possibility is to adjust the magnitude of each sinusoidal component to match the noise power at that frequency. In this way, the signal-to-noise ratio is kept constant for any frequency bin of the signal.

$$m(t) = \sum_{n=1}^{N} \frac{k}{\sqrt{\omega_n}} \cos\left(\omega_n t + \theta_n\right).$$

The constant k is selected to achieve the final desired signal power for the entire signal $m(t)$.

Chapter 7 Solutions

Solution 7.1-1

Let us define a signal $z(t) = 2x(t - 1)$, which has Fourier transform $Z(\omega) = 2e^{-j\omega}X(\omega)$. In this way, $y(t)$ is a simple 3-periodic replication of $z(t)$,

$$y(t) = \sum_{n=-\infty}^{\infty} z(t - 3n).$$

In this case, as shown in Sec. 7.1, the Fourier series coefficients Y_k are just a scaled (by $\frac{1}{T_0}$) and sampled (by $k\omega_0$) version of the Fourier transform $Z(\omega)$. Using Eq. (7.5), we therefore see that

$$Y_k = \frac{1}{T_0} Z(k\omega_0).$$

Noting $T_0 = 3$ and $\omega_0 = \frac{2\pi}{3}$ and substituting for $Z(\omega)$, the final result is

$$Y_k = \frac{2}{3} e^{-j2\pi k/3} X\left(\frac{2\pi k}{3}\right).$$

Solution 7.1-2

From Eq. (7.10),

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|\, e^{j\angle X(\omega)} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} |X(\omega)| \cos[\omega t + \angle X(\omega)]\, d\omega + j \int_{-\infty}^{\infty} |X(\omega)| \sin[\omega t + \angle X(\omega)]\, d\omega \right].$$

As shown by Eq. (7.12), a real signal $x(t)$ has a conjugate-symmetric spectrum $X(\omega)$. Consequently, $|X(\omega)|$ is an even function and $\angle X(\omega)$ is an odd function of ω. Therefore, the integrand in the second integral is an odd function of ω, and the integral evaluates to zero. Moreover, the integrand in the first integral is an even function of ω, and therefore

$$x(t) = \frac{1}{\pi} \int_0^{\infty} |X(\omega)| \cos[\omega t + \angle X(\omega)]\, d\omega.$$

This form is very similar to the compact trigonometric Fourier series of Eq. (6.9), except integrals replace summations.

Solution 7.1-3

By definition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t) \cos \omega t\, dt - j \int_{-\infty}^{\infty} x(t) \sin \omega t\, dt.$$

If $x(t)$ is an even function of t, $x(t) \sin \omega t$ is an odd function of t, and the second integral vanishes. Moreover, $x(t) \cos \omega t$ is an even function of t, and the first integral is twice the integral over the interval 0 to ∞. Thus,

$$X(\omega) = 2 \int_0^\infty x(t) \cos \omega t \, dt, \qquad \text{when } x(t) \text{ is even.}$$

If $x(t)$ is also real (in addition to being even), the integral for $X(\omega)$ is real, and

$$X(-\omega) = 2 \int_0^\infty x(t) \cos \omega t \, dt = X(\omega)$$

Hence, if $x(t)$ is a real and even function of t, then $X(\omega)$ is a real and even function of ω.

Now if $x(t)$ is an odd function of t, $x(t) \cos \omega t$ is an odd function of t, and the first integral vanishes. Moreover, $x(t) \sin \omega t$ is an even function of t, and the second integral is twice the integral over the interval 0 to ∞. Thus,

$$X(\omega) = -2j \int_0^\infty x(t) \sin \omega t \, dt, \qquad \text{when } x(t) \text{ is odd.}$$

If $x(t)$ is also real (in addition to being odd), the integral for $X(\omega)$ is imaginary, and

$$X(-\omega) = -2j \int_0^\infty x(t) \sin(-\omega t) \, dt = 2j \int_0^\infty x(t) \sin(\omega t) \, dt = -X(\omega).$$

Hence, if $x(t)$ is a real and odd function of t, then $X(\omega)$ is an imaginary and odd function of ω.

Solution 7.1-4

(a) Because $x(t) = x_o(t) + x_e(t)$ and $e^{-j\omega t} = \cos \omega t + j \sin \omega t$,

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^\infty [x_o(t) + x_e(t)] e^{-j\omega t} \, dt \\
&= \int_{-\infty}^\infty [x_o(t) + x_e(t)] \cos \omega t \, dt - j \int_{-\infty}^\infty [x_o(t) + x_e(t)] \sin \omega t \, dt.
\end{aligned}$$

Because $x_e(t) \cos \omega t$ and $x_o(t) \sin \omega t$ are even functions and $x_o(t) \cos \omega t$ and $x_e(t) \sin \omega t$ are odd functions of t, these integrals reduce to

$$X(\omega) = 2 \int_0^\infty x_e(t) \cos \omega t \, dt - 2j \int_0^\infty x_o(t) \sin \omega t \, dt.$$

From Prob. 7.1-4, we know

$$X_e(\omega) = 2 \int_0^\infty x_e(t) \cos \omega t \, dt \quad \text{and} \quad X_o(\omega) = -2j \int_0^\infty x_o(t) \sin \omega t \, dt$$

Combining, we obtain the result for real $x(t)$ that

$$x_e(t) \iff \text{Re}[X(\omega)]$$

and

$$x_o(t) \iff j \, \text{Im}[X(\omega)].$$

(b) (i) We can express $u(t)$ in terms of its even and odd components as follows

$$u(t) = \frac{1}{2}[u(t) + u(-t)] + \frac{1}{2}[u(t) - u(-t)]$$

$$= \underbrace{\frac{1}{2}}_{x_e(t)} + \underbrace{\frac{1}{2}\mathrm{sgn}(t)}_{x_o(t)}$$

and

$$X_e(\omega) = \pi\delta(\omega) \quad \text{and} \quad X_o(\omega) = \frac{1}{j\omega}.$$

Clearly, $X_e(\omega)$ is the real part and $X_o(\omega)$ is the odd part of $X(\omega)$.

(ii) We follow the same procedure for $x(t) = e^{-at}u(t)$. First, we note that

$$e^{-at}u(t) = \underbrace{\frac{1}{2}[e^{-at}u(t) + e^{-at}u(-t)]}_{x_e(t)} + \underbrace{\frac{1}{2}[e^{-at}u(t) - e^{-at}u(-t)]}_{x_o(t)}.$$

Also

$$X_e(\omega) = \frac{1}{2}\left[\frac{1}{j\omega + a} - \frac{1}{j\omega - a}\right] = \frac{2a}{\omega^2 + a^2}$$

and

$$X_o(\omega) = \frac{1}{2}\left[\frac{1}{j\omega + a} + \frac{1}{j\omega - a}\right] = \frac{2j\omega}{\omega^2 + a^2}.$$

Clearly, $X_e(\omega)$ is the real part and $X_o(\omega)$ is the odd part of $X(\omega)$.

Solution 7.1-5

(a)

$$X(\omega) = \int_0^T e^{-at}e^{-j\omega t}\,dt = \int_0^T e^{-(j\omega+a)t}\,dt = \frac{1 - e^{-(j\omega+a)T}}{j\omega + a}$$

(b)

$$X(\omega) = \int_0^T e^{at}e^{-j\omega t}\,dt = \int_0^T e^{-(j\omega-a)}\,dt = \frac{1 - e^{-(j\omega-a)T}}{j\omega - a}$$

Solution 7.1-6

(a)

$$X(\omega) = \int_0^1 4e^{-j\omega t}\,dt + \int_1^2 2e^{-j\omega t}\,dt = \frac{4 - 2e^{-j\omega} - 2e^{-j2\omega}}{j\omega}$$

(b)

$$X(\omega) = \int_{-\tau}^0 -\frac{t}{\tau}e^{-j\omega t}\,dt + \int_0^\tau \frac{t}{\tau}e^{-j\omega t}\,dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1]$$

This result could also be derived by observing that $x(t)$ is an even function. Using the results in Prob. 7.1-4, we therefore see that

$$X(\omega) = \frac{2}{\tau}\int_0^\tau t\cos\omega t\,dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1].$$

Solution 7.1-7

(a)

$$x(t) = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t} \, d\omega = \frac{1}{2\pi} \frac{e^{j\omega t}}{(jt)^3} [-\omega^2 t^2 - 2j\omega t + 2] \Big|_{-\omega_0}^{\omega_0}$$

$$= \frac{(\omega_0^2 t^2 - 2) \sin \omega_0 t + 2\omega_0 t \cos \omega_0 t}{\pi t^3}$$

(b) The derivation can be simplified by observing that $X(\omega)$ can be expressed as a sum of two gate functions $X_1(\omega)$ and $X_2(\omega)$ as shown in Fig. S7.1-7b. Therefore,

$$x(t) = \frac{1}{2\pi} \int_{-2}^{2} [X_1(\omega) + X_2(\omega)] e^{j\omega t} \, d\omega = \frac{1}{2\pi} \left\{ \int_{-2}^{2} e^{j\omega t} \, d\omega + \int_{-1}^{1} e^{j\omega t} \, d\omega \right\}$$

$$= \frac{\sin 2t + \sin t}{\pi t}.$$
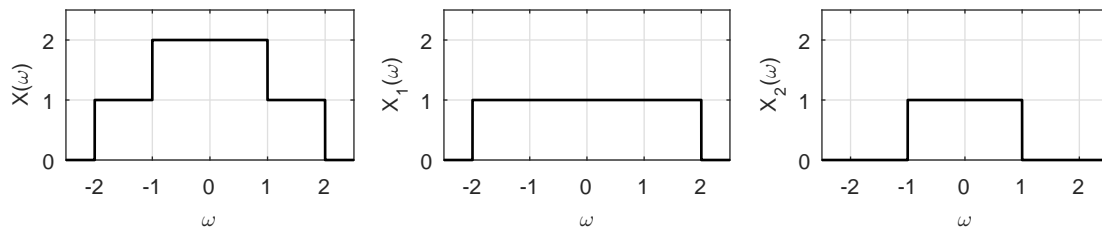
Figure S7.1-7b

Solution 7.1-8

(a)

$$x(t) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos \omega e^{j\omega t} \, d\omega$$

$$= \frac{e^{j\omega t}}{2\pi(1 - t^2)} (jt \cos \omega + \sin \omega) \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi(1 - t^2)} \cos \left(\frac{\pi t}{2} \right)$$

(b)

$$x(t) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} X(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi/2}^{\pi/2} X(\omega) \cos \omega t \, d\omega + j \int_{-\pi/2}^{\pi/2} X(\omega) \sin \omega t \, d\omega \right]$$

Because $X(\omega)$ is even function, the second integral on the right-hand side vanishes. Also the integrand of the first term is an even function. Therefore,

$$x(t) = \frac{1}{\pi} \int_{0}^{\pi/2} \frac{\omega}{\omega_0} \cos t\omega \, d\omega = \frac{1}{\pi\omega_0} \left[\frac{\cos t\omega + t\omega \sin t\omega}{t^2} \right] \Big|_{0}^{\omega_0}$$

$$= \frac{1}{\pi\omega_0 t^2} [\cos \omega_0 t + \omega_0 t \sin \omega_0 t - 1].$$

Solution 7.1-9

Since $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$, we see for $\omega = 0$ that

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt.$$

Since $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$, we see for $t = 0$ that

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega.$$

Because $\mathrm{sinc}\,(t) \leftrightarrow \pi\mathrm{rect}\left(\frac{\omega}{2}\right)$,

$$\pi\mathrm{rect}(0) = \pi = \int_{-\infty}^{\infty} \mathrm{sinc}(t)\, dt.$$

Since $\mathrm{sinc}^2(t) \leftrightarrow \pi\Delta\left(\frac{\omega}{4}\right)$, we see that

$$\pi\Delta(0) = \pi = \int_{-\infty}^{\infty} \mathrm{sinc}^2(t)\, dt.$$

Solution 7.2-1
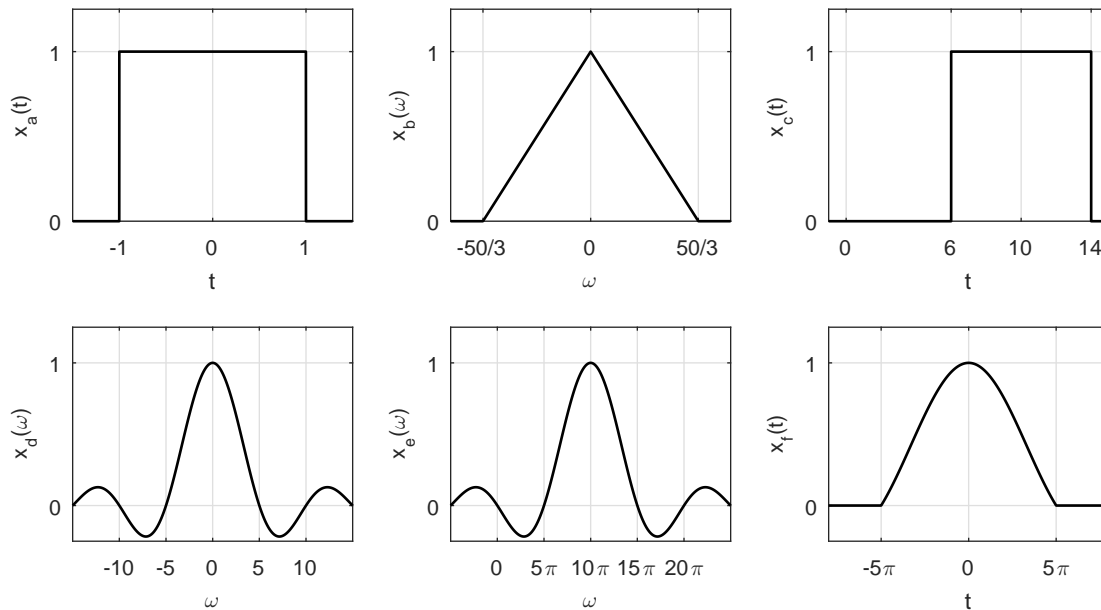
Figure S7.2-1

(a) As shown in Fig S7.2-1a, $x_a(t) = \mathrm{rect}\,(t/2)$ is a gate function centered at the origin and of width 2.

(b) As shown in Fig S7.2-1b, $x_b(\omega) = \Delta(3\omega/100)$ is a triangle function centered at the origin and of width $\frac{100}{3}$.

(c) As shown in Fig S7.2-1c, $x_c(t) = \mathrm{rect}\,((t-10)/8)$ is a gate function rect $\left(\frac{t}{8}\right)$ delayed by 10. In other words it is a gate pulse centered at $t = 10$ and of width 8.

(d) As shown in Fig S7.2-1d, $x_\mathrm{d}(\omega) = \mathrm{sinc}\,(\pi\omega/5)$ a sinc pulse centered at the origin and the first zero occurring at $\frac{\pi\omega}{5} = \pi$, that is at $\omega = 5$.

(e) As shown in Fig S7.2-1e, $x_\mathrm{e}(\omega) = \mathrm{sinc}\,((\omega/5) - 2\pi)$ is a sinc pulse $\mathrm{sinc}\,\left(\frac{\omega}{5}\right)$ delayed by 10π. For the sinc pulse $\mathrm{sinc}\,\left(\frac{\omega}{5}\right)$, the first zero occurs at $\frac{\omega}{5} = \pi$, that is at $\omega = 5\pi$. Therefore the function is a sinc pulse centered at $\omega = 10\pi$ and its zeros spaced at intervals of 5π.

(f) As shown in Fig S7.2-1f, $x_\mathrm{f}(t) = \mathrm{sinc}\,(t/5)\,\mathrm{rect}\,(t/10\pi)$ is a product of a gate pulse (centered at the origin) or width 10π and a sinc pulse (also centered at the origin) with zeros spaced at intervals of 5π. This results in the sinc pulse truncated beyond the interval $\pm 5\pi$ ($|t| \geq 5\pi$).

Solution 7.2-2

$$X(\omega) = \int_{4.5}^{5.5} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{4.5}^{5.5} = \frac{1}{j\omega}[e^{-j4.5\omega} - e^{-j5.5\omega}]$$

$$= \frac{e^{-j5\omega}}{j\omega}(e^{j\omega/2} - e^{-j\omega/2}) = \frac{e^{-j5\omega}}{j\omega}\left(2j\sin\frac{\omega}{2}\right)$$

$$= \mathrm{sinc}\left(\frac{\omega}{2}\right)e^{-j5\omega}$$

Figure S7.2-2 displays the resulting magnitude and phase spectra.
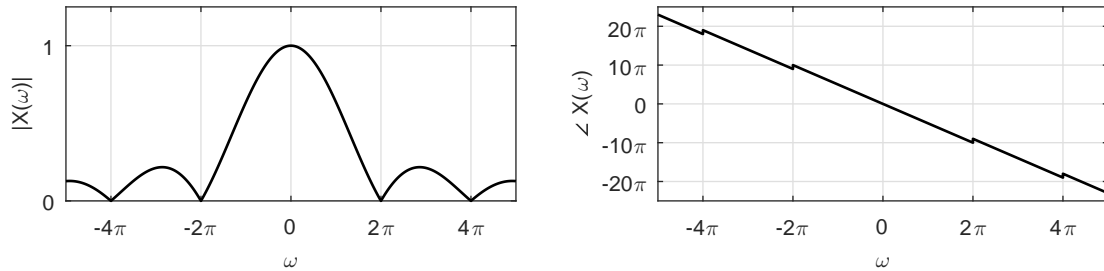
Figure S7.2-2

Solution 7.2-3

$$x(t) = \frac{1}{2\pi}\int_{10-\pi}^{10+\pi} e^{j\omega t}\, d\omega = \frac{e^{j\omega t}}{2\pi(jt)}\Big|_{10-\pi}^{10+\pi} = \frac{1}{j2\pi t}\left[e^{j(10+\pi)t} - e^{j(10-\pi)t}\right]$$

$$= \frac{e^{j10t}}{\pi t}[\sin \pi t] = \mathrm{sinc}(\pi t)e^{j10t}$$

Solution 7.2-4

(a)

$$x(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{-j\omega t_0} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{(2\pi)j(t-t_0)} e^{j\omega(t-t_0)}\Big|_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0(t-t_0)}{\pi(t-t_0)} = \frac{\omega_0}{\pi}\mathrm{sinc}[\omega_0(t-t_0)]$$

(b)

$$x(t) = \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} je^{j\omega t}\, d\omega + \int_{0}^{\omega_0} -je^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi t} e^{j\omega t}\Big|_{-\omega_0}^{0} - \frac{1}{2\pi t} e^{j\omega t}\Big|_{0}^{\omega_0} = \frac{1 - \cos\omega_0 t}{\pi t}$$

Solution 7.2-5

(a) When $a > 0$, we cannot find the Fourier transform of $e^{at}u(t)$ by setting $s = j\omega$ in the Laplace transform of $e^{at}u(t)$ because the ROC $\mathrm{Re}\{s\} > a$ does not include the ω-axis.

(b) In this case, the Laplace transform of $x(t)$ is

$$X(s) = \int_0^T e^{at} e^{-st} \, dt = \int_0^T e^{-(s-a)t} \, dt = \frac{1}{s-a} \left[1 - e^{-(s-a)T} \right].$$

Interestingly, because $x(t)$ has a finite duration, the ROC of $X(s)$ is the entire s-plane, which includes ω-axis. Hence, the Fourier transform is can be obtained from the Laplace transform as

$$X(\omega) = X(s)|_{s=j\omega} = \frac{1}{s-a} \left[1 - e^{-(j\omega-a)T} \right].$$

To verify this result, we directly find the Fourier transform of $x(t)$ as

$$X(\omega) = \int_0^T e^{at} e^{-j\omega t} \, dt = \int_0^T e^{-(j\omega-a)t} \, dt = \frac{1}{j\omega-a} \left[1 - e^{-(j\omega-a)T} \right].$$

Solution 7.3-1

(a) From Table 7.1,

$$\underbrace{u(t)}_{x(t)} \Longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{X(\omega)}.$$

Application of duality property yields

$$\underbrace{\pi\delta(t) + \frac{1}{jt}}_{X(t)} \Longleftrightarrow \underbrace{2\pi u(-\omega)}_{2\pi x(-\omega)}$$

or

$$\frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega).$$

Application of Eq. (7.27) yields

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega).$$

But $\delta(t)$ is an even function, that is $\delta(-t) = \delta(t)$, and

$$\frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right] \Longleftrightarrow u(\omega).$$

(b) From Table 7.1,

$$\underbrace{\cos\omega_0 t}_{x(t)} \Longleftrightarrow \underbrace{\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]}_{X(\omega)}.$$

Application of duality property yields

$$\underbrace{\pi[\delta(t+\omega_0) + \delta(t-\omega_0)]}_{X(t)} \Longleftrightarrow \underbrace{2\pi\cos(-\omega_0\omega)}_{2\pi x(-\omega)} = 2\pi\cos(\omega_0\omega).$$

Setting $\omega_0 = T$ yields

$$\delta(t+T) + \delta(t-T) \Longleftrightarrow 2\cos T\omega.$$

(c) From Table 7.1,

$$\underbrace{\sin \omega_0 t}_{x(t)} \Longleftrightarrow \underbrace{j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]}_{X(\omega)}.$$

Application of duality property yields

$$\underbrace{j\pi[\delta(t + \omega_0) - \delta(t - \omega_0)]}_{X(t)} \Longleftrightarrow \underbrace{2\pi \sin(-\omega_0\omega)}_{2\pi x(-\omega)} = -2\pi \sin(\omega_0\omega).$$

Setting $\omega_0 = T$ yields

$$\delta(t + T) - \delta(t - T) \Longleftrightarrow 2j \sin T\omega.$$

Solution 7.3-2

We can determine the desired Fourier transforms by applying various Fourier transform properties in sequence.

(a)

$$x(t) \Longleftrightarrow X(\omega) \qquad \text{(start)}$$

$$x(-2t) \Longleftrightarrow \frac{1}{2}X\left(-\frac{\omega}{2}\right) \qquad \text{(time scaling)}$$

$$x(-2(t - 3/2)) \Longleftrightarrow \frac{e^{-j3\omega/2}}{2}X\left(-\frac{\omega}{2}\right) \qquad \text{(time shifting)}$$

$$y(t) = \frac{1}{5}x(-2(t - 3/2)) \Longleftrightarrow \frac{e^{-j3\omega/2}}{10}X\left(-\frac{\omega}{2}\right) \qquad \text{(amplitude scaling)}$$

Thus,

$$Y(\omega) = \frac{e^{-j3\omega/2}}{10}X\left(-\frac{\omega}{2}\right).$$

(b)

$$x(t) \Longleftrightarrow X(\omega) \qquad \text{(start)}$$

$$x(-3t) \Longleftrightarrow \frac{1}{3}X\left(-\frac{\omega}{3}\right) \qquad \text{(time scaling)}$$

$$x^*(-3t) \Longleftrightarrow \frac{1}{3}X^*\left(\frac{\omega}{3}\right) \qquad \text{(conjugation)}$$

$$x^*(-3(t + 2)) \Longleftrightarrow \frac{e^{j2\omega}}{3}X^*\left(\frac{\omega}{3}\right) \qquad \text{(time shifting)}$$

$$y(t) = e^{2jt}x^*(-3(t + 2))) \Longleftrightarrow \frac{e^{j2(\omega-2)}}{3}X^*\left(\frac{\omega - 2}{3}\right) \qquad \text{(frequency shifting)}$$

Thus,

$$Y(\omega) = \frac{e^{j2(\omega-2)}}{3}X^*\left(\frac{\omega - 2}{3}\right).$$

Solution 7.3-3

We can determine the desired Fourier transforms by applying various Fourier transform properties in sequence.

(a)

$$x(t) \Longleftrightarrow X(\omega) \qquad \text{(start)}$$

$$x(-3t) \Longleftrightarrow \frac{1}{3}X\left(-\frac{\omega}{3}\right) \qquad \text{(time scaling)}$$

$$x(-3(t-2/3)) \Longleftrightarrow \frac{1}{3}e^{-j2\omega/3}X\left(-\frac{\omega}{3}\right) \qquad \text{(time shifting)}$$

$$y(t) = 4x(-3(t-2/3)) \Longleftrightarrow \frac{4}{3}e^{-j2\omega/3}X\left(-\frac{\omega}{3}\right) \qquad \text{(amplitude scaling)}$$

Thus,

$$y(t) = 4x(-3t+2).$$

(b)

$$x(t) \Longleftrightarrow X(\omega) \qquad \text{(start)}$$

$$x(-3t) \Longleftrightarrow \frac{1}{3}X\left(-\frac{\omega}{3}\right) \qquad \text{(time scaling)}$$

$$x^*(-3t) \Longleftrightarrow \frac{1}{3}X^*\left(\frac{\omega}{3}\right) \qquad \text{(conjugation)}$$

$$x^*(-3(t+2)) \Longleftrightarrow \frac{e^{j2\omega}}{3}X^*\left(\frac{\omega}{3}\right) \qquad \text{(time shifting)}$$

$$y(t) = e^{2jt}x^*(-3(t+2))) \Longleftrightarrow \frac{e^{j2(\omega-2)}}{3}X^*\left(\frac{\omega-2}{3}\right) \qquad \text{(frequency shifting)}$$

Thus,

$$y(t) = e^{2jt}x^*(-3t-6)).$$

Solution 7.3-4

From the problem statement, we know that

$$X(\omega) = \tfrac{1}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1).$$

(a) Inspecting Fig. P7.3-4, we know

$$x_1(t) = x(t+1) + x(-t+1).$$

Using the time-shift and time-reflection properties yields

$$X_1(\omega) = X(\omega)e^{j\omega} + X(-\omega)e^{-j\omega}.$$

(b) Inspecting Fig. P7.3-4, we know

$$x_2(t) = x(\tfrac{t}{2} + \tfrac{1}{2}) + x(-\tfrac{t}{2} + \tfrac{1}{2}).$$

Thus,

$$X_2(\omega) = 2X(2\omega)e^{j\omega} + 2X(-2\omega)e^{-j\omega}.$$

(c) Inspecting Fig. P7.3-4, we know

$$x_3(t) = x(\tfrac{t}{4} + \tfrac{1}{2}) + x(-\tfrac{t}{4} + \tfrac{1}{2}) + x(\tfrac{t}{2}) + x(-\tfrac{t}{2}).$$

Thus,

$$X_3(\omega) = 4X(4\omega)e^{j2\omega} + 4X(-4\omega)e^{-j2\omega} + 2X(2\omega) + 2X(-2\omega).$$

(d) Inspecting Fig. P7.3-4, we know

$$x_4(t) = \tfrac{4}{3}x(\tfrac{t}{2}+1) + \tfrac{4}{3}x(-\tfrac{t}{2}+1) - \tfrac{1}{3}x(\tfrac{t}{4}+\tfrac{1}{2}) - \tfrac{1}{3}x(-\tfrac{t}{4}+\tfrac{1}{2}).$$

Thus,

$$X_4(\omega) = \tfrac{8}{3}X(2\omega)e^{j2\omega} + \tfrac{8}{3}X(-2\omega)e^{-j2\omega} - \tfrac{4}{3}X(4\omega)e^{j2\omega} - \tfrac{4}{3}X(-4\omega)e^{-j2\omega}.$$

(e) Inspecting Fig. P7.3-4, we know

$$x_5(t) = x(t+\tfrac{1}{2}) + x(-t+\tfrac{1}{2}) + x(t+1.5) + x(-t+1.5).$$

Thus,

$$X_5(\omega) = X(\omega)e^{j\omega/2} + X(-\omega)e^{-j\omega/2} + X(\omega)e^{j1.5\omega} + (-\omega)e^{-j1.5\omega}.$$

Other forms are possible since the signal decompositions in terms of $x(t)$ are not unique. For example, we can represent $x_3(t)$ differently as $2\left[x(\tfrac{t}{4}+\tfrac{1}{2}) + x(-\tfrac{t}{2})\right]$. The evaluation of the expressions, however apparently different, must be the same.

Solution 7.3-5

Each case uses only the time-shifting property and Table 7.1 to find the Fourier transforms of the signals.

(a) Here,

$$x(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right).$$

From Table 7.1,

$$\text{rect}\left(\frac{t}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right).$$

Applying the time-shift property we obtain

$$\text{rect}\left(\frac{t\pm T/2}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)e^{\pm j\omega T/2}.$$

Thus,

$$X(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}]$$

$$= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2}$$

$$= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right).$$

(b) In this case,

$$x(t) = \sin t\, u(t) + \sin(t-\pi)u(t-\pi).$$

Now,

$$\sin t\, u(t) \Longleftrightarrow \tfrac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \tfrac{1}{1-\omega^2}$$

and

$$\sin(t-\pi)u(t-\pi) \Longleftrightarrow \left\{\tfrac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \tfrac{1}{1-\omega^2}\right\}e^{-j\pi\omega}.$$

Therefore

$$X(\omega) = \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\}(1+e^{-j\pi\omega}).$$

Recall that $g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0)$. Therefore, $\delta(\omega \pm 1)(1 + e^{-j\pi\omega}) = 0$, and

$$X(\omega) = \frac{1}{1 - \omega^2}(1 + e^{-j\pi\omega}).$$

(c) In this case,

$$x(t) = \cos t \left[u(t) - u\left(t - \frac{\pi}{2}\right)\right] = \cos tu(t) - \cos tu\left(t - \frac{\pi}{2}\right).$$

But $\sin\left(t - \frac{\pi}{2}\right) = -\cos t$. Therefore,

$$x(t) = \cos tu(t) + \sin\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right)$$

and

$$X(\omega) = \frac{\pi}{2}[\delta(\omega - 1) + \delta(\omega + 1)] + \frac{j\omega}{1 - \omega^2} + \left\{\frac{\pi}{2j}[\delta(\omega - 1) - \delta(\omega + 1)] + \frac{1}{1 - \omega^2}\right\}e^{-j\pi\omega/2}.$$

Also because $g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0)$,

$$\delta(\omega \pm 1)e^{-j\pi\omega/2} = \delta(\omega \pm 1)e^{\pm j\pi\omega/2} = \pm j\delta(\omega \pm 1).$$

Therefore,

$$X(\omega) = \frac{j\omega}{1 - \omega^2} + \frac{e^{-j\pi\omega/2}}{1 - \omega^2} = \frac{1}{1 - \omega^2}[j\omega + e^{-j\pi\omega/2}].$$

(d) Here,

$$x(t) = e^{-at}[u(t) - u(t - T)] = e^{-at}u(t) - e^{-at}u(t - T)$$
$$= e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t - T).$$

Thus,

$$X(\omega) = \frac{1}{j\omega + a} - \frac{e^{-aT}}{j\omega + a}e^{-j\omega T} = \frac{1}{j\omega + a}[1 - e^{-(a+j\omega)T}].$$

Solution 7.3-6

From the problem statement, we know that

$$\frac{\tau}{4\pi}\mathrm{sinc}^2\left(\frac{t\tau}{4}\right) \Longleftrightarrow \Lambda\left(\frac{\omega}{\tau}\right).$$

Setting $\tau = 1$ and using the duality property, we see that

$$y(t) = X(t) = \Lambda(t) \Longleftrightarrow Y(\omega)2\pi x(-\omega) = \frac{1}{2}\mathrm{sinc}^2\left(-\frac{\omega}{4}\right).$$

Since sinc is an even function, we obtain

$$Y(\omega) = \frac{1}{2}\mathrm{sinc}^2\left(\frac{\omega}{4}\right).$$

This result matches entry 19 of Table 7.1.

Solution 7.3-7

From time-shifting property

$$x(t \pm T) \Longleftrightarrow X(\omega)e^{\pm j\omega T}$$

Therefore

$$x(t + T) + x(t - T) \Longleftrightarrow X(\omega)e^{j\omega T} + X(\omega)e^{-j\omega T} = 2X(\omega)\cos\omega T$$

We can use this result to derive transforms of signals in Fig. P7.3-7.

(a) Define the gate pulse $x_\mathrm{a}(t)$ as shown in Fig. S7.3-7. From entry 17 of Table 7.1,

$$x_\mathrm{a}(t) = \mathrm{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\mathrm{sinc}(\omega).$$

Using $T = 3$, the signal in Fig. P7.3-7a is $x_\mathrm{a}(t+3) + x_\mathrm{a}(t-3)$, and

$$x_\mathrm{a}(t+3) + x_\mathrm{a}(t-3) \Longleftrightarrow 4\mathrm{sinc}(\omega)\cos 3\omega.$$

(b) Define the triangle pulse $x_\mathrm{b}(t)$ as shown in Fig. S7.3-7. From entry 19 of Table 7.1,

$$x_\mathrm{b}(t) = \Lambda\left(\frac{t}{2}\right) \Longleftrightarrow \mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

Using $T = 3$, the signal in Fig. P7.3-7b is $x_\mathrm{b}(t+3) + x_\mathrm{b}(t-3)$, and

$$x_\mathrm{b}(t+3) + x_\mathrm{b}(t-3) \Longleftrightarrow 2\mathrm{sinc}^2\left(\frac{\omega}{2}\right)\cos 3\omega.$$
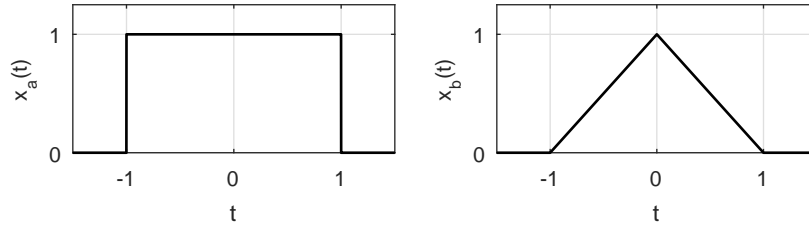
Figure S7.3-7

Solution 7.3-8

The frequency-shifting property states that $x(t)e^{\pm j\omega_0 t} \Longleftrightarrow X(\omega \mp \omega_0)$. Therefore,

$$x(t)\sin \omega_0 t = \frac{1}{2j}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}] = \frac{1}{2j}[X(\omega - \omega_0) + X(\omega - \omega_0)].$$

The time-shifting property states that $x(t \pm T) \Longleftrightarrow X(\omega)e^{\pm j\omega T}$. Therefore,

$$x(t+T) - x(t-T) \Longleftrightarrow X(\omega)e^{j\omega T} - X(\omega)e^{-j\omega T} = 2jX(\omega)\sin \omega T$$

and

$$\frac{1}{2j}[x(t+T) - x(t-T)] \Longleftrightarrow X(\omega)\sin T\omega.$$

The signal in Fig. S7.3-8 is $x(t+3) - x(t-3)$ where

$$x(t) = \mathrm{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\mathrm{sinc}(\omega).$$

Therefore,

$$x(t+3) - x(t-3) \Longleftrightarrow 2j[2\mathrm{sinc}(\omega)\sin 3\omega] = 4j\mathrm{sinc}(\omega)\sin 3\omega.$$

Solution 7.3-9

(a) The signal $x_a(t)$ in this case is a triangle pulse $\Lambda(\frac{t}{2\pi})$ multiplied by $\cos(10t)$. That is,

$$x_a(t) = \Lambda\left(\frac{t}{2\pi}\right)\cos(10t).$$

From pair 19 of Table 7.1 we know that $\Lambda(\frac{t}{2\pi}) \Longleftrightarrow \pi\operatorname{sinc}^2(\frac{\omega\pi}{2})$. Using the modulation property of Eq. (7.32), it follows that

$$x_a(t) = \Lambda\left(\frac{t}{2\pi}\right)\cos(10t) \Longleftrightarrow \frac{\pi}{2}\left\{\operatorname{sinc}^2\left[\pi\frac{\omega-10}{2}\right] + \operatorname{sinc}^2\left[\pi\frac{\omega+10}{2}\right]\right\} = X_a(\omega).$$

The Fourier transform in this case is a real and positive function and thus $X_a(\omega) = |X_a(\omega)|$, which is plotted using MATLAB and shown in Fig. S7.3-9.

```
>>  Xa = @(omega) pi/2*(sinc((omega-10)/2).^2+sinc((omega+10)/2).^2);
>>  omega = -20:.01:20; subplot(121); plot(omega,Xa(omega),'k'); grid on
>>  axis([-20 20 0 .6*pi]); xlabel('\omega'); ylabel('X_a(\omega)=|X_b(\omega)|');
>>  set(gca,'xtick',-20:10:20,'ytick',[0,pi/2],'yticklabel',{'0','\pi/2'});
```
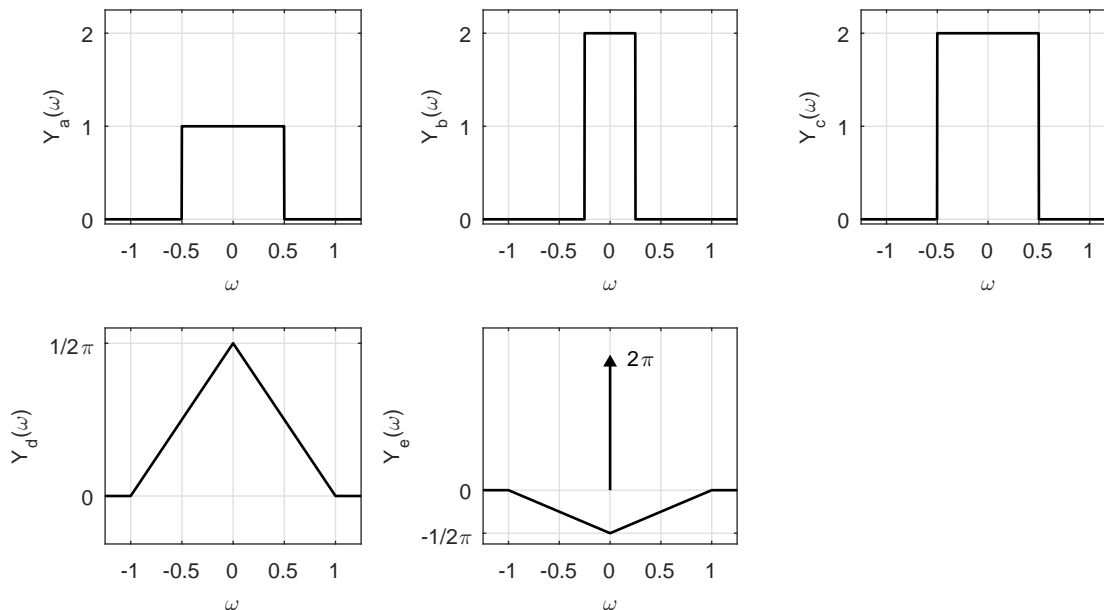

Figure S7.3-9

(b) The signal $x_b(t)$ is just $x_a(t)$ delayed by 2π. From time shifting property, its Fourier transform is the same as in part (a) multiplied by $e^{-j\omega(2\pi)}$. Therefore,

$$X_b(\omega) = X_a(\omega)e^{-j\omega(2\pi)} = \frac{\pi}{2}\left\{\operatorname{sinc}^2\left[\pi\frac{\omega-10}{2}\right] + \operatorname{sinc}^2\left[\pi\frac{\omega+10}{2}\right]\right\}e^{-j\omega(2\pi)}.$$

Since $X_b(\omega)$ differs from $X_a(\omega)$ only by a phase term, we see that $|X_b(\omega)| = |X_a(\omega)|$ (see Fig. S7.3-9). The multiplier $e^{-j2\pi\omega}$ represents a linear phase spectrum $\angle X_b(\omega) = -2\pi\omega$, which is plotted using MATLAB and is also shown in Fig. S7.3-9.

```
>>  Xb = @(omega) Xa(omega).*exp(-omega*2j*pi);
>>  subplot(122); plot(omega,unwrap(angle(Xb(omega)))+40*pi,'k'); grid on
>>  axis([-20 20 -40*pi 40*pi]); xlabel('\omega'); ylabel('\angle X_b(\omega)');
>>  set(gca,'xtick',-20:10:20,'ytick',[-40*pi:20*pi:40*pi]);
>>  set(gca,'yticklabel',{'-40\pi','-20\pi','0','20\pi','40\pi'});
```

Note: In the above solution, we first multiplied the triangle pulse $\Lambda(\frac{t}{2\pi})$ by $\cos(10t)$ and then delayed the result by 2π. This means the signal in (b) is expressed as $\Lambda(\frac{t-2\pi}{2\pi})\cos[10(t-2\pi)]$. Alternatively, we could have interchanged the operation in this particular case, that is, the triangle pulse $\Lambda(\frac{t}{2\pi})$ is first delayed by 2π and then the result is multiplied by $\cos(10t)$. In this alternate procedure, the signal in (b) is expressed as $\Lambda(\frac{t-2\pi}{2\pi})\cos(10t)$. This interchange of operation is permissible here only because the sinusoid $\cos(10t)$ executes integral number of cycles in the interval 2π. Because of this, both expressions are equivalent since $\cos[10(t-2\pi)] = \cos(10t)$.

(c) The signal $x_c(t)$ is identical to $x_b(t)$ except that the rectangular pulse $\Pi\left(\frac{t}{2\pi}\right)$ is applied instead of the triangular pulse $\Lambda(\frac{t}{2\pi})$. From pair 17 of Table 7.1 we know that

$$\mathrm{rect}\left(\frac{t}{2\pi}\right) \Longleftrightarrow 2\pi\,\mathrm{sinc}(\pi\omega).$$

Using the same argument as for part (b), we obtain

$$X_c(\omega) = \pi\{\mathrm{sinc}(\omega\pi + 10\pi) + \mathrm{sinc}(\omega\pi - 10\pi)\}e^{-j2\pi\omega}.$$

Solution 7.3-10

(a) Here,
$$X(\omega) = \mathrm{rect}\left(\frac{\omega-4}{2}\right) + \mathrm{rect}\left(\frac{\omega+4}{2}\right).$$

From pair 18 of Table 7.1 we know that

$$\tfrac{1}{\pi}\mathrm{sinc}(t) \Longleftrightarrow \mathrm{rect}\left(\frac{\omega}{2}\right).$$

Applying the frequency-shifting property, we see that

$$x(t) = \tfrac{2}{\pi}\mathrm{sinc}(t)\cos 4t.$$

(b) In this case,
$$X(\omega) = \Delta\left(\frac{\omega+4}{4}\right) + \Delta\left(\frac{\omega-4}{4}\right).$$

From pair 20 of Table 7.1 we know that

$$\tfrac{1}{\pi}\mathrm{sinc}^2(t) \Longleftrightarrow \Lambda(\tfrac{\omega}{4}).$$

Applying the frequency-shifting property, we see that

$$x(t) = \tfrac{2}{\pi}\mathrm{sinc}^2(t)\cos 4t$$

Solution 7.3-11

In this problem, $X(\omega) = \mathrm{rect}(\omega)$ is the Fourier transform of a signal $x(t)$.

(a) Represented in the frequency domain, $y_a(t) = x(t) * x(t)$ is

$$Y_a(\omega) = X(\omega) \cdot X(\omega) = \mathrm{rect}(\omega).$$

A sketch of $Y_a(\omega)$ is shown in Fig. S7.3-11a.

(b) Represented in the frequency domain, $y_{\mathrm{b}}(t) = x(t) * x(t/2)$ is

$$Y_{\mathrm{b}}(\omega) = X(\omega) \cdot 2X(2\omega) = 2\mathrm{rect}(2\omega).$$

A sketch of $Y_{\mathrm{b}}(\omega)$ is shown in Fig. S7.3-11b.

(c) Represented in the frequency domain, $y_{\mathrm{c}}(t) = 2x(t)$ is

$$Y_{\mathrm{c}}(\omega) = 2X(\omega) = 2\mathrm{rect}(\omega).$$

A sketch of $Y_{\mathrm{c}}(\omega)$ is shown in Fig. S7.3-11c.

(d) Represented in the frequency domain, $y_{\mathrm{d}}(t) = x^2(t)$ is

$$Y_{\mathrm{d}}(\omega) = \tfrac{1}{2\pi}X(\omega) * X(\omega) = \tfrac{1}{2\pi}\Lambda\left(\tfrac{\omega}{2}\right).$$

A sketch of $Y_{\mathrm{d}}(\omega)$ is shown in Fig. S7.3-11d.

(e) Represented in the frequency domain, $y_{\mathrm{e}}(t) = 1 - x^2(t)$ is

$$Y_{\mathrm{e}}(\omega) = 2\pi\delta(\omega) - \tfrac{1}{2\pi}X(\omega) * X(\omega) = 2\pi\delta(\omega) - \tfrac{1}{2\pi}\Lambda\left(\tfrac{\omega}{2}\right).$$

A sketch of $Y_{\mathrm{e}}(\omega)$ is shown in Fig. S7.3-11e.
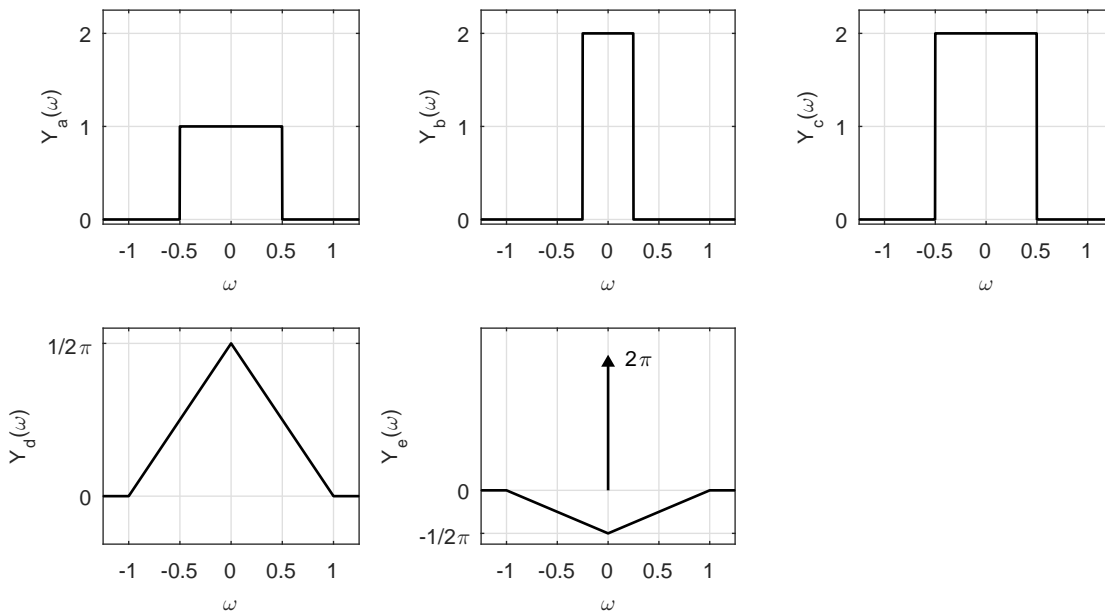
Figure S7.3-11

Solution 7.3-12

(a) Here,

$$e^{\lambda t}u(t) \iff \frac{1}{j\omega - \lambda} \quad \text{and} \quad u(t) \iff \pi\delta(\omega) + \frac{1}{j\omega}.$$

If $x(t) = e^{\lambda t}u(t)*u(t)$, then

$$X(\omega) = \left(\frac{1}{j\omega - \lambda}\right)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

$$= \frac{\pi\delta(\omega)}{j\omega - \lambda} + \left[\frac{1}{j\omega(j\omega - \lambda)}\right]$$

$$= -\frac{\pi}{\lambda}\delta(\omega) + \left[\frac{-\frac{1}{\lambda}}{j\omega} + \frac{\frac{1}{\lambda}}{j\omega - \lambda}\right] \qquad \text{because } g(x)\delta(x) = g(0)\delta(x)$$

$$= \frac{1}{\lambda}\left[\frac{1}{j\omega - \lambda} - \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right].$$

Taking the inverse transform of this equation yields

$$x(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)u(t).$$

(b) In this case,

$$e^{\lambda_1 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_2}.$$

If $x(t) = e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(t)$, then

$$X(\omega) = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_1} - \frac{\frac{1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_2}.$$

Therefore,

$$x(t) = \frac{1}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t})u(t).$$

(c) Here,

$$e^{\lambda_1 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(-t) \Longleftrightarrow \frac{1}{j\omega - \lambda_2}.$$

If $x(t) = e^{\lambda_1 t}u(t)*e^{\lambda_2 t}u(-t)$, then

$$X(\omega) = \frac{-1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_1} - \frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_2}.$$

Therefore,

$$x(t) = \frac{1}{\lambda_2 - \lambda_1}[e^{\lambda_1 t}u(t) + e^{\lambda_2 t}u(-t)].$$

Notice that $-\frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_2}$ inverts to $\frac{1}{\lambda_2 - \lambda_1}e^{\lambda_2 t}u(-t)$ and not $-\frac{1}{\lambda_2 - \lambda_1}e^{\lambda_2 t}u(t)$ because $\lambda_2 > 0$ (see pair 2 of Table 7.1).

(d) In this final case,

$$e^{\lambda_1 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_2}.$$

If $x(t) = e^{\lambda_1 t}u(-t)*e^{\lambda_2 t}u(-t)$, then

$$X(\omega) = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{-1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_1} - \frac{\frac{-1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_2}.$$

Therefore,

$$x(t) = \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} - e^{\lambda_2 t})u(-t).$$

The remarks at the end of part (c) apply here also.

Solution 7.3-13

From the frequency convolution property, we obtain

$$x^2(t) \iff \frac{1}{2\pi} X(\omega) * X(\omega)$$

Because of the width property of the convolution, the width of $X(\omega) * X(\omega)$ is twice the width of $X(\omega)$. Repeated application of this argument shows that the bandwidth of $x^n(t)$ is nB Hz(n times the bandwidth of $x(t)$).

Solution 7.3-14

(a) By direct integration, the Fourier transform is

$$X(\omega) = \int_{-T}^{0} e^{-j\omega t} dt - \int_{0}^{T} e^{-j\omega t} dt = -\frac{2}{j\omega}[1 - \cos \omega T] = \frac{j4}{\omega} \sin^2 \left(\frac{\omega T}{2} \right).$$

(b) To begin, we represent the signal as

$$x(t) = \text{rect} \left(\frac{t + T/2}{T} \right) - \text{rect} \left(\frac{t - T/2}{T} \right).$$

Using pair 17 of Table 7.1, we see that

$$\text{rect} \left(\frac{t}{T} \right) \iff T \text{sinc} \left(\frac{\omega T}{2} \right).$$

Applying the time shifting property yields

$$\text{rect} \left(\frac{t \pm T/2}{T} \right) \iff T \text{sinc} \left(\frac{\omega T}{2} \right) e^{\pm j\omega T/2}.$$

Putting everything together, the Fourier transform is

$$X(\omega) = T \text{sinc} \left(\frac{\omega T}{2} \right) [e^{j\omega T/2} - e^{-j\omega T/2}]$$

$$= 2jT \text{sinc} \left(\frac{\omega T}{2} \right) \sin \frac{\omega T}{2}$$

$$= \frac{j4}{\omega} \sin^2 \left(\frac{\omega T}{2} \right)$$

(c) Differentiating the signal yields

$$\frac{d}{dt} x(t) = \delta(t + T) - 2\delta(t) + \delta(t - T).$$

The Fourier transform of this equation yields

$$j\omega X(\omega) = e^{j\omega T} - 2 + e^{-j\omega T} = -2[1 - \cos \omega T] = -4 \sin^2 \left(\frac{\omega T}{2} \right).$$

Therefore,

$$X(\omega) = \frac{j4}{\omega} \sin^2 \left(\frac{\omega T}{2} \right).$$

Solution 7.3-15

(a) By definition,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{and} \qquad \frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Changing the order of differentiation and integration yields

$$\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} \frac{d}{d\omega}(x(t)e^{-j\omega t}) = \int_{-\infty}^{\infty} [-jtx(t)]e^{-j\omega t}dt.$$

The last part is clearly the Fourier transform of $-jtx(t)$. Thus,

$$-jtx(t) \iff \frac{d}{d\omega}X(\omega)$$

(b) From pair 1 of Table 7.1,

$$e^{-at}u(t) \iff \frac{1}{j\omega + a}.$$

Applying the results of part (a) yields

$$-jte^{-at}u(t) \iff \frac{d}{d\omega}\left(\frac{1}{j\omega + a}\right) = \frac{-j}{(j\omega + a)^2}.$$

Multiplying this result by j yields the desired transform pair,

$$te^{-at}u(t) \iff \frac{1}{(j\omega + a)^2}.$$

Solution 7.3-16

To begin, define

$$Y(\omega) = \frac{d}{d\omega}X(\omega) \qquad \text{and} \qquad Z(\omega) = \frac{d}{d\omega}Y(\omega) = \frac{d^2}{d\omega^2}X(\omega).$$

Both $Y(\omega)$ and $Z(\omega)$ are shown in Fig. S7.3-16.
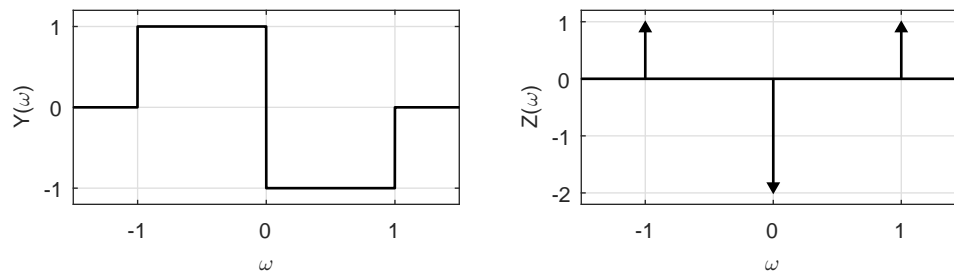
Figure S7.3-16

Using the frequency differentiation property, we see that

$$y(t) = -jtx(t) \qquad \text{and} \qquad z(t) = -jty(t) = -t^2x(t).$$

For $t \neq 0$, we therefore see that

$$x(t) = -\frac{z(t)}{t^2}.$$

Using $\delta(\omega) \Longleftrightarrow \frac{1}{2\pi}$ and the frequency-shifting property, we see that

$$Z(\omega) = \delta(\omega + 1) - 2\delta(\omega) + \delta(\omega - 1) \Longleftrightarrow \frac{1}{2\pi}\left(e^{-jt} - 2 + e^{jt}\right) = \frac{1}{\pi}\left(\cos(t) - 1\right) = z(t).$$

Using this result, we find $x(t)$ for $t \neq 0$ as

$$x(t) = \frac{1}{\pi t^2}\left(1 - \cos(t)\right).$$

For $t = 0$, we see that

$$x(0) = \frac{1}{2\pi}\int X(\omega)\,d\omega = \frac{1}{2\pi}\,(\text{area of } X(\omega)) = \frac{1}{2\pi}.$$

Thus,

$$x(t) = \begin{cases} \frac{1}{2\pi} & t = 0 \\ \frac{1}{\pi t^2}\left(1 - \cos(t)\right) & t \neq 0 \end{cases}.$$

Solution 7.3-17

As a preliminary step, using duality and $\delta(t) \Longleftrightarrow 1$, we know that $\frac{1}{2\pi} \Longleftrightarrow \delta(\omega)$. Next, define

$$Y(\omega) = \frac{d}{d\omega}X(\omega) \qquad \text{and} \qquad Z(\omega) = \frac{d}{d\omega}Y_1(\omega),$$

where $Y_1(\omega)$ is the non-delta part of $Y(\omega)$. Signals $X(\omega)$, $Y(\omega)$, and $Z(\omega)$ are all shown in Fig. S7.3-17.
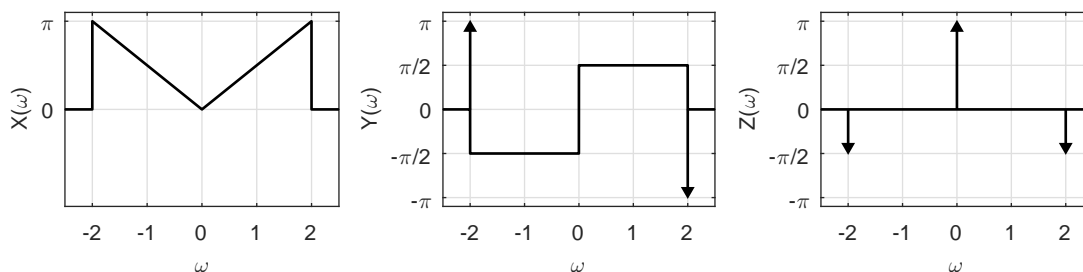
Figure S7.3-17

For $t \neq 0$, we know that

$$\frac{d}{d\omega}Y_1(\omega) = Z(\omega) = -\frac{\pi}{2}\delta(\omega + 2) + \pi\delta(\omega) - \frac{\pi}{2}\delta(\omega - 2).$$

Inverting, we obtain

$$-jty_1(t) = \frac{1}{2} - \frac{1}{4}\left(e^{j2t} + e^{-j2t}\right) \Longrightarrow y_1(t) = \frac{1}{j2t}\left(\cos(2t) - 1\right).$$

Next, we see that

$$\frac{d}{d\omega}X(\omega) = Y(\omega) = Y_1(\omega) + \pi\delta(\omega + 2) - \pi\delta(\omega - 2).$$

Inverting, we obtain

$$-jtx(t) = \frac{1}{j2t}\left(\cos(2t) - 1\right) + \frac{1}{2}\left(e^{-j2t} - e^{j2t}\right).$$

Solving for $x(t)$, we obtain

$$x(t) = \frac{1}{2t^2}\left(\cos(2t) - 1\right) + \frac{1}{t}\sin(2t), \qquad t \neq 0.$$

Since $\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\,d\omega = \frac{1}{2\pi}\cdot 2\cdot\frac{1}{2}\cdot 2\pi = 1$, we know that $x(0) = 1$. Putting everything together yields

$$x(t) = \begin{cases} 1 & t = 0 \\ \frac{1}{2t^2}\left(\cos(2t) - 1\right) + \frac{1}{t}\sin(2t) & t \neq 0 \end{cases}.$$

Solution 7.4-1

In the frequency domain, the zero-state response is $Y(\omega) = X(\omega)H(\omega)$, where the system frequency response is

$$H(\omega) = \frac{1}{j\omega + 1}.$$

(a) For input $x(t) = e^{-2t}u(t)$,

$$X(\omega) = \frac{1}{j\omega + 2}$$

$$Y(\omega) = \frac{1}{(j\omega + 1)(j\omega + 2)} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

(b) For input $x(t) = e^{-t}u(t)$,

$$X(\omega) = \frac{1}{j\omega + 1}$$

$$Y(\omega) = \frac{1}{(j\omega + 1)^2}$$

$$y(t) = te^{-at}u(t)$$

(c) For input $x(t) = e^{t}u(-t)$,

$$X(\omega) = \frac{1}{j\omega - 1}$$

$$Y(\omega) = \frac{-1}{(j\omega + 1)(j\omega - 1)} = \frac{1/2}{j\omega + 1} - \frac{1/2}{j\omega - 1}$$

$$y(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

(d) For input $x(t) = u(t)$,

$$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$Y(\omega) = \frac{1}{j\omega + 1}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \pi\delta(\omega) + \frac{1}{j\omega(j\omega + 1)} = \pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{j\omega + 1}$$

$$y(t) = (1 - e^{-t})u(t)$$

Solution 7.4-2

(a) Here,

$$X(\omega) = \frac{1}{j\omega + 1} \quad \text{and} \quad H(\omega) = \frac{-1}{j\omega - 2}.$$

Thus,

$$Y(\omega) = \frac{-1}{(j\omega - 2)(j\omega + 1)} = \frac{1}{3}\left[\frac{1}{j\omega + 1} - \frac{1}{j\omega - 2}\right]$$

and

$$y(t) = \frac{1}{3}[e^{-t}u(t) + e^{2t}u(-t)].$$

(b) In this case,

$$X(\omega) = \frac{-1}{j\omega - 1} \quad \text{and} \quad H(\omega) = \frac{-1}{j\omega - 2}.$$

Thus,

$$Y(\omega) = \frac{1}{(j\omega - 1)(j\omega - 2)} = \frac{-1}{j\omega - 1} - \frac{-1}{j\omega - 2}$$

and

$$y(t) = [e^t - e^{2t}]u(-t)].$$

Solution 7.4-3

(a) We begin by noting that

$$x(t) = \underbrace{1}_{\text{dc}} + \underbrace{2\cos(5\pi t)}_{2.5 \text{ Hz}} + \underbrace{3\sin(8\pi t)}_{4 \text{ Hz}}.$$

Given the underlying frequencies of 0, 2.5 and 4 Hz, the fundamental period is $T_0 = 2$ and the fundamental radian frequency is

$$\omega_0 = \frac{2\pi}{T_0} = \pi \text{rad/s}.$$

(b) Using Tables, the Fourier transform of $x(t) = 1 + 2\cos(5\pi t) + 3\sin(8\pi t)$ is easily shown as

$$X(\omega) = 2\pi\delta(\omega) + 2\pi\delta(\omega + 5\pi) + 2\pi\delta(\omega - 5\pi) + 3j\pi\delta(\omega + 8) - 3j\pi\delta(\omega - 8).$$

The magnitude spectrum $|X(\omega)|$ is shown in Fig. S7.4-3a.
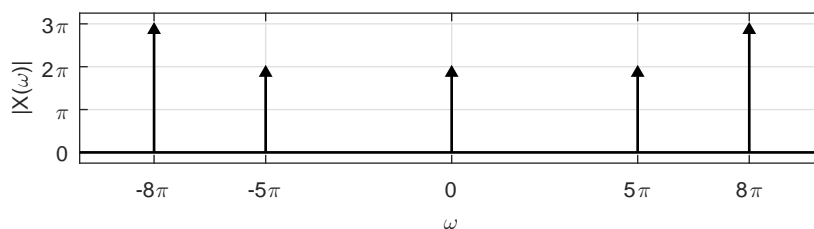
Figure S7.4-3a

(c) Using pair 18 of Table 7.1 and the time-scaling property, we see that

$$8\text{sinc}(4\pi t) \Longleftrightarrow 8\left(\frac{1}{4}\right)\text{rect}\left(\frac{\omega}{8\pi}\right) = 2\text{rect}\left(\frac{\omega}{8\pi}\right).$$

Using the frequency-shifting (modulation) property, we find that

$$h(t) = 8\text{sinc}(4\pi t)\cos(2\pi t) \Longleftrightarrow \text{rect}\left(\frac{\omega - 2\pi}{8\pi}\right) + \text{rect}\left(\frac{\omega + 2\pi}{8\pi}\right) = H(\omega).$$

Using this expression, we use MATLAB to plot the system magnitude response (see Fig. S7.4-3b). Notice that $H(\omega) = |H(\omega)|$.

```
>>  u = @(t) 1.0*(t>=0); rec = @(t) u(t+0.5)-u(t-0.5);
>>  H = @(w) rec((w-2*pi)/(8*pi))+rec((w+2*pi)/(8*pi));
>>  w = -10*pi:20*pi/10001:10*pi;
>>  plot(w,abs(H(w)),'k'); xlabel('\omega'); ylabel('|H(\omega)|');
>>  set(gca,'xtick',-10*pi:4*pi:10*pi,'ytick',0:2); grid on;
>>  set(gca,'xticklabel',{'-10\pi','-6\pi','-2\pi','2\pi','6\pi','10\pi'});
>>  axis([-10*pi 10*pi -.25 2.25]); grid on
```
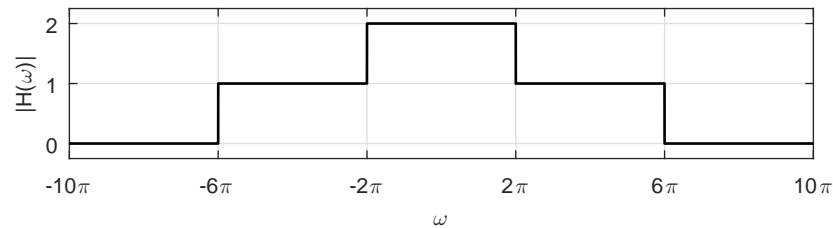

Figure S7.4-3b

(d) Distortionless systems have constant magnitude response, which Fig. S7.4-3b clearly shows is not true. Thus,

since $|H(\omega)|$ is not a constant, the system is not distortionless .

Since the magnitude response is constant over select frequency bands, however, it is possible to consider the system distortionless over those bands. For example, the system is distortionless over $0 \leq \omega < 2\pi$ and also over $2\pi < \omega < 6\pi$.

(e) By inspection of Figs S7.4-3a and S7.4-3b, we see that the 8π rad/s content is eliminated, the 5π content is passed with unit gain, and the dc component is amplified by 2. Thus,

$$y(t) = 2 + 2\cos(5\pi t).$$

Solution 7.4-4

(a) The signal $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - \pi n)$ is a periodic delta train with period $T_0 = \pi$. Thus,

$$\omega_0 = \frac{2\pi}{T_0} = 2.$$

(b) To find the Fourier transform of $x(t)$, we follow Ex. 7.8. First, the Fourier series coefficients of $x(t)$ are $D_n = \frac{1}{T_0} = \pi$. From Eq. (7.22), the Fourier transform of $x(t)$ is therefore

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$= 2 \sum_{n=-\infty}^{\infty} \delta(\omega - 2n).$$

Notice, the Fourier transform of a delta train is just another delta train!

(c) The system impulse response is given as $h(t) = \sin(3t)\mathrm{sinc}^2(t)$. From Table 7.1,

$$\sin(3t) \Longleftrightarrow j\pi\left[\delta(\omega + 3) - \delta(\omega - 3)\right] \qquad \text{and} \qquad \mathrm{sinc}^2(t) \Longleftrightarrow \pi\Delta\left(\frac{\omega}{4}\right).$$

Using the frequency convolution property, we find that

$$h(t) = \sin(3t)\text{sinc}^2(t) \iff \frac{1}{2\pi}\left[j\pi\delta(\omega+3) - j\pi\delta(\omega-3)\right] * \pi\Delta\left(\frac{\omega}{4}\right) = H(\omega).$$

Thus,

$$H(\omega) = \frac{j\pi}{2}\Delta\left(\frac{\omega+3}{4}\right) - \frac{j\pi}{2}\Delta\left(\frac{\omega-3}{4}\right).$$

Figure S7.4-4c shows the resulting magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$.
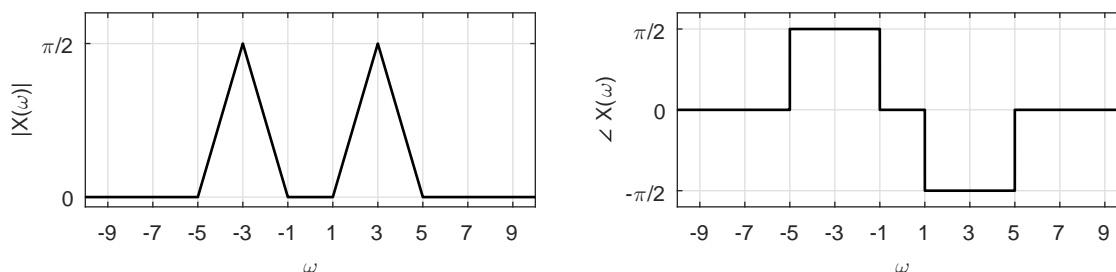
Figure S7.4-4c

(d) Distortionless systems have constant magnitude response and linear phase response. Figure S7.4-4c shows that the magnitude response is not constant. Thus,

since $|H(\omega)|$ is not a constant, the system is not distortionless .

(e) The input $x(t)$ is comprised of component frequencies $\omega = 0, \pm2, \pm4, \pm6, \ldots$ rad/s. Inspection of Fig. S7.4-4c confirms that only the components at $\omega = \pm2$ and $\omega = \pm4$ will pass through the system (with gains $\frac{\pi}{4}$ and phases $\pm\frac{\pi}{2}$). In the frequency domain, the output is

$$Y(\omega) = X(\omega)H(\omega) = 2\delta(w+4)\frac{j\pi}{4} + 2\delta(w+2)\frac{j\pi}{4} + 2\delta(w-2)\frac{-j\pi}{4} + 2\delta(w-4)\frac{-j\pi}{4}.$$

Inverting, we obtain

$$\begin{aligned}
y(t) &= \frac{1}{2\pi}\left(\frac{j\pi}{2}e^{-j4t} + \frac{j\pi}{2}e^{-j2t} - \frac{j\pi}{2}e^{j2t} - \frac{j\pi}{2}e^{j4t}\right) \\
&= \frac{1}{2}\left(\frac{e^{j4t} - e^{-j4t}}{2j} + \frac{e^{j2t} - e^{-j2t}}{2j}\right) \\
&= \frac{1}{2}\sin(2t) + \frac{1}{2}\sin(4t).
\end{aligned}$$

Solution 7.4-5

(a) Consulting Table 7.1, we see that

$$X_1(\omega) = \text{sinc}(\tfrac{\omega}{20000}) \quad \text{and} \quad X_2(\omega) = 1.$$

Figure S7.4-5a shows $X_1(\omega)$ and $X_2(\omega)$, plotted as a function of hertzian frequency $f = \frac{\omega}{2\pi}$.

(b) Figure S7.4-5b shows $H_1(\omega)$ and $H_2(\omega)$, also plotted as a function of hertzian frequency $f = \frac{\omega}{2\pi}$.
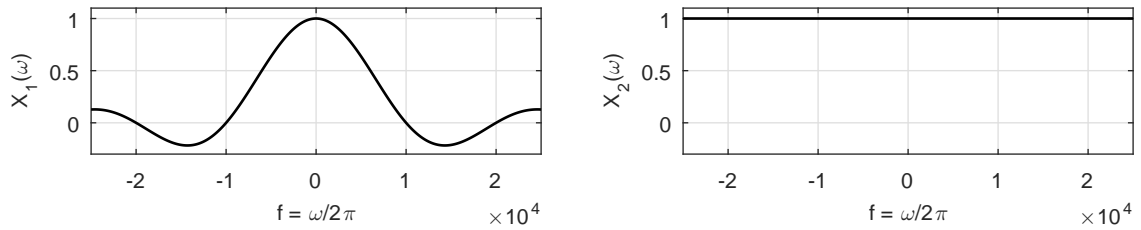
Figure S7.4-5a
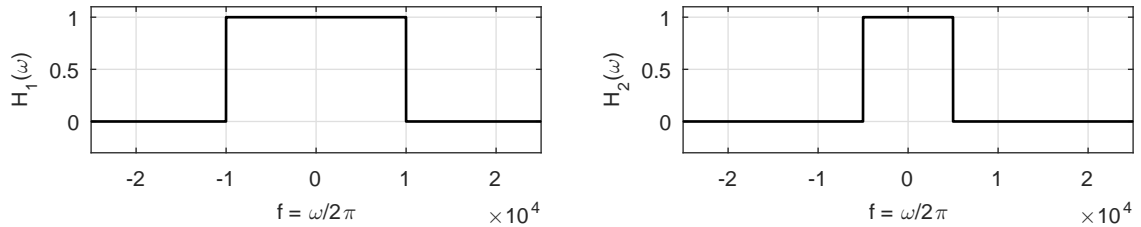
Figure S7.4-5b
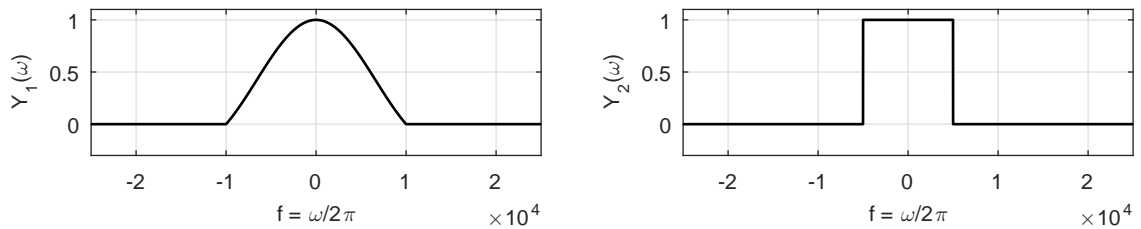
Figure S7.4-5c

(c) The outputs of the two filters are given as

$$Y_1(\omega) = X_1(\omega)H_1(\omega) \qquad \text{and} \qquad Y_2(\omega) = X_2(\omega)H_2(\omega).$$

Figure S7.4-5c shows $Y_1(\omega)$ and $Y_2(\omega)$ as functions of $f = \frac{\omega}{2\pi}$.

(d) Because $y(t) = y_1(t)y_2(t)$, the frequency convolution property yields $Y(\omega) = Y_1(\omega) * Y_2(\omega)$. From the width property of convolution, it follows that the bandwidth of $Y(\omega)$ is the sum of bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$. Because the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are 10 and 5 kHz, respectively, the bandwidth of $Y(\omega)$ is 15 kHz.

Solution 7.4-6

Since $h(t) = \text{rect}\,(t/10^{-3})$ and $p(t) = \Delta(t/10^{-6})$,

$$H(\omega) = 10^{-3}\text{sinc}(\tfrac{\omega}{2000}) \quad \text{and} \quad P(\omega) = 0.5 \times 10^{-6}\text{sinc}^2(\tfrac{\omega}{4 \times 10^6}).$$

The two spectra are sketched in Fig. S7.4-6. It is clear that the main lobe of $H(\omega)$ is at least a thousand times narrower than $P(\omega)$, and we may consider $P(\omega)$ to be nearly constant of value $P(0) = 10^{-6}/2$ over the entire band of $H(\omega)$. Hence,

$$Y(\omega) = P(\omega)H(\omega) \approx P(0)H(\omega) = 0.5 \times 10^{-6}H(\omega) \Rightarrow y(t) = 0.5 \times 10^{-6}h(t).$$

Recall that $h(t)$ is the unit impulse response of the system. Hence, the output $y(t)$ is equal to the system response to an input $0.5 \times 10^6\delta(\omega) = A\delta(\omega)$.
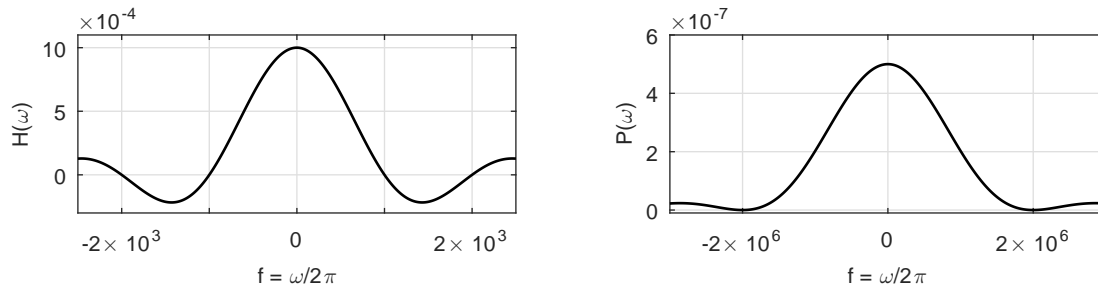
Figure S7.4-6

Solution 7.4-7

Since $h(t) = \text{rect}\,(t/10^{-3})$ and $p(t) = \Delta(t)$,

$$H(\omega) = 10^{-3}\text{sinc}(\tfrac{\omega}{20000}) \quad \text{and} \quad P(\omega) = 0.5\text{sinc}^2(\tfrac{\omega}{4}).$$

The two spectra are shown in Fig. S7.4-7. It is clear that the main lobe of $P(\omega)$ is at least five hundred times narrower than $H(\omega)$, and we may consider $H(\omega)$ to be nearly constant of value $H(0) = 10^{-3}$ over the entire band of $P(\omega)$. Hence,

$$Y(\omega) = P(\omega)H(\omega) \approx P(\omega)H(0) = 10^{-3}P(\omega) \Rightarrow y(t) = 10^{-3}p(t).$$

Note that the dc gain of the system is $k = H(0) = 10^{-3}$. Hence, the output is nearly $kP(t)$.
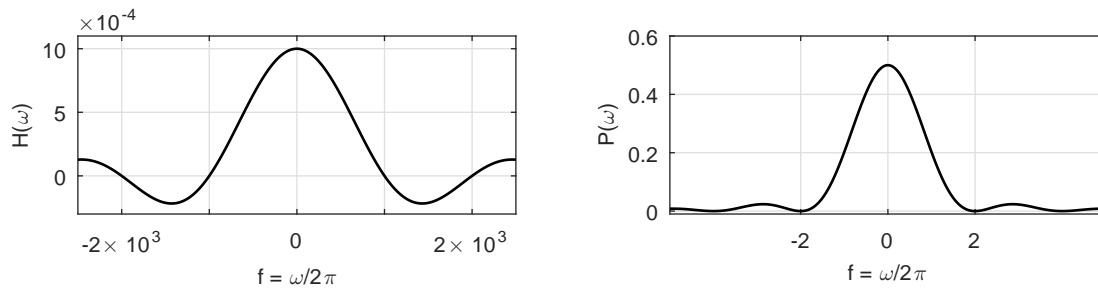
Figure S7.4-7

Solution 7.4-8

Every signal can be expressed as a sum of even and odd components [see Eq. (1.17)]. Hence, $h(t)$ can be expressed as a sum of its even and odd components as

$$h(t) = h_e(t) + h_o(t),$$

where $h_e(t) = \frac{1}{2}[h(t)u(t) + h(-t)u(-t)]$ and $h_o(t) = \frac{1}{2}[h(t)u(t) - h(-t)u(-t)]$. From these equations, we make an important observation that

$$h_e(t) = h_o(t)\text{sgn}(t) \quad \text{and} \quad h_o(t) = h_e(t)\text{sgn}(t). \tag{7.4-8c}$$

Provided that $h(t)$ has no impulse at the origin. This result applies only if $h(t)$ is causal. A graphical demonstration of this result may be seen in text Fig. 1.24.

Moreover, we have proved in Prob. 7.1-4 that the Fourier transform of a real and even signal is a real and even function of ω, and the Fourier transform of a real and odd signal is an imaginary odd function of ω. Therefore, if $H(\omega) = R(\omega) + jX(\omega)$, then

$$h_e(t) \Longleftrightarrow R(\omega) \quad \text{and} \quad h_o(t) \Longleftrightarrow jX(\omega). \tag{7.4-8d}$$

Combining Eqs. (7.4-8c) and (7.4-8d) and using the convolution property, we obtain

$$\mathcal{F}\{h_e(t)\} = R(\omega) = \mathcal{F}\{h_o(t)\text{sgn}(t)(t)\} = \frac{1}{2\pi}jX(\omega)*\frac{2}{j\omega}.$$

Thus,

$$R(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{X(y)}{\omega - y}dy.$$

Similarly,

$$\mathcal{F}\{h_o(t)\} = jX(\omega) = \mathcal{F}\{h_e(t)\text{sgn}(t)(t)\} = \frac{1}{2\pi}R(\omega)*\frac{2}{j\omega}.$$

Thus,

$$jX(\omega) = \frac{1}{j\pi}\int_{-\infty}^{\infty}\frac{R(y)}{\omega - y}dy$$

or

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{\omega - y}dy.$$

Solution 7.5-1

Here,
$$H(\omega) = e^{-k\omega^2}e^{-j\omega t_0}.$$

Using pair 22 in Table 7.1 and the time-shifting property, we get

$$h(t) = \frac{1}{\sqrt{4\pi k}}e^{-(t-t_0)^2/4k}.$$

Since the impulse response is noncausal, the filter is unrealizable. Also,

$$\int_{-\infty}^{\infty}\frac{|\ln|H(\omega)||}{\omega^2 + 1}d\omega = \int_{-\infty}^{\infty}\frac{k\omega^2}{\omega^2+1}d\omega = \infty.$$

Since the integral is not finite, $H(\omega)$ does not satisfy the Paley-Wiener criterion, and the filter is therefore unrealizable.

Since $h(t)$ is a Gaussian function delayed by t_0, it looks as shown in Fig. S7.5-1. Choosing $t_0 == 3\sqrt{2k}$, $h(0) = e^{-4.5} = 0.011$, or 1.1% of its peak value. Hence $t_0 = 3\sqrt{2k}$ is a reasonable choice to make the filter approximately realizable.
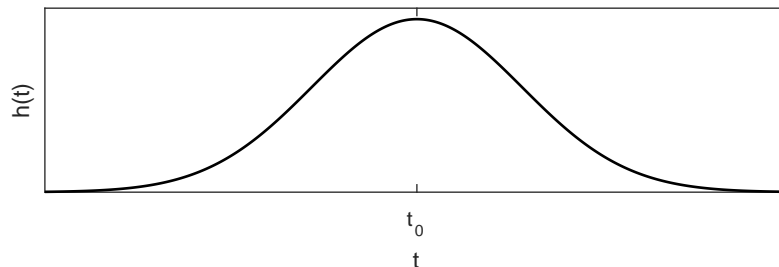
Figure S7.5-1

Solution 7.5-2

Here,

$$H(\omega) = \frac{2 \times 10^5}{\omega^2 + 10^{10}} e^{-j\omega t_0}.$$

Using pair 3 in Table 7.1 and time-shifting property, we get

$$h(t) = e^{-10^5 |t - t_0|}.$$

The impulse response is noncausal, and the filter is unrealizable.

The exponential decays to 1% of the peak value when

$$0.01 = e^{-10^5 |0 - t_0|} \Rightarrow t_0 = \frac{-\ln(0.01)}{10^5} = 4.6052 \times 10^{-5}.$$

Hence, $t_0 = 46$ μs is a reasonable choice to make this filter approximately realizable.

Solution 7.5-3

The unit impulse response is the inverse Fourier transform of $H(\omega)$.

(a) For $H(\omega) = 10^{-6} \operatorname{sinc}(10^{-6}\omega)$, we see that

$$h_a(t) = 0.5 \operatorname{rect}\left(\frac{t}{2 \times 10^{-6}}\right).$$

Notice that $h_a(t)$ is a rectangular pulse that starts at $t = -10^{-6}$. Since $h_a(t) \neq 0$ for all $t < 0$, the system is both noncausal and unrealizable. Delaying $h_a(t)$ by 1 μs, however, makes the system both causal and realizable. This adjustment will not change anything in the system behavior except a 1 μs delay in the system response.

(b) For $H(\omega) = 10^{-4} \Delta(\omega/40{,}000\pi)$, we see that

$$h_b(t) = \operatorname{sinc}^2(10{,}000\pi t).$$

Since $h_b(t)$ starts before $t = 0$, the system is both noncausal and unrealizable. Furthermore, $h_b(t)$ is non-zero all the way to $-\infty$. Clearly, this system cannot be made realizable with a finite time delay. Because $h_b(t)$ decays rapidly, however, we can truncate it at a suitable point, such as $t = 10^{-4}$ when $h_b(t)$ is relatively small, and then delay the truncated impulse response by 10^{-4}. The resulting filter is causal and realizable, but only an approximation of the desired behavior (due to the truncation) with 100 μs of delay (to ensure a causal, realizable system).

(c) For $H(\omega) = 2\pi \, \delta(\omega)$, we see that
$$h_c(t) = 1.$$

Since $h_c(t)$ starts before $t = 0$, the system is both noncausal and unrealizable. Further, since $h_c(t)$ never decays, this filter cannot be suitably realized with any amount of delay.

Solution 7.5-4

(a) Comparing the plots of $x_1(t)$ and $x_2(t)$, we see that the carrier of $x_2(t)$ is half the frequency of the carrier of $x_1(t)$. The content centered at $f = \pm 4$ for $X_1(f)$ therefore moves to $f = \pm 2$ for $X_2(f)$, as shown in Fig. S7.5-4.

(b) Comparing the plots of $X_1(f)$ and $X_3(f)$, we see that $X_3(f) = 2X_1(2f)$. Using the scaling property, we see that
$$x_3(t) = x_1\left(\tfrac{t}{2}\right).$$
Signal $x_3(t)$ is also shown in Fig. S7.5-4.

(c) Since $x_4(t) = x_1(t) + x_2(t)$, we know that $X_4(f) = X_1(f) + X_2(f)$. The entire spectrum $X_1(f)$ is outside the 3 Hz passband of the filter while the entire spectrum $X_2(f)$ is inside the 3 Hz passband of the filter. Thus, the $x_1(t)$ component is filtered out while the $x_2(t)$ component is preserved. Hence,

$$y_4(t) = x_2(t).$$

Signal $y_4(t)$ is shown in Fig. S7.5-4.
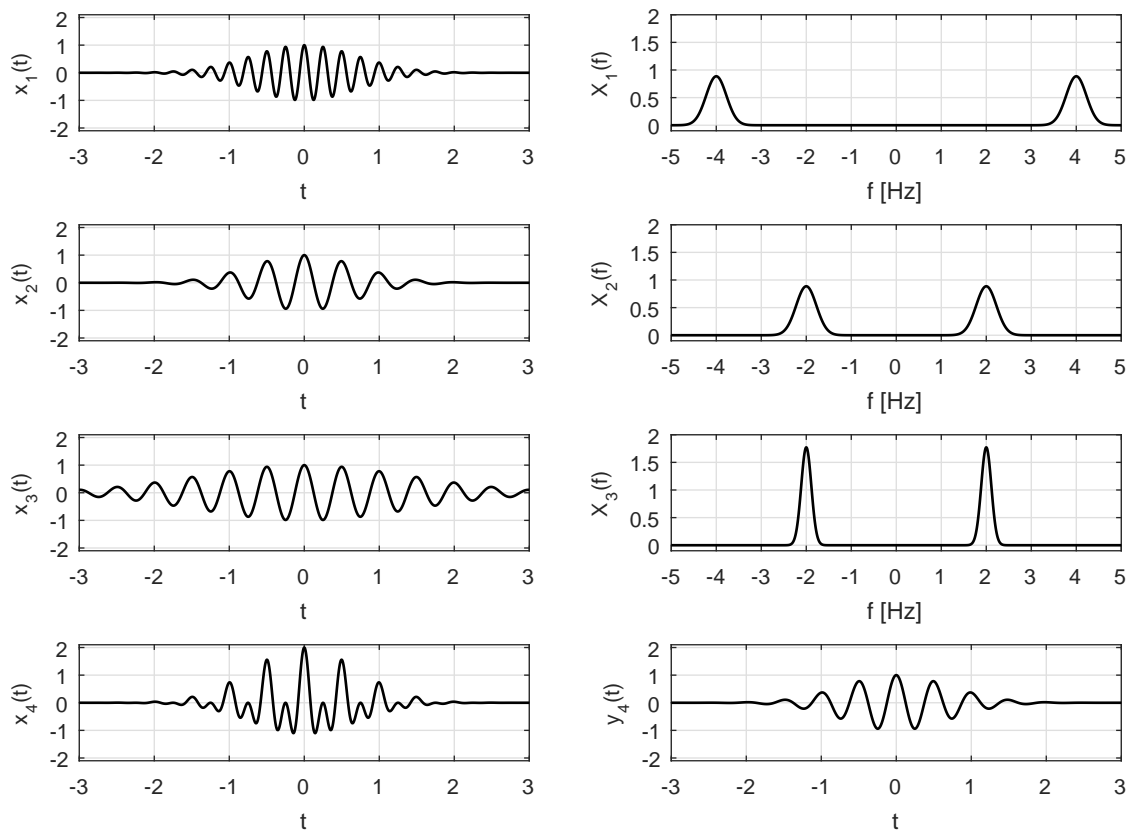
Figure S7.5-4

Solution 7.6-1

In this problem, we have

$$x(t) = \frac{1}{2\pi}\text{sinc}(t/2) \qquad \text{and} \qquad X(\omega) = \text{rect}(\omega).$$

Define $z(t) = \text{sinc}(t) = 2\pi x(2t)$. Thus,

$$Z(\omega) = \frac{2\pi}{2}X\left(\frac{\omega}{2}\right) = \pi\text{rect}\left(\frac{\omega}{2}\right).$$

Parseval's theorem [see Eq. (7.45)] states

$$E_z = \int_{-\infty}^{\infty} |z(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z(\omega)|^2 \, d\omega.$$

Thus,

$$E_z = \int_{-\infty}^{\infty} \text{sinc}^2(t) \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi^2\text{rect}^2\left(\frac{\omega}{2}\right) \, d\omega = \frac{\pi}{2}\int_{-1}^{1} 1 \, d\omega = \pi.$$

Since time-shifting does not affect energy, we see that the energy of $z(t) = \text{sinc}(t)$ equals the energy of $z(t-2) = \text{sinc}(t-2)$. Thus,

$$\int_{-\infty}^{\infty} \text{sinc}^2(t-2)\, dt = E_z = \pi.$$

Solution 7.6-2

To begin, we see that

$$E_x = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-t^2/\sigma^2} dt.$$

Letting $\frac{t}{\sigma} = \frac{x}{\sqrt{2}}$ and consequently $dt = \frac{\sigma}{\sqrt{2}}dx$,

$$E_x \frac{1}{2\pi\sigma^2} \frac{\sigma}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2}dx.$$

Using the fact that $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$, we see that

$$E_x = \frac{\sqrt{2\pi}}{2\sqrt{2\pi}\sigma} = \frac{1}{2\sigma\sqrt{\pi}}.$$

We can also verify this result using the frequency domain. Using pair 22 in Table 7.1, we see that

$$X(\omega) = e^{-\sigma^2\omega^2/2}.$$

Parseval's theorem requires that

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2\omega^2}\, d\omega.$$

Letting $\sigma\omega = \frac{x}{\sqrt{2}}$ and consequently $d\omega = \frac{1}{\sigma\sqrt{2}}dx$,

$$E_x = \frac{1}{2\pi} \frac{1}{\sigma\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2}dx = \frac{\sqrt{2\pi}}{2\pi\sigma\sqrt{2}} = \frac{1}{2\sigma\sqrt{\pi}}.$$

Solution 7.6-3

Consider a signal

$$x(t) = \text{sinc}(kt) \quad \text{and} \quad X(\omega) = \tfrac{\pi}{k}\text{rect}(\tfrac{\omega}{2k}).$$

Using Parseval's theorem [Eq. (7.45)], we see that

$$E_x = \int_{-\infty}^{\infty} \text{sinc}^2(kt)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tfrac{\pi^2}{k^2} \left[\text{rect}\left(\tfrac{\omega}{2k}\right)\right]^2 d\omega$$

$$= \frac{\pi}{2k^2} \int_{-k}^{k} d\omega = \tfrac{\pi}{k}.$$

Solution 7.6-4

If $x^2(t) \Longleftrightarrow A(\omega)$, then the output $Y(\omega) = A(\omega)H(\omega)$, where $H(\omega)$ is the lowpass filter transfer function (Fig. S7.6-4). Because this filter band $\Delta f \to 0$, we may express it as an impulse function of area $4\pi\Delta f$. Thus,

$$H(\omega) \approx [4\pi\Delta f]\delta(\omega) \quad \text{and} \quad Y(\omega) \approx [4\pi A(\omega)\Delta f]\delta(\omega) = [4\pi A(0)\Delta f]\delta(\omega).$$

Here we used the property $g(x)\delta(x) = g(0)\delta(x)$. This yields

$$y(t) \approx 2A(0)\Delta f.$$

Next, because $x^2(t) \iff A(\omega)$, we have

$$A(\omega) = \int_{-\infty}^{\infty} x^2(t)e^{-j\omega t}dt \quad \text{so that} \quad A(0) = \int_{-\infty}^{\infty} x^2(t)dt = E_x.$$

Hence, $y(t) \approx 2E_x \Delta f$.
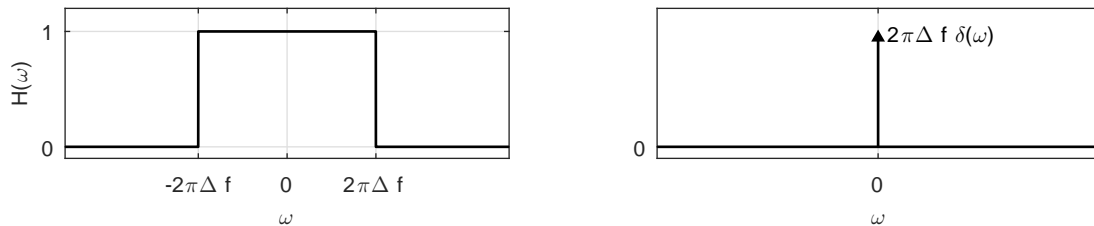
Figure S7.6-4

Solution 7.6-5

Recall that

$$x_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t} \, d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} x_1(t)e^{j\omega t} \, dt = X_1(-\omega).$$

Therefore,

$$\int_{-\infty}^{\infty} x_1(t)x_2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(t) \left[\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t} \, d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega) \left[\int_{-\infty}^{\infty} x_1(t)e^{j\omega t} \, dt \right] d\omega = \frac{1}{2\pi} \int X_1(-\omega)X_2(\omega) \, d\omega.$$

Interchanging the roles of $x_1(t)$ and $x_2(t)$ in the previous development, we also see that

$$\int_{-\infty}^{\infty} x_1(t)x_2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega) \, d\omega.$$

Solution 7.6-6

In the generalized Parseval's theorem in Prob. 7.6-5, if we identify $g_1(t) = \text{sinc}(Wt - m\pi)$ and $g_2(t) = \text{sinc}(Wt - n\pi)$, then

$$G_1(\omega) = \frac{\pi}{W}\text{rect}\left(\frac{\omega}{2W}\right)e^{\frac{jm\pi\omega}{W}} \quad \text{and} \quad G_2(\omega) = \frac{\pi}{W}\text{rect}\left(\frac{\omega}{2W}\right)e^{\frac{jn\pi\omega}{W}}.$$

Therefore,

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)dt = \frac{1}{2\pi}\left(\frac{\pi}{W}\right)^2 \int_{-\infty}^{\infty} \left[\text{rect}\left(\frac{\omega}{2W}\right)\right]^2 e^{\frac{j(n-m)\pi\omega}{W}} \, d\omega.$$

But $\text{rect}\left(\frac{\omega}{2W}\right) = 1$ for $|\omega| \leq W$, and is 0 otherwise. Hence

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)dt = \frac{\pi}{2W^2} \int_{-W}^{W} e^{\frac{j(n-m)\pi\omega}{W}} \, d\omega = \begin{cases} 0 & n \neq m \\ \frac{\pi}{W} & n = m \end{cases}.$$

In evaluating the integral, we used the fact that $e^{\pm j2\pi k} = 1$ when k is an integer.

Solution 7.6-7

(a) The 95% essential bandwidth is the frequency band, centered at 0, that includes 95% of the energy of $x(t)$. That is, the 95% essential bandwidth is the value B satisfying

$$\frac{1}{2\pi} \int_{-B}^{B} |X(\omega)|^2 \, d\omega = 0.95 E_x = 0.95 \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.$$

(b) For $X(\omega) = \text{rect}(\omega)$, we see that

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \, d\omega = \frac{1}{2\pi}.$$

To determine the 95% essential bandwidth, we need to find B that satisfies

$$\frac{1}{2\pi} \int_{-B}^{B} 1 \, d\omega = \frac{B}{\pi} = 0.95 \frac{1}{2\pi}.$$

Solving for B, we see that the 95% essential bandwidth is

$$B = \frac{19}{40} = 0.475.$$

In this case, the 95% bandwidth is just 95% of the true bandwidth of $\frac{1}{2}$ — a behavior that happens because $X(\omega)$ is uniformly spread over the entire signal bandwidth.

(c) For $X(\omega) = \Delta(\omega)$, we see that

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\frac{1}{2}} (1 - 2\omega)^2 \, d\omega = \frac{1}{\pi} \left(-\frac{1}{6} \right) (1 - 2\omega)^3 \Big|_{0}^{\frac{1}{2}} = \frac{1}{6\pi}.$$

Here and later, we simplify calculations by utilizing the even symmetry of $X(\omega)$.

To determine the 95% essential bandwidth, we need to find B that satisfies

$$\frac{1}{\pi} \int_{0}^{B} (1 - 2\omega)^2 \, d\omega = \frac{1}{\pi} \left(-\frac{1}{6} \right) (1 - 2\omega)^3 \Big|_{0}^{B} = \frac{1}{6\pi} \left(1 - [1 - 2B]^3 \right) = 0.95 \frac{1}{6\pi}.$$

This requires that
$$(1 - 2B)^3 = 0.05 \qquad \text{or} \qquad 1 - 2B = (0.05)^{1/3}.$$

Solving for B, we see that the 95% essential bandwidth is

$$B = \frac{1 - 0.05^{1/3}}{2} \approx 0.315798.$$

In this case, the 95% bandwidth is approximately 63% of the true bandwidth of $\frac{1}{2}$. This result is much small than the essential bandwidth for the rectangular case of part (b) because the triangular spectrum has energy more concentrated at lower frequencies.

Solution 7.6-8

For $X(\omega) = e^{-|\omega|}$, we see that

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2|\omega|} \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} e^{-2\omega} \, d\omega = \frac{e^{-2\omega}}{-2\pi} \Big|_{0}^{\infty} = \frac{1}{2\pi}.$$

Here and later, we simplify calculations by utilizing the even symmetry of $X(\omega)$.

To determine the 95% essential bandwidth, we need to find B that satisfies

$$\frac{1}{\pi}\int_0^B e^{-2\omega}\,d\omega = \frac{1}{\pi}\frac{e^{-2\omega}}{-2}\bigg|_0^B = \frac{1}{\pi}\left(\frac{1-e^{-2B}}{2}\right) = 0.95\frac{1}{2\pi}.$$

This requires that

$$1 - e^{-2B} = 0.95 \qquad \text{or} \qquad e^{-2B} = 0.05.$$

Solving for B, we see that the 95% essential bandwidth is

$$B = \frac{\ln(0.05)}{-2} \approx 1.497866.$$

Solution 7.6-9

Applying the duality property of Eq. (7.25) to pair 3 of Table 7.1 yields

$$\frac{2a}{t^2+a^2} \iff 2\pi e^{-a|\omega|}.$$

The signal energy is given by

$$E_x = \frac{1}{\pi}\int_0^\infty |2\pi e^{-a\omega}|^2 d\omega = 4\pi\int_0^\infty e^{-2a\omega}d\omega = \frac{2\pi}{a}.$$

The energy contained within the band $(0$ to $W)$ is

$$E_W = 4\pi\int_0^W e^{-2a\omega}d\omega = \frac{2\pi}{a}[1 - e^{-2aW}].$$

If $E_W = 0.99E_x$, then

$$e^{-2aW} = 0.01 \Rightarrow W = \frac{2.3025}{a}\ \text{rad/s} = \frac{0.366}{a}\ \text{Hz}.$$

Solution 7.7-1

(a) For (i), we see that $m(t) = \cos 1000t$, and for (ii), we see that $m(t) = 2\cos 1000t + \cos 2000t$. For (iii), $m(t) = \cos 1000t \cos 3000t$. Using trigonometric properties, we express the signal as $m(t) = \frac{1}{2}[\cos 2000t + \cos 4000t]$. The message spectra for these cases follow from pair 9 of Table 7.1 and are shown in Fig. S7.7-1a.
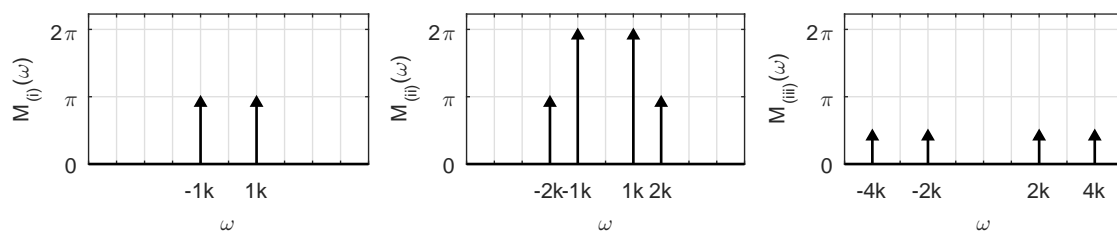
Figure S7.7-1a

(b), (c)

(i) For $m(t) = \cos 1000t$:

$$\phi_{DSB-SC}(t) = m(t)\cos 10000t = \cos 1000t \cos 10000t$$

$$= \underbrace{\frac{1}{2}\cos 9000t}_{\text{LSB}} + \underbrace{\frac{1}{2}\cos 11000t}_{\text{USB}}.$$

(ii) For $m(t) = 2\cos 1000t + \cos 2000t$:

$$\phi_{DSB-SC}(t) = m(t)\cos 10,000t = [2\cos 1000t + \cos 2000t]\cos 10000t$$

$$= \cos 9000t + \cos 11000t + \frac{1}{2}[\cos 8000t + \cos 12000t]$$

$$= \underbrace{\cos 9000t + \frac{1}{2}\cos 8000t}_{\text{LSB}} + \underbrace{\cos 11000t + \frac{1}{2}\cos 12000t}_{\text{USB}}$$

(iii) For $m(t) = \cos 1000t \cos 3000t$:

$$\phi_{DSB-SC}(t) = m(t)\cos 10000t = \frac{1}{2}[\cos 2000t + \cos 4000t]\cos 10000t$$

$$= \frac{1}{2}[\cos 8000t + \cos 12000t] + \frac{1}{2}[\cos 6000t + \cos 14000t]$$

$$= \underbrace{\frac{1}{2}\cos 8000t + \frac{1}{2}\cos 6000t}_{\text{LSB}} + \underbrace{\frac{1}{2}\cos 12000t + \frac{1}{2}\cos 14000t}_{\text{USB}}$$

Figure S7.7-1b shows the corresponding DSB-SC spectra.
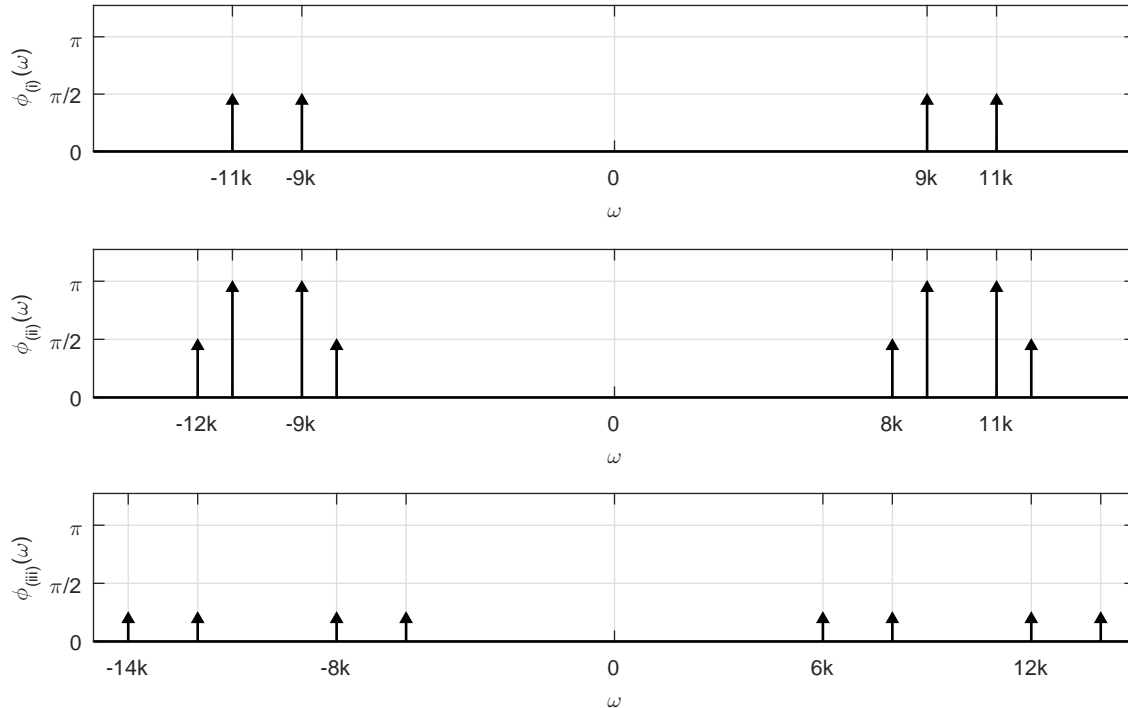
Figure S7.7-1b

(d) The following table identifies the frequencies in the baseband and the corresponding frequencies in the DSB-SC, USB, and LSB spectra.

Case	Baseband frequency	DSB frequency	LSB frequency	USB frequency
(i)	1,000	9,000 and 11,000	9,000	11,000
(ii)	1,000	9,000 and 11,000	9,000	11,000
	2,000	8,000 and 12,000	8,000	12,000
(iii)	2,000	8,000 and 12,000	8,000	12,000
	4,000	6,000 and 14,000	6,000	14,000

Solution 7.7-2

(a) Since $M(\omega) = \frac{1}{1000}\Delta\left(\frac{\omega}{2\pi 6000}\right)$,

$$\text{signal } m(t) \text{ has a bandwidth of 3000 Hz.}$$

(b) Fig. S7.7-2b sketches the DSB-SC spectrum with $\omega_c = 2\pi 100000$.
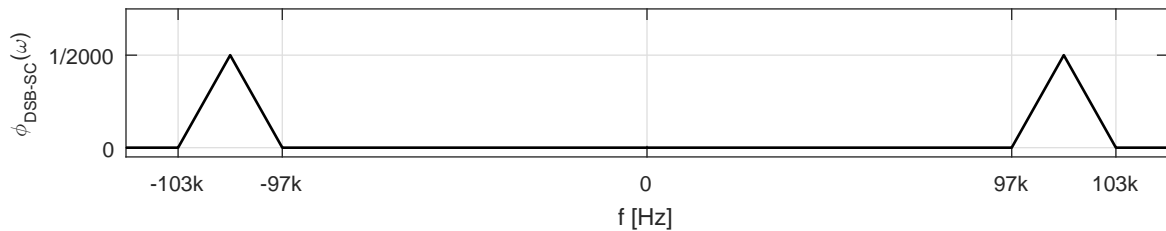
Figure S7.7-2b

(c) Fig. S7.7-2c sketches the AM spectrum with $\mu = 1$ and $\omega_c = 2\pi 100000$. From entry 20 of Table 7.1, we see that $m(t) = 3\text{sinc}^2(3000\pi t)$. Thus, the peak amplitude of the message signal is $m_{\text{peak}} = 3$. Since $\mu = 1$, we see that $A = m_{\text{peak}} = 3$.
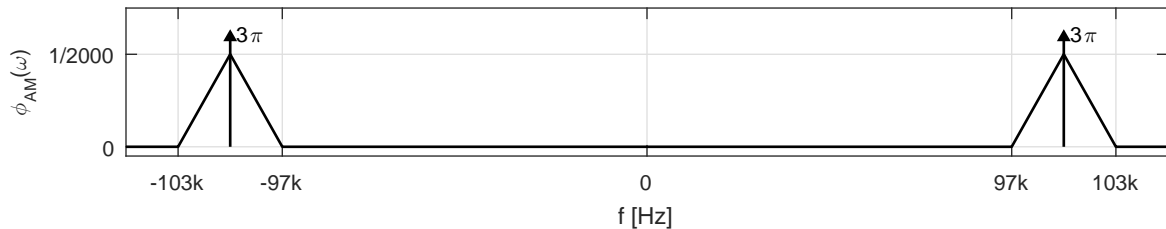
Figure S7.7-2c

(d) Fig. S7.7-2d sketches the USB spectrum with $\omega_c = 2\pi 100000$.
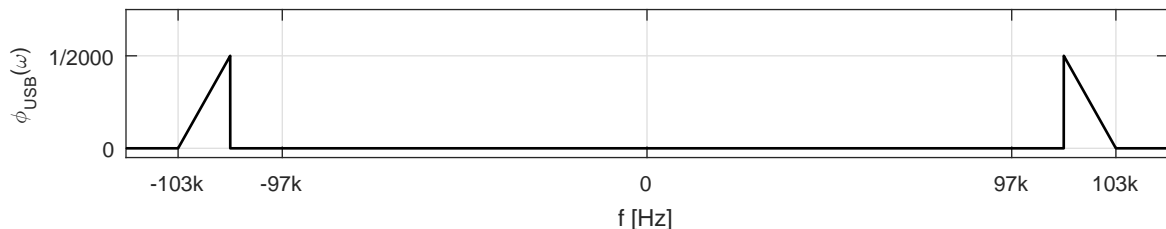
Figure S7.7-2d
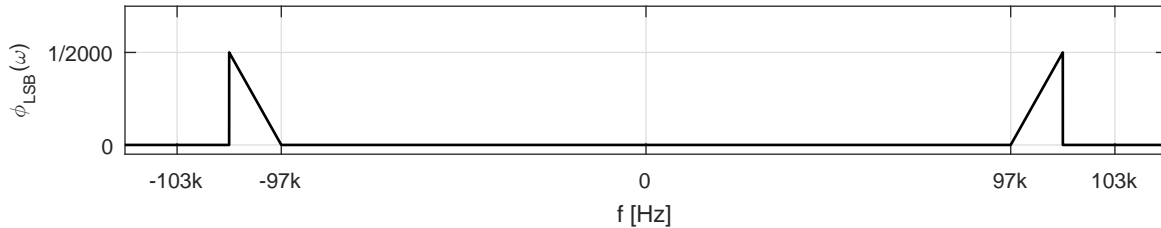
Figure S7.7-2e

(e) Fig. S7.7-2e sketches the LSB spectrum with $\omega_c = 2\pi 100000$.

(f) Figure S7.7-2f shows the positive-frequency portion of the FDM signal spectrum. From this picture, we see that

$$\omega_1 = 2\pi(103,000), \quad \omega_2 = 2\pi(114,000), \quad \omega_3 = 2\pi(122,000), \quad \omega_4 = 2\pi(133,000).$$

Also from Fig. S7.7-2f, we see that the end hertzian frequency of the FDM signal is 133 kHz.
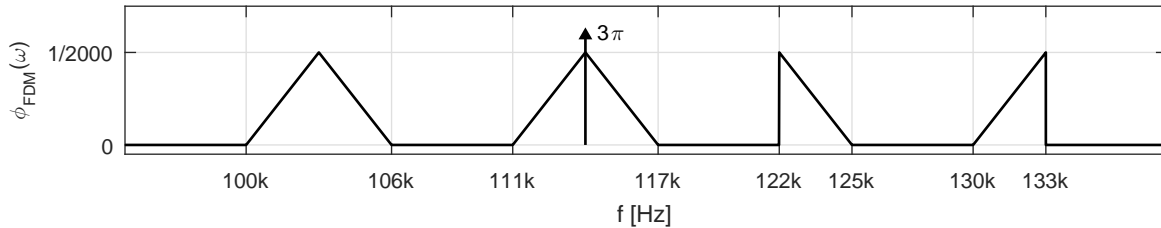
Figure S7.7-2f

Solution 7.7-3

(a) The signal at point b is

$$x_a(t) = m(t)\cos^3\omega_c t = m(t)\left[\frac{3}{4}\cos\omega_c t + \frac{1}{4}\cos 3\omega_c t\right].$$

The term $\frac{3}{4}m(t)\cos\omega_c t$ is the desired modulated signal, whose spectrum is centered at $\pm\omega_c$. The remaining term $\frac{1}{4}m(t)\cos 3\omega_c t$ is the unwanted term, which represents the modulated signal with carrier frequency $3\omega_c$ with spectrum centered at $\pm\omega_c$ as shown in Fig. S7.7-3. The bandpass filter centered at $\pm\omega_c$ allows to pass the desired term $\frac{3}{4}m(t)\cos\omega_c t$, but suppresses the unwanted term $\frac{1}{4}m(t)\cos 3\omega_c t$. Hence, this system works as desired with the output $\frac{3}{4}m(t)\cos\omega_c t$. Clearly, this method of DSB-SC generation results in $k = \frac{3}{4}$.

(b) Figure S7.7-3 shows the spectra at points b and c.
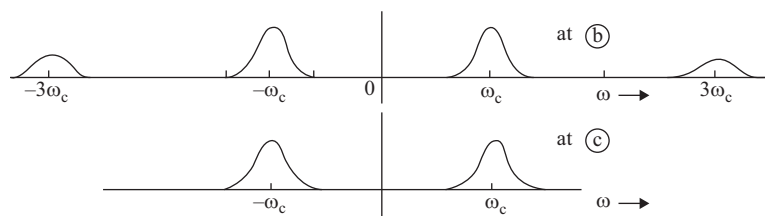
Figure S7.7-3

(c) The minimum usable value of ω_c is $2\pi B$ in order to avoid spectral folding at dc.

(d) Here,

$$m(t)\cos^2\omega_c t = \frac{m(t)}{2}[1 + \cos 2\omega_c t] = \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos 2\omega_c t.$$

The signal at point b consists of the baseband signal $\frac{1}{2}m(t)$ and a modulated signal $\frac{1}{2}m(t)\cos 2\omega_c t$, which has a carrier frequency $2\omega_c t$, not the desired value $\omega_c t$. Both the components will be suppressed by the filter, whose center center frequency is ω_c. Hence, this system will not do the desired job.

(e) The reader may verify that the identity for $\cos^n\omega_c t$ contains a term $\cos\omega_c t$ when n is odd. This is not true when n is even. Hence, the system works for a carrier $\cos^n\omega_c t$ only when n is odd.

Solution 7.7-4

This signal is identical to that in Fig. 6.6a with period T_0 (instead of 2π). We find the Fourier series for this signal as

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t + \cdots\right].$$

Hence, $y(t)$, the output of the multiplier is

$$y(t) = m(t)x(t) = m(t)\left[\frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t + \cdots\right)\right].$$

The bandpass filter suppresses the signals $m(t)$ and $m(t)\cos n\omega_c t$ for all $n \neq 1$. Hence, the bandpass filter output is

$$km(t)\cos\omega_c t = \frac{2}{\pi}m(t)\cos\omega_c t.$$

Solution 7.7-5

(a) Figure S7.7-5 shows the signal spectra at points a, b, and c.
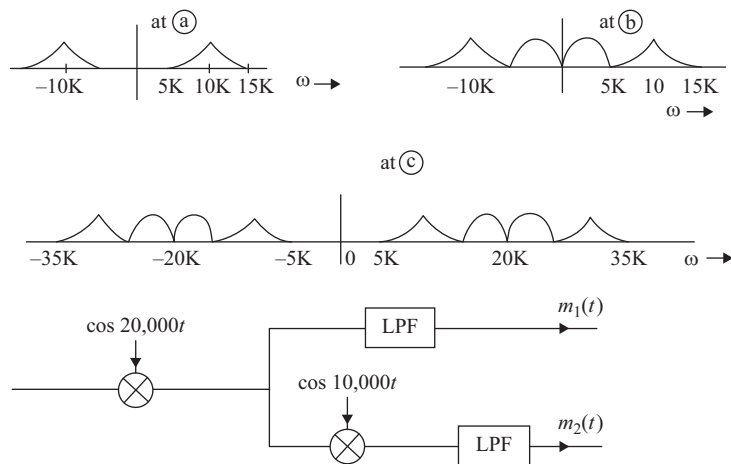
Figure S7.7-5

(b) From the spectrum at point c, it is clear that the channel bandwidth must be at least 30,000 rad/s (from 5000 to 35,000 rad/s).

(c) Figure S7.7-5 shows the receiver to recover $m_1(t)$ and $m_2(t)$ from the received modulated signal.

Solution 7.7-6

(a) Figure S7.7-6 shows the output signal spectrum $Y(\omega)$.

(b) Observe that $Y(\omega)$ is the same as $M(\omega)$ with the frequency spectrum inverted; that is, the high frequencies are shifted to lower frequencies and vice versa. Thus, the scrambler in Fig. S7.7-6 inverts the frequency spectrum. To get back the original spectrum $M(\omega)$, we need to invert the spectrum $Y(\omega)$ once again. This can be done by passing the scrambled signal $y(t)$ through the same scrambler.
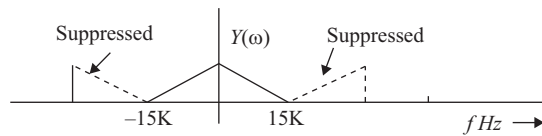
Figure S7.7-6

Solution 7.7-7

Here, the AM signal is $x_a(t) = [A + m(t)] \cos \omega_c t$. Hence,

$$x_b(t) = [A + m(t)] \cos^2 \omega_c t = \frac{1}{2}[A + m(t)] + \frac{1}{2}[A + m(t)] \cos 2\omega_c t.$$

The first term is a lowpass signal because its spectrum is centered at $\omega = 0$. The lowpass filter allows this term to pass, but suppresses the second term, whose spectrum is centered at $\pm 2\omega_c$. Hence the output of the lowpass filter is

$$y(t) = A + m(t).$$

When the signal is passed through a dc block, the dc term A is suppressed yielding the output $m(t)$. This shows that the system can demodulate an AM signal regardless of the value of A. Since the demodulator needs the original (phase-aligned) carrier signal, this is synchronous (or coherent) demodulation.

Solution 7.7-8

(a) $\mu = 0.5 = \frac{m_p}{A} = \frac{10}{A} \Rightarrow A = 20$

(b) $\mu = 1.0 = \frac{m_p}{A} = \frac{10}{A} \Rightarrow A = 10$

(c) $\mu = 2.0 = \frac{m_p}{A} = \frac{10}{A} \Rightarrow A = 5$

(d) $\mu = \infty = \frac{m_p}{A} = \frac{10}{A} \Rightarrow A = 0$

Since $A = 0$ for part (d), the $\mu = \infty$ represents the DSB-SC case. Figure S7.7-8 shows the resulting AM waveforms using a relatively low carrier frequency of 20 kHz.
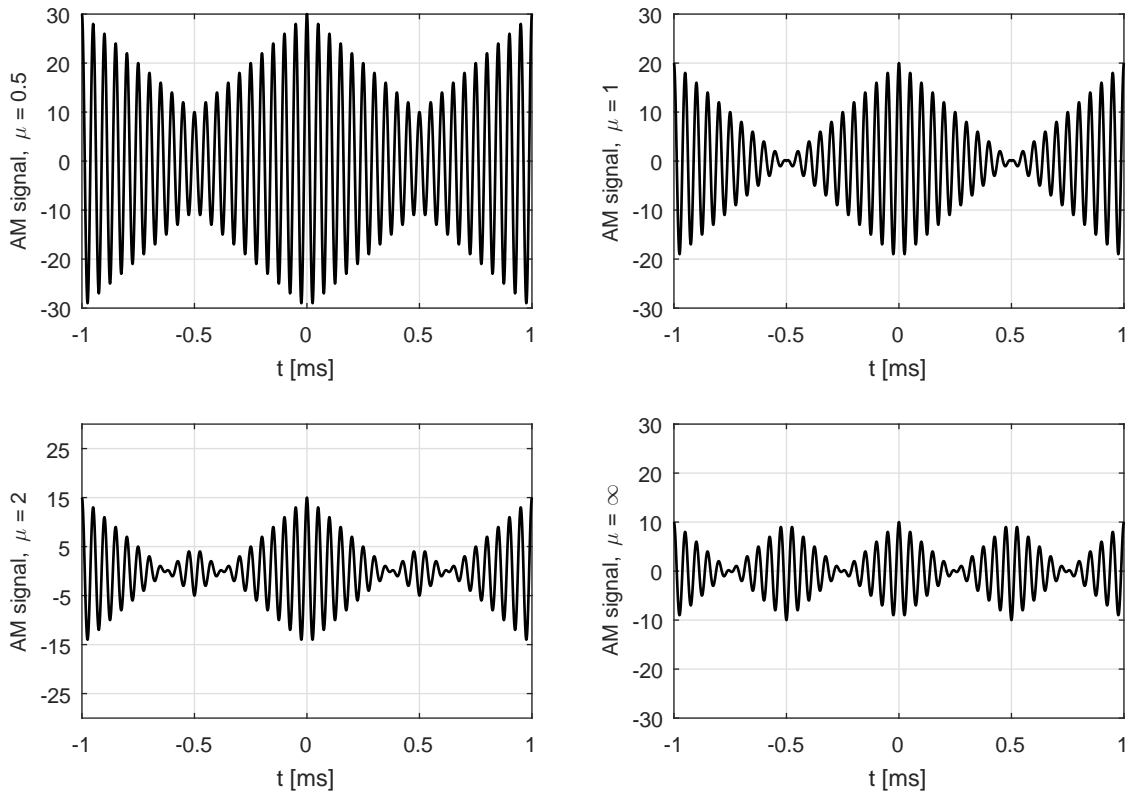
Figure S7.7-8

Solution 7.9-1

(a) A plot of $x(t)$ over $-2 \le t \le 2$ is shown in the first graph of Fig. S7.9-1b.

(b) Here, we use time-differentiation and other Fourier transform properties as well as $\delta(t) \Longleftrightarrow 1$ to determine $X(\omega)$. To begin, we assume $\omega \neq 0$ and define

$$y(t) = \frac{d}{dt}x(t) \qquad \text{and} \qquad z(t) = \frac{d}{dt}y_1(t),$$

where $y_1(t)$ is the non-delta part of $y(t)$. Signals $x(t)$, $y(t)$, and $z(t)$ are shown in Fig. S7.9-1b.
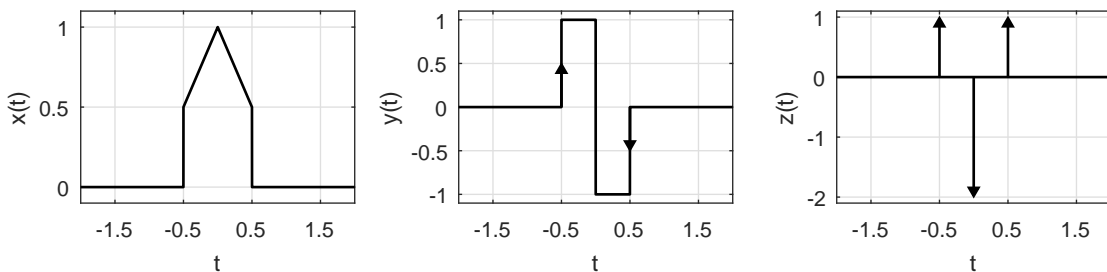
Figure S7.9-1b

By inspection, we see that

$$z(t) = \delta(t + 0.5) - 2\delta(t) + \delta(t - 0.5) \Longleftrightarrow Z(\omega) = e^{j\omega/2} - 2 + e^{-j\omega/2} = 2\cos\left(\frac{\omega}{2}\right) - 2.$$

Since $z(t) = \frac{d}{dt}y_1(t)$, we know that $Z(\omega) = j\omega Y_1(\omega)$. Substituting and solving for $Y_1(\omega)$, we obtain

$$Y_1(\omega) = \frac{2\cos\left(\frac{\omega}{2}\right) - 2}{j\omega}.$$

Next, we see that

$$y(t) = y_1(t) + \frac{1}{2}\delta(t + 0.5) - \frac{1}{2}\delta(t - 0.5)$$

and

$$Y(\omega) = Y_1(\omega) + \frac{1}{2}e^{j\omega/2} - \frac{1}{2}e^{-j\omega/2} = \frac{2\cos\left(\frac{\omega}{2}\right) - 2}{j\omega} + j\sin\left(\frac{\omega}{2}\right).$$

Since $y(t) = \frac{d}{dt}x(t)$, we know that $Y(\omega) = j\omega X(\omega)$. Substituting and solving for $X(\omega)$, we obtain

$$X(\omega) = \frac{2 - 2\cos\left(\frac{\omega}{2}\right)}{\omega^2} + \frac{\sin\left(\frac{\omega}{2}\right)}{\omega} = \frac{4\sin^2\left(\frac{\omega}{4}\right)}{\omega} + \frac{\sin\left(\frac{\omega}{2}\right)}{\omega}.$$

For $\omega = 0$ (the dc part), we can graphically see that $X(0) = \int x(t)dt = \frac{3}{4}$. Putting everything together and simplifying yields

$$X(\omega) = \begin{cases} \frac{3}{4} & \omega = 0 \\ \frac{1}{4}\mathrm{sinc}^2\left(\frac{\omega}{4}\right) + \frac{1}{2}\mathrm{sinc}\left(\frac{\omega}{2}\right) & \omega \neq 0 \end{cases}.$$

Since the $\omega \neq 0$ expression evaluates to $\frac{3}{4}$ at $\omega = 0$, the expression for $X(\omega)$ is further simplified to

$$X(\omega) = \frac{1}{4}\mathrm{sinc}^2\left(\frac{\omega}{4}\right) + \frac{1}{2}\mathrm{sinc}\left(\frac{\omega}{2}\right).$$

(c) Following the approach taken in Ex. 7.17, we next use MATLAB to verify the correctness of $X(\omega)$ by using it to synthesize a 3-periodic replication of the original time-domain signal $x(t)$.

```
>>  snc = @(t) sinc(t/pi); % Conform MATLAB sinc to textbook notation
>>  X = @(omega) 1/4*(snc(omega/4)).^2+1/2*snc(omega/2);
>>  T0 = 3; omega0 = 2*pi/T0; D = @(n) X(n*omega0)/T0;
>>  t = -T0:.001:T0; xN = D(0)*ones(size(t)); N = 100;
>>  for n = 1:N,
>>      xN = xN+real(D(n)*exp(1j*omega0*n*t)+D(-n)*exp(-1j*omega0*n*t));
>>  end
>>  plot(t,xN,'k'); xlabel('t'); ylabel('x_{100}(t)'); grid on;
>>  axis([-T0 T0 -.1 1.1]); set(gca,'ytick',0:.25:1,'xtick',-3:.5:3);
```

The resulting MATLAB-synthesized waveform, shown in Fig. S7.9-1c, approaches a 3-periodic replication of the original waveform $x(t)$ and thereby confirms the correctness of the Fourier transform $X(\omega)$ computed in part (b).
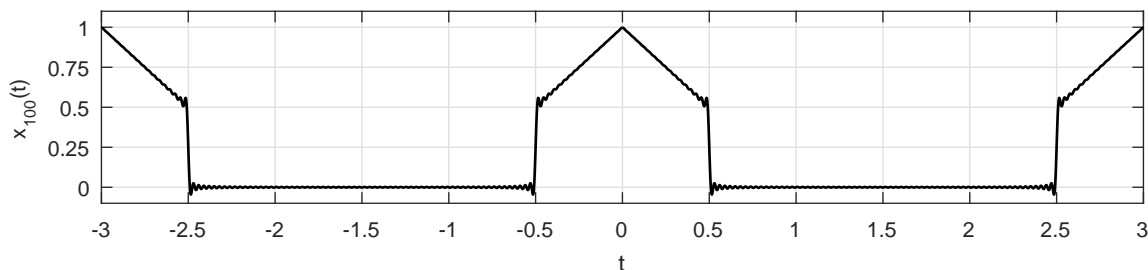
Figure S7.9-1c

Solution 7.9-2

(a) A plot of $x(t)$ over $-5 \leq t \leq 5$ is shown in the first graph of Fig. S7.9-2b.

(b) Here, we use time-differentiation and other Fourier transform properties as well as $\delta(t) \Longleftrightarrow 1$ to determine $X(\omega)$. To begin, we assume $\omega \neq 0$ and define

$$y(t) = \frac{d}{dt}x(t) \quad \text{and} \quad z(t) = \frac{d}{dt}y_1(t),$$

where $y_1(t)$ is the non-delta part of $y(t)$. Signals $x(t)$, $y(t)$, and $z(t)$ are shown in Fig. S7.9-2b.
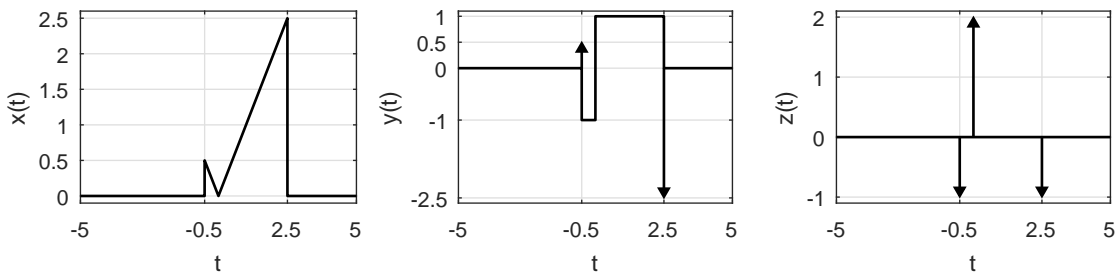
Figure S7.9-2b

By inspection, we see that

$$z(t) = -\delta(t+0.5) + 2\delta(t) - \delta(t-2.5) \Longleftrightarrow Z(\omega) = -e^{j\omega/2} + 2 - e^{-j5\omega/2} = 2 - 2e^{-j\omega}\cos\left(\frac{3\omega}{2}\right).$$

Since $z(t) = \frac{d}{dt}y_1(t)$, we know that $Z(\omega) = j\omega Y_1(\omega)$. Substituting and solving for $Y_1(\omega)$, we obtain

$$Y_1(\omega) = \frac{2 - 2e^{-j\omega}\cos\left(\frac{3\omega}{2}\right)}{j\omega}.$$

Next, we see that

$$y(t) = y_1(t) + \frac{1}{2}\delta(t+0.5) - \frac{5}{2}\delta(t-2.5)$$

and

$$Y(\omega) = Y_1(\omega) + \frac{1}{2}e^{j\omega/2} - \frac{5}{2}e^{-j5\omega/2} = \frac{2 - 2e^{-j\omega}\cos\left(\frac{3\omega}{2}\right)}{j\omega} + \frac{1}{2}e^{j\omega/2} - \frac{5}{2}e^{-j5\omega/2}.$$

Since $y(t) = \frac{d}{dt}x(t)$, we know that $Y(\omega) = j\omega X(\omega)$. Substituting and solving for $X(\omega)$, we obtain

$$X(\omega) = \frac{2e^{-j\omega}\cos\left(\frac{3\omega}{2}\right) - 2}{\omega^2} + \frac{\frac{1}{2}e^{j\omega/2} - \frac{5}{2}e^{-j5\omega/2}}{j\omega}.$$

For $\omega = 0$ (the dc part), we can graphically see that $X(0) = \int x(t)dt = \frac{25}{8} + \frac{1}{8} = \frac{13}{4}$. Putting everything together and simplifying yields

$$X(\omega) = \begin{cases} \frac{13}{4} & \omega = 0 \\ \frac{2e^{-j\omega}\cos\left(\frac{3\omega}{2}\right) - 2}{\omega^2} + \frac{\frac{1}{2}e^{j\omega/2} - \frac{5}{2}e^{-j5\omega/2}}{j\omega} & \omega \neq 0 \end{cases}.$$

(c) Following the approach taken in Ex. 7.17, we next use MATLAB to verify the correctness of $X(\omega)$ by using it to synthesize a 10-periodic replication of the original time-domain signal $x(t)$.

```
>>  X = @(omega) (2*exp(-1j*omega)*cos(3*omega/2)-2)./(omega.^2)+...
>>      (0.5*exp(1j*omega/2)-2.5*exp(-5j*omega/2))./(1j*omega);
>>  T0 = 10; omega0 = 2*pi/T0; D = @(n) X(n*omega0)/T0; D0 = 13/4/T0;
>>  t = -T0:.001:T0; xN = D0*ones(size(t)); N = 100;
>>  for n = 1:N,
>>      xN = xN+real(D(n)*exp(1j*omega0*n*t)+D(-n)*exp(-1j*omega0*n*t));
>>  end
>>  plot(t,xN,'k'); xlabel('t'); ylabel('x_{100}(t)');
>>  grid on; axis([-T0 T0 -.3 2.8]);
>>  set(gca,'ytick',0:.5:2.5,'xtick',[-10 -7.5 -.5 2.5 9.5]);
```

The resulting MATLAB-synthesized waveform, shown in Fig. S7.9-2c, approaches a 10-periodic replication of the original waveform $x(t)$ and thereby confirms the correctness of the Fourier transform $X(\omega)$ computed in part (b).
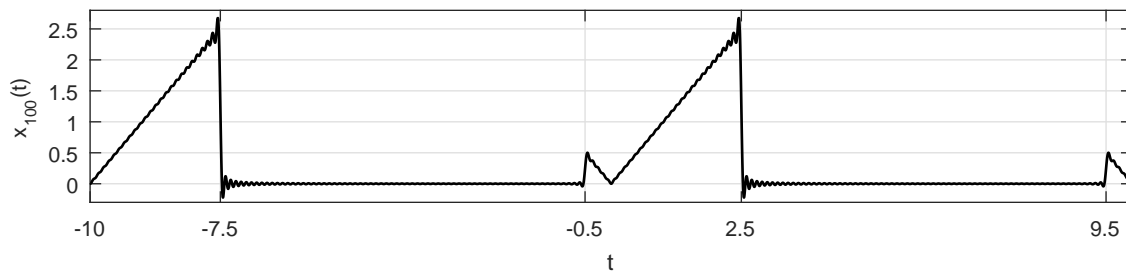
Figure S7.9-2c

Solution 7.9-3

This problem considers the continuous-time aperiodic signal $x(t) = \text{rect}(t)$ with Fourier transform $X(\omega) = \text{sinc}(\omega/2)$, and the signal $y(t) = (1 - |t - 1|)(u(t) - u(t - 2))$ with Fourier transform $Y(\omega)$.

(a) We see that $y(t) = x(t) * x(t - 1)$, so

$$Y(\omega) = e^{-j\omega}X^2(\omega) = e^{-j\omega}\text{sinc}^2\left(\frac{\omega}{2}\right).$$

(b) If $f(t)$ is T_0-periodically replicated to produce $g(t)$, then we know from Eq. (7.5) that $G_k = \frac{1}{T_0}F\left(k\frac{2\pi}{T_0}\right)$, where k is used to index frequency. Thus, $V_k = Y(2\pi k/3)$ implies that $v(t)$ is a 3-periodic replication of $3y(t)$. A sketch of $v(t)$ is shown in Fig. S7.9-3b.

```
>>  y = @(t) t.*((t>=0)&(t<1))+(2-t).*((t>=1)&(t<2));
>>  v = @(t) 3.*y(mod(t,3)); t = -4.5:.001:4.5;
>>  plot(t,v(t),'k'); xlabel('t'); ylabel('v(t)'); grid on
>>  set(gca,'xtick',-4:4,'ytick',0:3); axis([-4.5 4.5 -.1 3.1]);
```

(c) Using the Fourier series coefficients $V_k = Y(2\pi k/3)$, we use MATLAB to synthesize and plot $v(t)$. The result, shown in Fig. S7.9-3c using up to the 10th harmonic, is close to the signal $v(t)$ shown in Fig. S7.9-3b. As more harmonics are included, the two waveforms become indistinguishable from one another.

```
>>  snc = @(t) sinc(t/pi); % Conform MATLAB sinc to textbook notation
>>  V = @(k) exp(-1j*2*pi*k/3).*(snc(k*pi/3)).^2; omega0 = 2*pi/3;
>>  vK = zeros(size(t)); K = 10;
>>  for k = -K:K,
>>      vK = vK+V(k)*exp(1j*k*omega0*t);
```
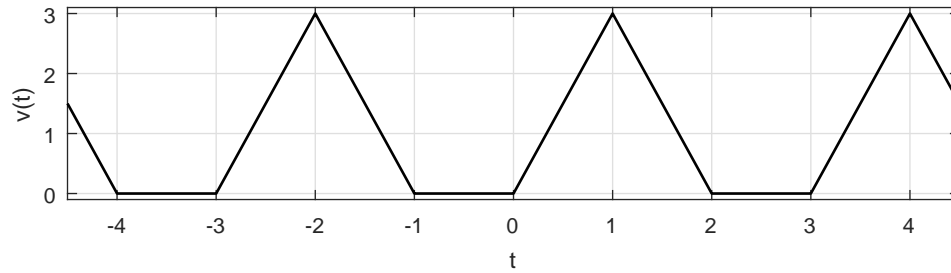
Figure S7.9-3b

```
>>  end
>>  plot(t,real(vK),'k'); xlabel('t'); ylabel('v_{10}(t)'); grid on;
>>  axis([-4.5 4.5 -.3 3.3]); set(gca,'xtick',-4:4,'ytick',0:3);
```
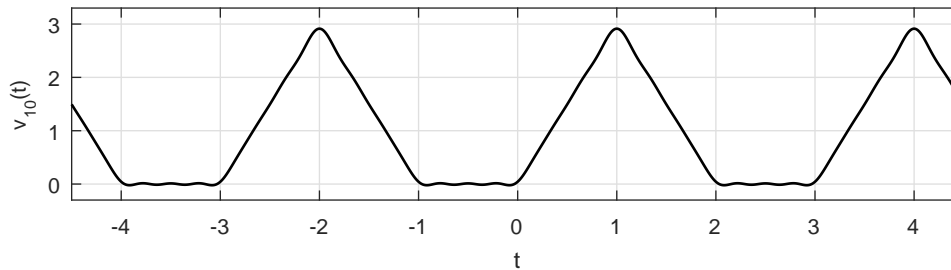

Figure S7.9-3c

(d) When we upsample V_k by factor 2 to create W_k, the corresponding time-domain relationship is $w(t) = v(2t)$. Thus, $P_k = V_k + W_k$ produces the time-domain waveform

$$p(t) = v(t) + v(2t).$$

Figure S7.9-3d shows the resulting wavform.

```
>>  p = @(t) v(t) + v(2*t);
>>  plot(t,p(t),'k'); xlabel('t'); ylabel('p(t)'); grid on
>>  set(gca,'xtick',-4:.5:4,'ytick',0:1.5:4.5); axis([-4.5 4.5 -.2 4.8]);
```
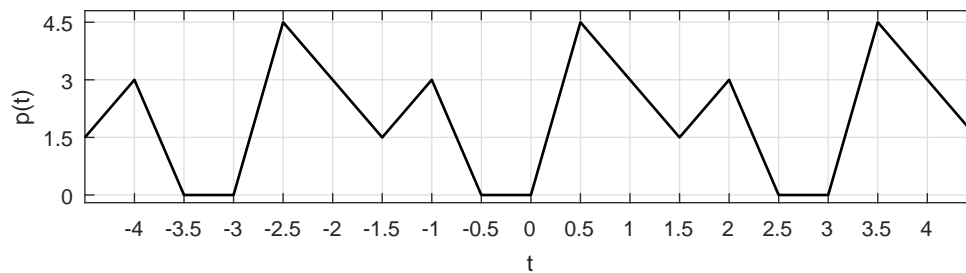

Figure S7.9-3d

(e) Using a truncated FS, we next use MATLAB to synthesize and plot $p(t)$. The result, shown in Fig. S7.9-3e using up to the 10th harmonic, is close to the signal $p(t)$ shown in Fig. S7.9-3d. As more harmonics are included, the two waveforms become indistinguishable from one another.

```
>>  P = @(k) V(k).*(mod(k,1)==0)+V(k/2).*(mod(k/2,1)==0);
>>  pK = zeros(size(t)); K = 10;
>>  for k = -K:K,
>>       pK = pK+P(k)*exp(1j*k*omega0*t);
>>  end
>>  plot(t,real(pK),'k'); xlabel('t'); ylabel('p_{10}(t)'); grid on;
>>  set(gca,'xtick',-4:.5:4,'ytick',0:1.5:4.5); axis([-4.5 4.5 -.2 4.8]);
```
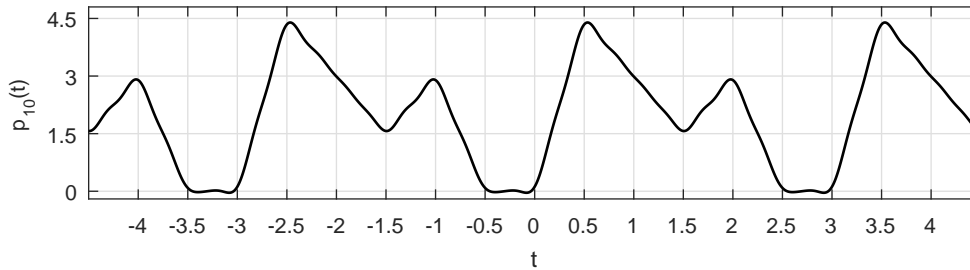

Figure S7.9-3e

Solution 7.9-4

The signal $x(t) = e^{-at}u(t)$ has Fourier Transform given by $X(\omega) = \frac{1}{j\omega+a}$ and energy $E_x = \frac{1}{2a}$. Using this information, MATLAB program CH7MP2 is modified.

```
function [W,E_W] = CH7MP2mod1(a,beta,tol)
% CH7MP2mod1.m
% Function M-file computes essential bandwidth W for exp(-at)u(t).
% INPUTS:   a = decay parameter of x(t)
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  W = essential bandwidth [rad/s]
%           E_W = Energy contained in bandwidth W
W = 0; step = a;        % Initial guess and step values
X_squared = @(omega,a) 1./(omega.^2+a.^2);
E = beta/(2*a);         % Desired energy in W
relerr = (E-0)/E;       % Initial relative error is 100 percent
while(abs(relerr)>tol),
    if (relerr>0),      % W too small, so...
        W=W+step;       % ... increase W by step
    elseif (relerr<0),  % W too large, so...
        step = step/2;  % ... decrease step size and then W.
        W = W-step;
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
    relerr = (E - E_W)/E;
end
```

(a) Setting $a = 1$ and using 95% signal energy results in

```
>>  [W,E_W] = CH7MP2mod1(1,.95,1e-9)
    W = 12.7062
    E_W = 0.4750
```

Thus,

$$W_1 = 12.7062.$$

From the text example, the essential bandwidth corresponding to 95% signal energy is derived as $W = 12.706a$ radians per second. For $a = 1$, this corresponds nicely with the computed value of $W_1 = 12.7062$.

(b) Setting $a = 2$ and using 90% signal energy results in

```
>>  [W,E_W] = CH7MP2mod1(2,.90,1e-9)
    W = 12.6275
    E_W = 0.2250
```

Thus,
$$W_2 = 12.6275.$$

(c) Setting $a = 3$ and using 75% signal energy results in

```
>>  [W,E_W] = CH7MP2mod1(3,.75,1e-9)
    W = 7.2426
    E_W = 0.1250
```

Thus,
$$W_3 = 7.2426.$$

Solution 7.9-5

To solve this problem, program CH7MP2 is modified to solve for the pulse width to achieve a desired essential bandwidth, rather than solving for the essential bandwidth that corresponds to a desired pulse.

```
function [tau,E_W] = CH7MP2mod2(W,beta,tol)
% CH7MP2mod2.m
% Function M-file computes the width of a square pulse to achieve a
% desired essential bandwidth W.
% INPUTS:    W = essential bandwidth [rad/s]
%            beta = fraction of signal energy desired in W
%            tol = tolerance of relative energy error
% OUTPUTS:   tau = pulse width
%            E_W = Energy contained in bandwidth W
tau = 1; step = 1;      % Initial guess and step values
snc = @(t) sinc(t/pi);  % Conform MATLAB sinc to textbook notation
X_squared = @(omega,tau) (tau*snc(omega*tau/2)).^2;
E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],tau);
E = beta*tau;           % Desired energy in W
relerr = (E-E_W)/E;
while(abs(relerr) > tol),
    if (relerr>0),      % tau too small, so...
        tau=tau+step;   % ... increase tau by step
     elseif (relerr<0), % tau too large, so...
        step = step/2; % ... decrease step size and then tau.
        tau = tau-step;
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],tau);
    E = beta*tau; % Desired energy in W
    relerr = (E - E_W)/E;
end
```

(a) Set $W = 2\pi5$ and select 95% signal energy.

```
>>  [tau,E_W] = CH7MP2mod2(2*pi*5,.95,1e-9)
        tau = 0.4146
        E_W = 0.3939
```

Thus,

$$\tau_1 = 0.4146.$$

(b) Set $W = 2\pi 10$ and select 90% signal energy.

```
>>  [tau,E_W] = CH7MP2mod2(2*pi*10,.90,1e-9)
        tau = 0.0849
        E_W = 0.0764
```

Thus,

$$\tau_2 = 0.0849.$$

(c) Set $W = 2\pi 20$ and select 75% signal energy.

```
>>  [tau,E_W] = CH7MP2mod2(2*pi*20,.75,1e-9)
        tau = 0.0236
        E_W = 0.0177
```

Thus,

$$\tau_3 = 0.0236.$$

Solution 7.9-6

To solve this problem, program CH7MP2 is modified to solve for the decay parameter a to achieve a desired essential bandwidth, rather than solving for the essential bandwidth that corresponds to a desired decay parameter.

```
function [a,E_W] = CH7MP2mod3(W,beta,tol)
% CH7MP2mod3.m
% Function M-file computes decay parameter a needed to achieve a
% desired essential bandwidth W.
% INPUTS:   W = essential bandwidth [rad/s]
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  a = decay parameter
%           E_W = Energy contained in bandwidth W
a = 1; step = 1;        % Initial guess and step values
X_squared = inline('1./(omega.^2+a.^2)','omega','a');
E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
E = beta/(2*a);         % Desired energy in W
relerr = (E - E_W)/E;
while(abs(relerr) > tol),
    if (relerr<0),        % a too small, so...
        a=a+step;         % ... increase a by step
     elseif (relerr>0), % a too large, so...
         step = step/2; % ... decrease step size and then a.
         a = a-step;
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
    E = beta/(2*a);     % Desired energy in W
    relerr = (E - E_W)/E;
end
```

(a) Set $W = 2\pi 5$ and select 95% signal energy.

```
>>  [a,E_W] = CH7MP2mod3(2*pi*5,.95,1e-9)
    a = 2.4725
    E_W = 0.1921
```

Thus,
$$a_1 = 2.4725.$$

(b) Set $W = 2\pi 10$ and select 90% signal energy.

```
>>  [a,E_W] = CH7MP2mod3(2*pi*10,.90,1e-9)
    a = 9.9524
    E_W = 0.0452
```

Thus,
$$a_2 = 9.9524.$$

(c) Set $W = 2\pi 20$ and select 75% signal energy.

```
>>  [a,E_W] = CH7MP2mod3(2*pi*20,.75,1e-9)
    a = 52.0499
    E_W = 0.0072
```

Thus,
$$a_3 = 52.0499.$$

Solution 7.9-7

Call the desired unit-amplitude, unit duration triangle function $x(t)$. First, notice that $x(t)$ can be constructed by convolving two rectangular process, each of width $\tau = 0.5$ and height $A = \sqrt{2}$. The energy of $x(t)$ is $E_x = 2\int_{t=0}^{0.5}(2t)^2 dt = 1/3$. Furthermore, using the convolution-in-time property and spectrum of a rectangular pulse, we know that $X(\omega) = \left(\frac{\sqrt{2}}{2}\mathrm{sinc}(\omega/4)\right)^2$.

Next, program CH7MP2 is modified to solve for the essential bandwidths of this signal for various signal energies.

```
function [W,E_W] = CH7MP2mod4(beta,tol)
% CH7MP2mod4.m
% Function M-file computes essential bandwidth W for a unit-amplitude, unit
% duration triangle function.
% INPUTS:   beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  W = essential bandwidth [rad/s]
%           E_W = Energy contained in bandwidth W
W = 0; step = 1;        % Initial guess and step values
snc = @(t) sinc(t/pi);  % Conform MATLAB sinc to textbook notation
X_squared = @(omega) (sqrt(2)/2*snc(omega /4)).^4;
E = beta/3;             % Desired energy in W
relerr = (E-0)/E;       % Initial relative error is 100 percent
while(abs(relerr) > tol),
    if (relerr>0),      % W too small, so...
        W=W+step;       % ... increase W by step
     elseif (relerr<0), % W too large, so...
        step = step/2; % ... decrease step size and then W.
        W=W-step;
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W);
    relerr = (E - E_W)/E;
end
```

Use 95% signal energy to compute the essential bandwidth:

```
>>  [W,E_W] = CH7MP2mod4(.95,1e-9)
    W = 6.2877
    E_W = 0.3167
```

Use 90% signal energy to compute the essential bandwidth:

```
>>  [W,E_W] = CH7MP2mod4(.9,1e-9)
    W = 5.3350
    E_W = 0.3000
```

Use 75% signal energy to compute the essential bandwidth:

```
>>  [W,E_W] = CH7MP2mod4(.75,1e-9)
    W = 3.7872
    E_W = 0.2500
```

Thus, the essential bandwidths are

$$W_{0.95} = 6.2877 \text{rad/s}, \ W_{0.90} = 5.3350 \text{rad/s}, \ W_{0.75} = 3.7872 \text{rad/s}.$$

Solution 7.9-8

Following the spectral sampling example in Sec. 7.9.3, the first 10 Fourier series coefficients of a $1/3$ duty-cycle square wave are

$$D_n = \frac{\tau}{T_0} \text{sinc}\left(\frac{n\pi\tau}{T_0}\right).$$

(a) Setting $T_0 = 2\pi$ and $\tau = 2\pi/3$ yields

$$D_n = \frac{1}{3} \text{sinc}\left(\frac{n\pi}{3}\right).$$

MATLAB is used to evaluate and plot the first ten coefficients.

```
>>  snc = @(t) sinc(t/pi); % Conform MATLAB sinc to textbook notation
>>  tau = 2*pi/3; T_0 = 2*pi; n = [0:10];
>>  D_n = tau/T_0*snc(n*pi*tau/T_0);
>>  stem(n,D_n,'k.'); xlabel('n'); ylabel('D_n');
>>  axis([-0.5 10.5 -0.1 0.35]); grid on;
```
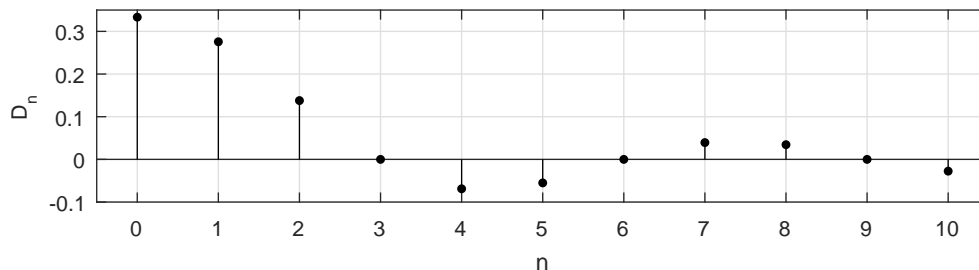

Figure S7.9-8a

(b) Setting $T_0 = \pi$ and $\tau = \pi/3$ yields

$$D_n = \frac{1}{3} \text{sinc}\left(\frac{n\pi}{3}\right).$$

Notice, the coefficients D_n depend only on the duty-cycle of the signal, not the period. Since the duty cycle is fixed, the coefficients D_n are identical to those determined in part (a) [see Fig. S7.9-8a].

Solution 7.9-9

By definition of the FT, we see that

$$X_{(\omega)} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-t^2}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-(t^2+j\omega t+(j\omega/2)^2-(j\omega/2)^2)}dt$$

$$= e^{(j\omega/2)^2}\int_{-\infty}^{\infty} e^{-(t+j\omega/2)^2}dt.$$

Substituting $t'/\sqrt{2} = t$ and $dt'/\sqrt{2} = dt$ yields

$$X(\omega) = \frac{e^{-\omega^2/4}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-(t'+j\omega^2/\sqrt{2})^2}dt'.$$

However, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(t-\alpha)^2}{2}}dt = 1$ for any a, so $\int_{-\infty}^{\infty} e^{\frac{-(t'+j\omega^2)^2}{2}}dt' = \sqrt{2\pi}$. Thus,

$$X(\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

MATLAB is used to plot $x(t)$ and $X(\omega)$.

```
>>   t = linspace (-5,5,1001); x = exp(-t.^2);
>>   subplot(211); plot(t,x,'k'); xlabel('t'); ylabel('x(t)');
>>   omega = linspace (-5,5,1001); X = sqrt(pi)*exp(-omega.^2/4);
>>   subplot(212); plot(t,X,'k'); xlabel('\omega'); ylabel('X(\omega)');
```
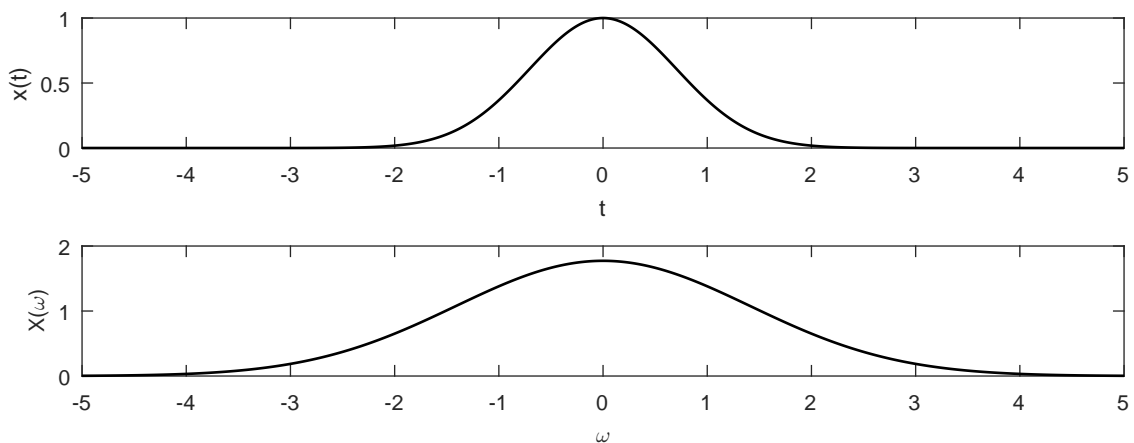

Figure S7.9-9

Figure S7.9-9 confirms that $X(\omega)$ is just a scaled and stretched version of $x(t)$. This is something remarkable; the Fourier Transform a Gaussian pulse is itself a Gaussian pulse!

Chapter 8 Solutions

Solution 8.1-1

given f_s is the Nyquist rate for signal $x(t)$, we determine the Nyquist rate for each of the following signals.

(a) For $y_a(t) = \frac{d}{dt}x(t)$, we know that $Y_a(\omega) = j\omega X(\omega)$. Thus, the bandwidth of $y_a(t)$ equals the bandwidth of $x(t)$, and

$$\text{the Nyquist rate for signal } y_a(t) \text{ is } f_s.$$

(b) For $y_b(t) = x(t)\cos(2\pi f_0 t)$, we know that $Y_b(\omega) = \frac{1}{2}X(\omega + 2\pi f_0) + \frac{1}{2}X(\omega - 2\pi f_0)$. Thus, we see that the spectrum of $y_b(t)$ is just the (scaled) spectrum of $x(t)$ shifted by $\pm f_0$. Since $x(t)$ has bandwidth $\frac{f_s}{2}$, the bandwidth of $y_b(t)$ is $\frac{f_s}{2} + f_0$, and

$$\text{the Nyquist rate for signal } y_b(t) \text{ is } f_s + 2f_0.$$

(c) In this case, $y_c(t) = x(t+a) + x(t-b)$, for real constants a and b. Since a time shift simply scales a signal's spectrum by a complex exponential, the bandwidth of $y_c(t)$ equals that of $x(t)$, and

$$\text{the Nyquist rate for signal } y_c(t) \text{ is } f_s.$$

(d) For $y_d(t) = x(at)$, where a is real and positive, we know that $Y_d(\omega) = \frac{1}{a}X\left(\frac{\omega}{a}\right)$. Thus, the bandwidth of $y_d(t)$ equals a times the bandwidth of $x(t)$, and

$$\text{the Nyquist rate for signal } y_d(t) \text{ is } af_s.$$

Solution 8.1-2

The bandwidths of $x_1(t)$ and $x_2(t)$ are 100 kHz and 150 kHz, respectively. Therefore, the Nyquist sampling rate for $x_1(t)$ is 200 kHz, and the Nyquist sampling for $x_2(t)$ is 300 kHz. Now, $x_1{}^2(t) \Longleftrightarrow \frac{1}{2\pi}X_1(\omega) * X_1(\omega)$, and from the width property of convolution the bandwidth of $x_1{}^2(t)$ is twice the bandwidth of $x_1(t)$. Similarly, the bandwidth of $x_2{}^3(t)$ is three times the bandwidth of $x_2(t)$, and the bandwidth of $x_1(t)x_2(t)$ is the sum of the bandwidths of $x_1(t)$ and $x_2(t)$. Therefore, the Nyquist rate for $x_1{}^2(t)$ is 400 kHz, the Nyquist rate for $x_2{}^3(t)$ is 900 kHz, and the Nyquist rate for $x_1(t)x_2(t)$ is 500 kHz.

Solution 8.1-3

Here, signal $x(t)$ has a bandwidth of $B = 1000$ Hz and Nyquist frequency $f_s = 2000$ Hz. For a positive integer N, we seek the Nyquist rate for the signal $y(t) = x^N(t)$. Using the properties of the Fourier transform, we know that

$$y(t) = x^2(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * X(\omega).$$

From the width property of convolution, $y(t) = x^2(t)$ has twice the bandwidth of $x(t)$. Continuing,

$$y(t) = x^3(t) \iff \frac{1}{4\pi^2}X(\omega) * X(\omega) * X(\omega).$$

Thus, $y(t) = x^3(t)$ has thrice the bandwidth of $x(t)$. Generalizing, we see that $y(t) = x^N(t)$ has N times the bandwidth of $x(t)$, and

$$\text{the Nyquist rate for signal } y(t) = x^N(t) \text{ is } Nf_\text{s} = N(2000) \text{ Hz.}$$

Solution 8.1-4

(a) From pair 20 of Table 7.1 we know that

$$x_\text{a}(t) = \text{sinc}^2(100\pi t) \iff \frac{1}{100}\Delta\left(\frac{\omega}{2\pi 200}\right).$$

Thus, the bandwidth of this signal is 200π rad/s or 100 Hz, and the Nyquist sampling rate and interval are

$$f_\text{s} = 2(100) = 200 \text{ Hz (samples/s)} \quad \text{and} \quad T_\text{s} = \frac{1}{200} \text{ s/sample (5 ms/sample).}$$

(b) Multiplying a signal by a constant does not change the signal's bandwidth. Since $x_\text{b}(t) = 0.01x_\text{a}(t)$, signal $x_\text{b}(t)$ has the same 100 Hz bandwidth as does $x_\text{a}(t)$, and the Nyquist sampling rate and interval are

$$f_\text{s} = 2(100) = 200 \text{ Hz (samples/s)} \quad \text{and} \quad T_\text{s} = \frac{1}{200} \text{ s/sample (5 ms/sample).}$$

(c) From pairs 18 and 20 of Table 7.1 we know that

$$\text{sinc}(100\pi t) + 3\text{sinc}^2(60\pi t) \iff \frac{1}{100}\text{rect}\left(\frac{\omega}{200\pi}\right) + \frac{1}{20}\Delta\left(\frac{\omega}{240\pi}\right).$$

The bandwidth of $\text{rect}\left(\frac{\omega}{200\pi}\right)$ is 50 Hz and that of $\Delta\left(\frac{\omega}{240\pi}\right)$ is 60 Hz. The bandwidth of the sum is the higher of the two, that is, 60 Hz. The Nyquist sampling rate and interval are

$$f_\text{s} = 2(60) = 120 \text{ Hz (samples/s)} \quad \text{and} \quad T_\text{s} = \frac{1}{120} \text{ s/sample (8.33 ms/sample).}$$

(d) From pair 18 of Table 7.1 and we know that

$$\text{sinc}(50\pi t) \iff \frac{1}{50}\text{rect}\left(\frac{\omega}{100\pi}\right) \quad \text{and} \quad \text{sinc}(100\pi t) \iff \frac{1}{100}\text{rect}\left(\frac{\omega}{200\pi}\right).$$

The two signals have bandwidths 25 Hz and 50 Hz, respectively. Furthermore, $x_1(t)x_2(t) \iff \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$. From the width property of convolution, the width of $X_1(\omega) * X_2(\omega)$ is the sum of the widths of the signals convolved. Therefore, the bandwidth of $\text{sinc}(50t)\text{sinc}(100t)$ is $25 + 50 = 75$ Hz, and the Nyquist sampling rate and interval are

$$f_\text{s} = 2(75) = 150 \text{ Hz (samples/s)} \quad \text{and} \quad T_\text{s} = \frac{1}{150} \text{ s/sample (6.67 ms/sample).}$$

Solution 8.1-5

(a) By inspection, we know that

$$|X(\omega)| = 3\pi \left[\delta(\omega + 6\pi) + \delta(\omega - 6\pi)\right]$$
$$+ \pi \left[\delta(\omega + 18\pi) + \delta(\omega - 18\pi)\right]$$
$$+ 2\pi \left[\delta(\omega + [28 - \epsilon]\pi) + \delta(\omega - [28 - \epsilon]\pi)\right].$$

Figure S8.1-5a shows $|X(\omega)|$ as a function of $\frac{\omega}{2\pi}$, which is frequency in Hz. Clearly, signal $x(t)$ has bandwidth of $14-\epsilon$ Hz and the minimum sampling rate is thus

$$F_{\mathrm{s}} = 2(14) = 28 \text{ Hz (samples/s)}.$$
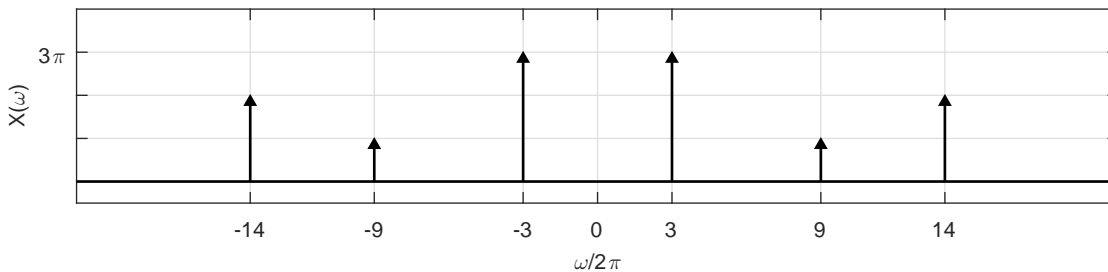
Figure S8.1-5a

(b) Sampling at 25% greater than the 28 Hz Nyquist rate results in $F_{\mathrm{s}} = 35$ Hz. Figure S8.1-5b shows the resulting spectrum over the range $|\omega/2\pi| < 50$ Hz. This is the spectrum of Fig. S8.1-5a scaled by $1/T = 35$ and periodically replicated every 35 Hz. To reconstruct $x(t)$, we pass the sampled signal through an ideal lowpass filter with gain $T = \frac{1}{F_{\mathrm{s}}} = \frac{1}{35}$ and cutoff frequency anywhere between $(14 + \beta)$ Hz to $(21 - \beta)$ Hz where β is a small positive number.
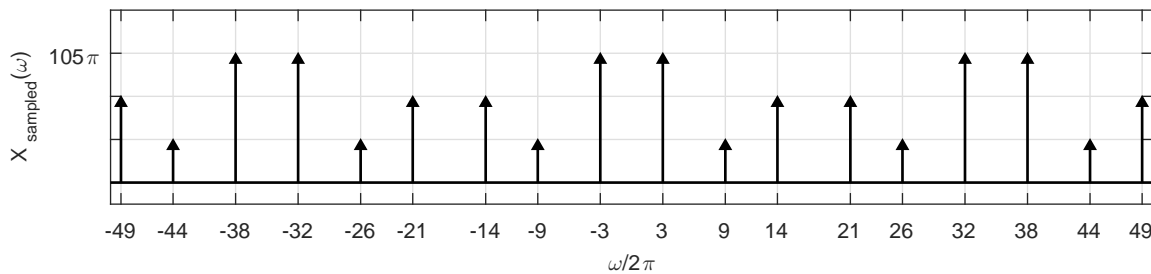
Figure S8.1-5b

Solution 8.1-6

(a) Using the results of Ex. 7.3,

$$\delta_T(t) \Longleftrightarrow \omega_{\mathrm{s}} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\omega_{\mathrm{s}}\right), \qquad \omega_{\mathrm{s}} = \frac{2\pi}{T}.$$

Hence

$$\overline{x}(t) = x(t)\delta_T(t) \Longleftrightarrow \omega_s X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$= \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega) * \delta(\omega - n\omega_s)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s).$$

This result matches the (sampling theorem) result of Eq. (8.2).

(b) Here, the sampling impulse train is given by

$$s(t) = \sum_n \delta(t - nT - \tau).$$

This is same as $\sum_n \delta(t - nT)$ right-shifted by τ. Hence

$$s(t) \Longleftrightarrow \omega_s \sum_n \delta(\omega - n\omega_s)e^{-jn\omega_s\tau}$$

and

$$x(t)s(t) \Longleftrightarrow \frac{\omega_s}{2\pi} X(\omega) * \sum_n \delta(\omega - n\omega_s)e^{-jn\omega_s\tau}$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)e^{-jn\omega_s\tau}.$$

Solution 8.1-7

In this problem, signal $x(t)$ is bandlimited to 12 kHz where the band between 10 and 12 kHz is corrupted by excessive noise. When a signal is sampled at rate f_s Hz, the spectrum is replicated every f_s Hz. To determine the minimum sampling rate for $x(t)$, we need to determine an f_s such that the spectral content between -10 and 10 kHz remains uncorrupted. To achieve this, we require that the lowest frequency of the first replicate just avoids the highest frequency of useful signal content. That is, we require that $f_s - 12 \geq 10$ or

$$f_s = 22 \text{ kHz (samples/s) minimum sampling rate for unfiltered signal } x(t).$$

This rate is lower than the Nyquist rate of $2(12) = 24$ kHz. The reason we can tolerate sub-Nyquist sampling is that the resulting aliasing only occurs between 10 and 12 kHz, a band already unusable due to noise. The useful signal content up to 10 kHz is untouched. Figure S8.1-7a illustrates this a situation, where the gray shading is used to show the corrupted band.

By filtering $x(t)$ with an ideal lowpass filter with 10-kHz cutoff frequency, we can obtain a signal $y(t)$ that contains only useful signal and no noise. This filtered signal, which is bandlimited to 10 kHz, must be sampled at no lower than its Nyquist rate, or

$$f_s = 2(10) = 20 \text{ kHz (samples/s) minimum sampling rate for filtered signal } y(t).$$

Figure S8.1-7b illustrates this a situation.
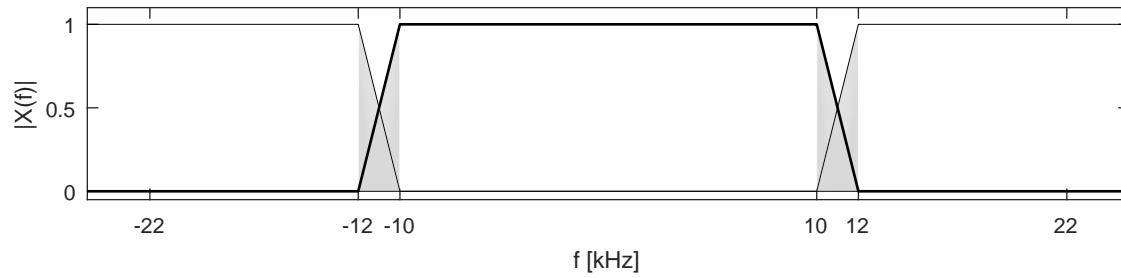
Figure S8.1-7a

Figure S8.1-7b

Solution 8.1-8

Figure S8.1-8 shows the signal $x(t) = \Lambda\left(\frac{t-1}{2}\right)$ when sampled at rates of 10, 2, and 1 Hz, respectively.
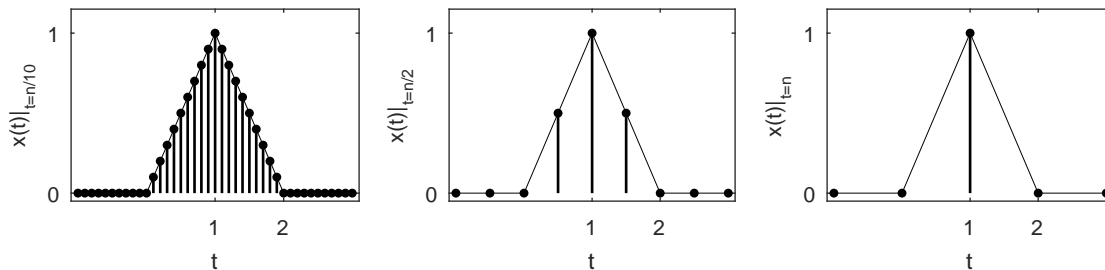
Figure S8.1-8

Using pair 19 of Table 7.1 and the Fourier transform shift property, we see that

$$x(t) = \Delta\left(\frac{t-1}{2}\right) \Longleftrightarrow \operatorname{sinc}^2\left(\frac{\omega}{2}\right)e^{-j\omega} = X(\omega).$$

This spectrum decays rapidly with frequency, so even modest sampling rates are sufficient to adequately represent this signal. To determine a suitable sampling rate, we need to determine the bandwidth B that contains a majority of the signal energy. This can be accomplished by finding the essential bandwidth B of signal $x(t)$ using, say, a 99% energy criterion, and then doubling that frequency to obtain f_s. Guided by Sec. 7.9.2 and using MATLAB's `quad` and `fminsearch` functions, we compute the 99% energy criterion bandwidth of signal $x(t)$. The program uses the fact that $x(t)$ has energy $E_x = \frac{2}{3}$.

```
>>  snc = @(t) sinc(t/pi); % Conform MATLAB sinc to textbook notation
>>  Xsquared = @(omega) (snc(omega/2)).^4;
>>  Ex = @(B) quad(Xsquared,-B,B)/(2*pi); % Energy within bandwidth B
>>  ObjFun = @(B) abs(.99*(2/3)-Ex(B)); B = fminsearch(ObjFun,1)
    B = 4.0807
```

From this program, we know that 99% of the signal energy is below 4.0807 rad/s or 0.6495 Hz. Using this 99% energy-criterion bandwidth, a reasonable sampling rate is thus

$$f_s = 2(0.6495) = 1.2989 \text{ Hz.}$$

By this calculation, the first two cases of Fig. S8.1-8 both represent significant oversampling of the signal $x(t)$.

Solution 8.1-9

Using pairs 9 and 20 of Table 7.1, the spectrum of $x(t) = 5\text{sinc}^2(5\pi t) + \cos(20\pi t)$ is given by

$$X(\omega) = \Delta \left(\frac{\omega}{20\pi} \right) + \pi[\delta(\omega + 20\pi) + \delta(\omega - 20\pi)].$$

This spectrum has impulses at ± 10 Hz and the signal bandwidth is 10 Hz. The Nyquist rate of $x(t)$ is thus 20 Hz.

(a) Let $x_a(t) = x(t)\tilde{\delta}(t)$ be a $f_s = 10$ Hz sampled version of $x(t)$. Since the sample rate is only half of the Nyquist rate, it is not possible to reconstruct $x(t)$ from $x_a(t)$. This fact is clarified by viewing the magnitude spectrum $|X_a(\omega)|$, shown in Fig. S8.1-9a. Compared with the original magnitude spectrum $|X(\omega)|$, we see that $|X_a(\omega)|$ contains an impulse function at dc as well as erroneous content between 5 and 10 Hz. The impulses are (relatively speaking) twice as large as they should be, a consequence of constructive interference between spectral replicates. There is no lowpass reconstruction filter that can reconstruct $x(t)$ from $x_a(t)$.

(b) Let $x_b(t) = x(t)\tilde{\delta}(t)$ be a $f_s = 20$ Hz sampled version of $x(t)$. The magnitude spectrum $|X_b(\omega)|$ is shown in Fig. S8.1-9b. Notice that the impulses are, relatively speaking, twice as large as expected, a consequence of constructive interference between spectral replicates at the folding frequency. Since the sample rate equals the Nyquist rate, it may be possible to reconstruct $x(t)$ from $x_b(t)$. The only potential reconstruction problem is due to the impulse functions at the folding frequency of 10 Hz. Fortunately, $\cos(20\pi t)$ proves to be the exception to the difficulty and can, at least theoretically, be reconstructed. Such reconstruction, however, requires an ideal lowpass filter $H(\omega) = \frac{1}{20}\Pi(\omega/2\pi 20)$ with cutoff frequency exactly at 10 Hz. Furthermore, the gain at 10 Hz must be exactly $\frac{1}{40}$, which is half the regular passband gain of $\frac{1}{20}$. This half-strength gain compensates for the doubling in impulse size that occurred during spectral replication. These stringent characteristics are difficult to manage. If the filter cutoff frequency is even a little too low, the reconstructed signal will lack the needed $\cos(20\pi t)$ term. If the filter cutoff frequency is even a little too large, the reconstructed signal will have a $2\cos(20\pi t)$ that is twice as large as desired. Since such a filter would be very difficult to create in practice, it would be wise to use a greater-than-Nyquist sampling rate for this signal.

(c) Next, sine replaces cosine and $x(t) = 5\text{sinc}^2(5\pi t) + \sin(20\pi t)$. Let $x_c(t) = x(t)\tilde{\delta}(t)$ be a $f_s = 20$ Hz sampled version of this new $x(t)$. As shown by the magnitude spectrum $|X_c(\omega)|$ in Fig. S8.1-9c, cancellation during spectral replication completely removes the impulses associated with the $\sin(20\pi t)$ term (another way to see how the $\sin(20\pi t)$ term is lost is by simple substitution $t = n/20$, which produces $\sin(\pi n) = 0$). Since the sine term is lost, no reconstruction filter that can reconstruct $x(t)$ from $x_c(t)$.

(d) In this part, we again use $x(t) = 5\text{sinc}^2(5\pi t) + \sin(20\pi t)$. Let $x_d(t) = x(t)\tilde{\delta}(t)$ be a $f_s = 21$ Hz sampled version of $x(t)$. The magnitude spectrum $|X_d(\omega)|$ is shown in Fig. S8.1-9d. In this case, the sampling frequency exceeds the Nyquist rate and $x(t)$ can be obtained by lowpass filtering $x_d(t)$. The lowpass filter would need a cutoff frequency between 10 and 11 Hz and a passband gain of $\frac{1}{21}$.
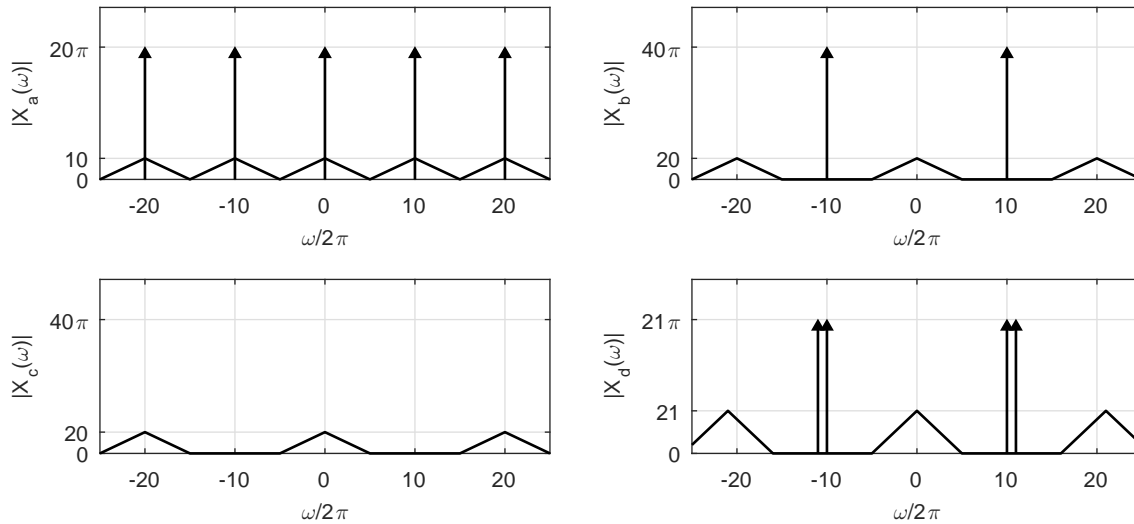
Figure S8.1-9

Solution 8.1-10

(a) The spectrum $X(\omega)$ is bandpass in nature with bandwidth 10 Hz centered at 25 Hz. The highest frequency is 30 Hz. If we use a 60 Hz sampling frequency, the resulting spectrum $X_a(\omega)$ is just $X(\omega)$ scaled by 60 and replicated every 60 Hz, as shown in Fig. S8.1-10.

```
>>  tri = @(f) (1-2*f).*((f>=0)&(f<=1/2))+(2*f+1).*((f>=-1/2)&(f<0));
>>  X = @(f) tri((f-25)/10)+tri((f+25)/10);
>>  Xa = @(f) 60*(X(f)+X(f-60)+X(f+60)+X(f-120)+X(f+120));
>>  f = linspace(-100,100,10001); subplot(311); plot(f,Xa(f)); grid on;
>>  axis([-100 100 0 65]); xlabel('\omega/2\pi'); ylabel('X_a(\omega)');
```

In this case, the original spectrum $X(\omega)$ remains intact and can be recovered with a bandpass filter that has 10 Hz bandwidth centered at 25 Hz and passband gain of $\frac{1}{60}$. The upper cutoff frequency of this reconstruction filter needs to be nearly perfect, however, in order to remove the adjacent replicate found starting at 30 Hz.

(b) When we use a sampling frequency of 20 Hz, the resulting spectrum $X_b(\omega)$ is just $X(\omega)$ scaled by 20 and replicated every 20 Hz, also shown in Fig. S8.1-10.

```
>>  tri = @(f) (1-2*f).*((f>=0)&(f<=1/2))+(2*f+1).*((f>=-1/2)&(f<0));
>>  X = @(f) tri((f-25)/10)+tri((f+25)/10);
>>  Xb = @(f) 20*(X(f)+X(f-20)+X(f+20)+X(f-40)+X(f+40)+X(f-60)+X(f+60));
>>  f = linspace(-35,35,10001); subplot(312); plot(f,Xb(f)); grid on;
>>  axis([-35 35 0 22]); xlabel('\omega/2\pi'); ylabel('X_b(\omega)');
```

Rather fortuitously in this case, none of the replicates overlap, they only just touch. As a result the original spectrum $X(\omega)$ remains intact and can be recovered with a bandpass filter that has 10 Hz bandwidth centered at 25 Hz and passband gain of $\frac{1}{20}$. The reconstruction filter needs to be nearly perfect on both sides, however, in order to remove the adjacent replicates found just outside the 20 to 30 Hz band of desired signal content.

(c) When we use a sampling frequency of 20 Hz on signal $y(t)$, the resulting spectrum $Y_c(\omega)$ is $Y(\omega)$ scaled by 20 and replicated every 20 Hz, as shown in Fig. S8.1-10.

```
>>  tri = @(f) (1-2*f).*((f>=0)&(f<=1/2))+(2*f+1).*((f>=-1/2)&(f<0));
>>  Y = @(f) tri((f-23)/10)+tri((f+23)/10);
>>  Yc = @(f) 20*(Y(f)+Y(f-20)+Y(f+20)+Y(f-40)+Y(f+40)+Y(f-60)+Y(f+60));
>>  f = linspace(-35,35,10001); subplot(313); plot(f,Yc(f)); grid on;
>>  axis([-35 35 0 22]); xlabel('\omega/2\pi'); ylabel('Y_c(\omega)');
```

Unfortunately in this case, the replication of $Y(\omega)$ that results from sampling causes overlapping components. The original spectrum $Y(\omega)$ is destroyed by these overlapping pieces and cannot be recovered.
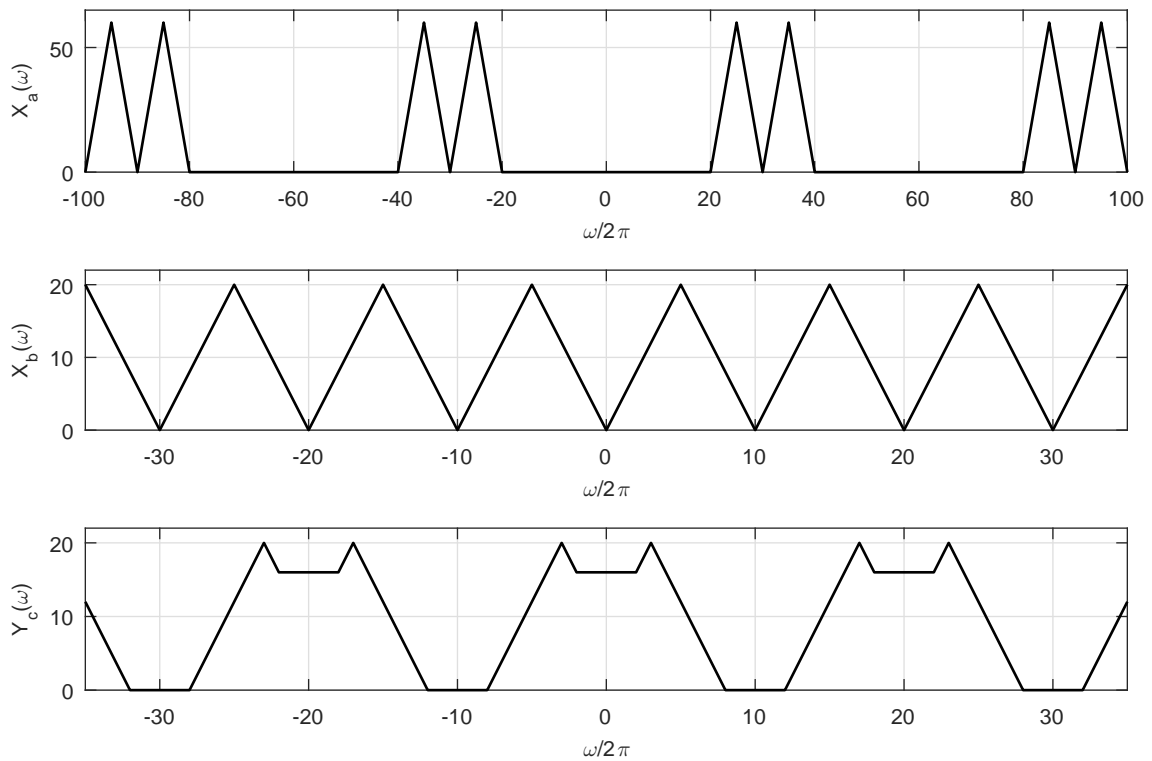
Figure S8.1-10

Solution 8.1-11

This problem is trivial when worked out in the frequency-domain. The sampled signal spectrum is given by

$$X_{\text{sampled}}(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n2\pi f_s).$$

We repeat the spectrum periodically with period $(f_1 + f_2)$ Hz, as shown in the top plot of Fig. S8.1-11. The amplitude at the origin is $\frac{1}{T} = f_1 + f_2$. From Fig. S8.1-11, it is obvious that the resulting spectrum $X_{\text{sampled}}(\omega)$ is constant for all ω and has a value $f_1 + f_2$, as shown in the bottom plot of Fig. S8.1-11. Hence,

$$X_{\text{sampled}}(\omega) = f_1 + f_2 \iff (f_1 + f_2)\delta(t) = x_{\text{sampled}}(t).$$

Clearly, all the samples of $x_{\text{sampled}}(t)$ at a rate $f_s = f_1 + f_2$ are zero except the sample at $t = 0$, which has (impulse) amplitude $f_1 + f_2$.
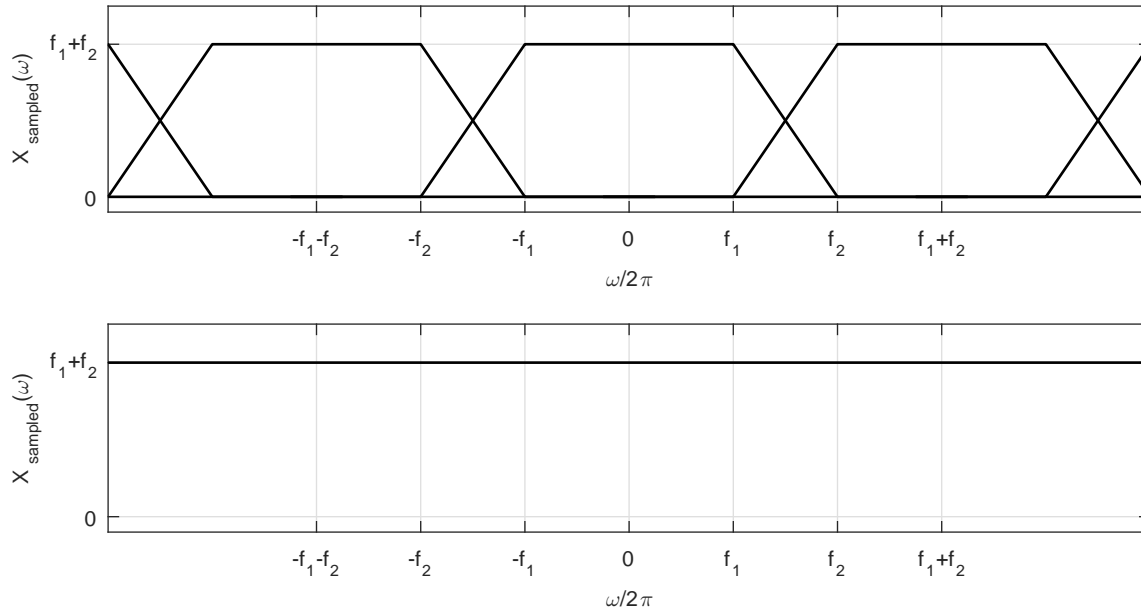
Figure S8.1-11

Solution 8.1-12

This problem considers two cases of practical sampling achieved by multiplying a signal $x(t)$ by a periodic train of pulses $p_T(t)$, where

$$p_T(t) = \sum_{k=-\infty}^{\infty} p(t - kT)$$

The two pulses are under consideration are

$$p_a(t) = -\tfrac{1}{4}u(t) + \tfrac{5}{4}u(t - \tfrac{2T}{20}) + \\ -\tfrac{5}{4}u(t - \tfrac{3T}{20}) + \tfrac{1}{4}u(t - \tfrac{5T}{20})$$

and

$$p_b(t) = e^{-t/T}\left[u(t) - u(t - 1.5T)\right]$$

The desired sampling rate is $f_s = 100$ Hz (or $T = 0.01$).

(a) Figure S8.1-12a shows $p_T(t)$ using $p_a(t)$ over $0 \le t \le 4T$.
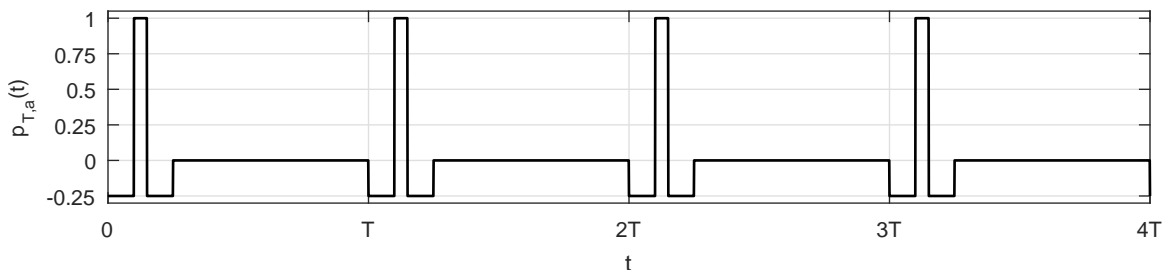
Figure S8.1-12a

(b) Figure S8.1-12b shows $p_T(t)$ using $p_b(t)$ over $0 \le t \le 4T$.
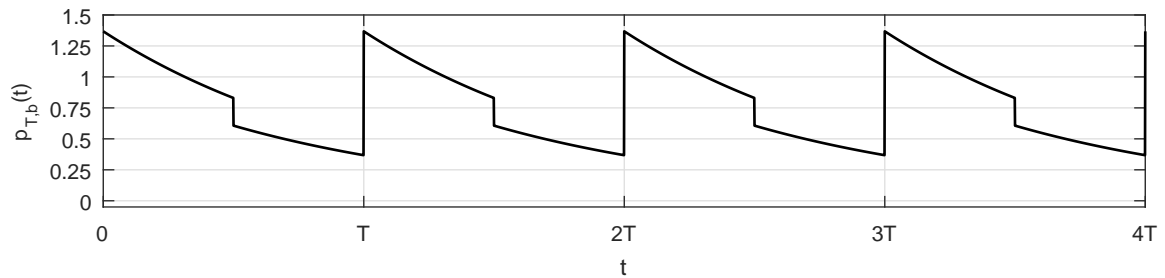
Figure S8.1-12b

(c) Superficially, pulse $p_a(t)$ produces a more delta-like sampling train than does $p_b(t)$, which is wider than the desired sampling interval T. However, $p_a(t)$ has zero area, which makes it unsuitable as a sampling function (see Drill 8.2). The area of pulse $p_b(t)$, on the other hand, is non-zero, which is desirable of a sampling pulse. Thus,

$$p_b(t) \text{ is more suitable than } p_a(t) \text{ as a sampling pulse.}$$

Solution 8.1-13

In this problem, we set to demonstrate the important principle in communication theory that the maximum information rate is 2 pieces of information per second per Hz.

A knowledge of the maximum rate of information that can be transmitted over a channel of bandwidth B Hz is of fundamental importance in digital communication. We can demonstrate a scheme which allows error-free transmission of $2B$ independent pieces of information per second over a channel of bandwidth B Hz. Recall that a continuous-time signal $x(t)$ of bandwidth B Hz can be constructed from its Nyquist samples (which are at a rate of $2B$ Hz) using the interpolation formula of Eq. (8.6). Letting the kth piece of information equal to $x(kT)$, the kth Nyquist sample, and using the interpolation formula of Eq. (8.6), we can construct a signal $x(t)$ that is bandlimited to B Hz. Clearly, this signal can be transmitted error-free over the channel of bandwidth B Hz. Moreover, the $2B$ pieces of information are readily obtained (error-free) from this signal by taking its Nyquist samples.

This theoretical rate of communication assumes a noise-free channel. In practice, channel noise is unavoidable, and consequently, this rate will cause some detection errors.

We shall prove the converse of this result using the method of *reductio ad absurdum*. Independent pieces imply that each piece of information can be any one of the (uncountably) infinite number of amplitudes. To prove the converse, let us assume that a scheme exists that can transmit more than $2B$ independent pieces of information/s. If this were the case, the interpolation formula implies that we can transmit a signal of bandwidth higher than B Hz over a channel of bandwidth B Hz, which is not possible.

Solution 8.1-14

In this problem, $x(t) = \text{sinc}(4\pi t)$ and $X(\omega) = \frac{1}{4}\text{rect}(\omega/8\pi)$. The bandwidth of $x(t)$ is $B = 2$ Hz, and its Nyquist rate is 4 Hz.

(a) The spectrum $X(\omega) = \frac{1}{4}\text{rect}(\omega/8\pi)$ is rectangular of width 8π rad/s or 4 Hz. To find the sampled signal spectrum $X_a(\omega)$, we multiply $X(\omega)$ by $\frac{1}{T} = 4$ and repeat it periodically with period $f_s = 4$ Hz (8π rad/s). As shown in Fig. S8.1-14, the resulting spectrum $X_a(\omega)$ equals one for all ω.

(b) To reconstruct $x(t)$ from $X_a(\omega)$, we pass $X_a(\omega)$ through an ideal lowpass filter of gain $T = \frac{1}{4}$ and bandwidth $f_s/2 = 2$ Hz. The input is $X_a(\omega) = 1$. Hence the output spectrum is $\frac{1}{4}\text{rect}(\omega/8\pi)$ and the output is indeed $x(t) = \text{sinc}(4\pi t)$, as expected.

(c) If we sample $x(t)$ at a rate $f_s = 2$ Hz $(T = 1/2)$, the sampled signal spectrum is

$$X_c(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - 2\pi n f_s) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 4\pi n}{8\pi}\right).$$

Thus $X_c(\omega)$ is obtained by repeating $\frac{1}{2}\text{rect}\left(\frac{\omega}{8\pi}\right)$ with period $\omega = 4\pi$ rad/s (2 Hz), as shown in Fig. S8.1-14. Each replicated rectangle has amplitude $1/2$ and overlaps half with the rectangle before and half with the rectangle after. When summed together, the result is a constant value $X_c(\omega) = 1$. When this $X_c(\omega)$ is applied to the filter of part (b), the output spectrum is again $\frac{1}{4}\text{rect}(\omega/8\pi)$, corresponding to $x(t) = \text{sinc}(4\pi t)$.

(d) We get an identical result using $f_s = 1$ Hz, where $X(\omega)$ repeats every 2π rad/s (1 Hz), but the amplitude is only $1/4$. For all ω, there are 4 overlapping rectangle components, the sum of which again yields sampled signal spectrum $X_d(\omega) = 1$. When $X_d(\omega)$ is applied to the filter of part (b), the output spectrum is again $\frac{1}{4}\text{rect}(\omega/8\pi)$, corresponding to $x(t) = \text{sinc}(4\pi t)$.

(e) When sampled at a rate $f_s = \frac{4}{N}$ Hz (N integer), $X(\omega)$ is repeated every $\frac{4}{N}$ Hz with amplitude $\frac{1}{N}$. For any ω, there are exactly N replicated rectangles, the sum of which is always one. Thus, the sampled signal spectrum $X_{\text{sampled}}(\omega) = 1$ for all sampling rates $f_s = \frac{4}{N}$ Hz. When such $X_{\text{sampled}}(\omega)$ are applied to the filter of part (b), the output spectrum is always $\frac{1}{4}\text{rect}(\omega/8\pi)$, corresponding to $x(t) = \text{sinc}(4\pi t)$.

(f) The signal $x(t) = \text{sinc}(4\pi t)$ is shown in Fig. S8.1-14. Observe that $x(t) = 0$ at $t = \frac{n}{4}$ for all integer values of n. This means sampling $x(t)$ at a rate $f_s = \frac{4}{N}$ $(T = \frac{N}{4})$ yields all zero valued samples except at $t = 0$. That is, for $T = N/4$,

$$x(nT) = \text{sinc}(nN\pi) = \begin{cases} 1 & n = 0 \\ 0 & n = \pm1, \pm2, \pm3, \dots \end{cases}.$$

Since $x(t)$ sampled at any rate $f_s = \frac{4}{N}$ are identical regardless of N, so too will the signals reconstructed through the filter of part (b) be identical, namely $x(t) = \text{sinc}(4\pi t)$.
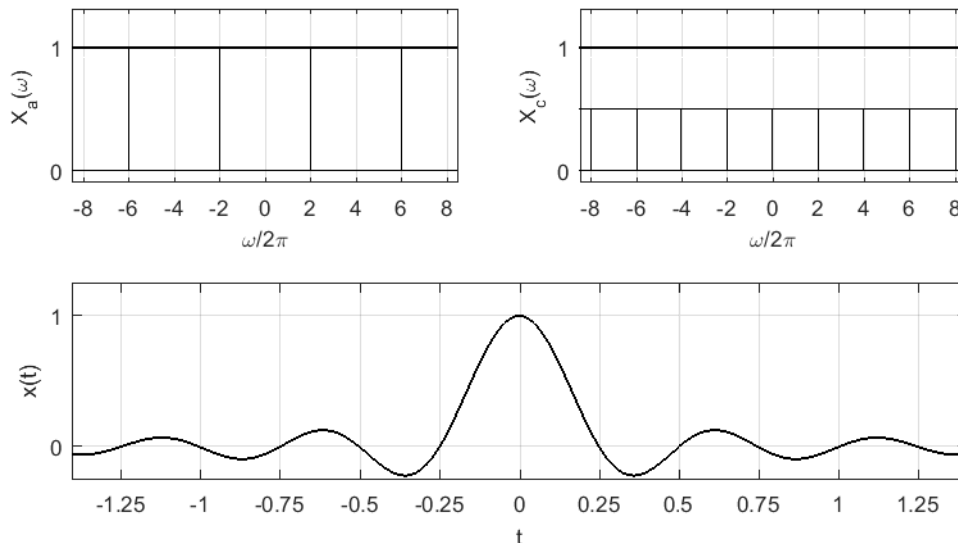
Figure S8.1-14

Solution 8.2-1

From pair 18 of Table 7.1, we know that

$$x(t) = \text{sinc}(200\pi t) \iff \frac{1}{200}\text{rect}\left(\frac{\omega}{400\pi}\right) = X(\omega).$$

Pulse sampling signal $x(t)$ produces signal $x_P(t) = x(t)p_T(t)$ with spectrum

$$X_P(\omega) = \sum_{n=-\infty}^{\infty} P_n X(\omega - n\omega_s),$$

where P_n are the (exponential-form) Fourier series coefficients of $p_T(t)$.

Pulse train $p_T(t)$ has period 4 ms and thus fundamental frequency $\omega_0 = 2\pi/0.004 = 500\pi$. Using Eq. (6.19), the Fourier series coefficients are computed as

$$\begin{aligned}
\tilde{P}_n &= \frac{1}{0.004}\int_{-0.0004}^{0.0004} e^{j500\pi nt}\, dt \\
&= 250\frac{e^{j500\pi nt}}{j500\pi n}\Bigg|_{t=-0.0004}^{0.0004} \\
&= \frac{e^{j0.2\pi n} - e^{-j0.2\pi n}}{n\pi(2j)} \\
&= 0.2\text{sinc}(0.2\pi n).
\end{aligned}$$

Thus, the pulse-sampled signal $x_P(t)$ has spectrum

$$X_P(\omega) = \sum_{n=-\infty}^{\infty} 0.2\text{sinc}(0.2\pi n)\frac{1}{200}\text{rect}\left(\frac{\omega - n\omega_s}{400\pi}\right).$$

This spectrum is readily plotted with MATLAB.

```
>>  omegas = 500*pi; omega = linspace(-2*pi*1000,2*pi*1000,10001);
>>  snc = @(t) sinc(t/pi); u = @(t) 1.0.*(t>=0);
>>  rect = @(omega) u(omega+0.5)-u(omega-0.5); Xp = zeros(size(omega));
>>  for n = -4:4,
>>      Xp = Xp+sinc(0.2*n)/1000*rect((omega-n*omegas)/(400*pi));
>>  end
>>  plot(omega/(2*pi),Xp); xlabel('\omega/2\pi'); ylabel('X_P(\omega)');
>>  axis([-1000 1000 0 .0011]); grid on; set(gca,'xtick',-750:250:750);
```
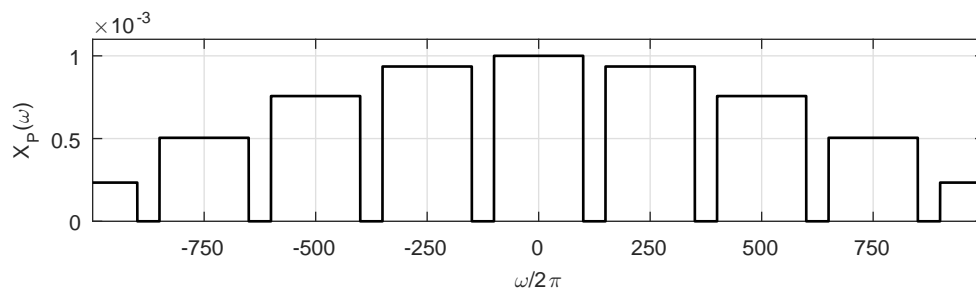

Figure S8.2-1

As shown in Fig. S8.2-1, there is no overlap between replicates, and $X(\omega)$ can be recovered by using an ideal lowpass filter of bandwidth 100 Hz. An ideal lowpass filter of standard unit gain (and

bandwidth 100 Hz) will pass the $n = 0$ term of $X_{\tilde{p}}(\omega)$ and suppress all the other terms. Hence, the output $y(t)$ is

$$y(t) = 0.2\, x(t).$$

The 0.2 gain occurs because of the $0.2\text{sinc}(0.2\pi n)$ term and can be easily corrected by using an ideal lowpass filter with gain 5 instead of unity.

Because the spectrum $X_P(\omega)$ has a zero value in the band from 100 to 150 Hz, we can use an ideal lowpass filter of bandwidth B Hz where $100 < B < 150$. But if $B > 150$ Hz, the filter will pick up the unwanted spectral components from the $n \neq 0$ replicates, and the output will be distorted.

Solution 8.2-2

When an input of $x(t) = \delta(t)$ is applied to the system of Fig. P8.2-2, the output of the summer is $\delta(t) - \delta(t - T)$. The integral of $\delta(t) - \delta(t - T)$ is simply $u(t) - u(t - T)$. Thus, the impulse response of the block diagram of Fig. P8.2-2 is

$$h(t) = u(t) - u(t - T) = \text{rect}\left(\frac{t - T/2}{T}\right).$$

This impulse response matches the ZOH impulse response of Eq. (8.5) delayed by $T/2$.

To illustrate the ZOH nature of this system, notice that an impulse sampled signal $x_{\text{sampled}}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$ produces an output

$$y(t) = \hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\text{rect}\left(\frac{t - nT - T/2}{T}\right).$$

From this equation we see that when an impulse-sampled signal, such as that shown left in Fig. S8.2-2, is applied to this causal ZOH system, the output is staircase approximation of the signal where each stair extends to the right of the impulse that generated it, such as shown right in Fig. S8.2-2. The constraint of causality delays the output by $T/2$, which is why the staircase approximation is not centered but to the right of the input delta functions.
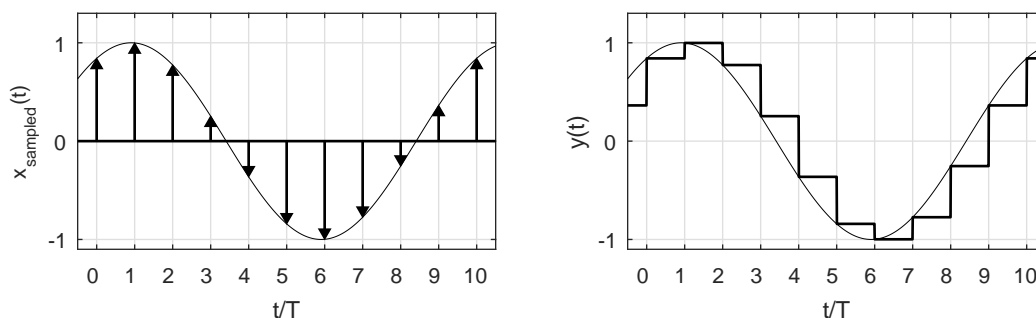
Figure S8.2-2

Solution 8.2-3

(a) When an impulse sampled signal $x_{\text{sampled}}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$ is applied to a FOH system with impulse response $h(t) = \Delta\left(\frac{t}{2T}\right)$, the output is

$$y(t) = \hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\Delta\left(\frac{t - nT}{2T}\right).$$

Each impulse of the input produces a $2T$-wide triangle scaled by $x(nT)$. Each pair of adjacent triangles overlap over an interval T, and the sum produces a linear interpolation of the samples

$x(nT)$. To see this, consider an impulse sampled signal $x_{\text{sampled}}(t)$ such as the one shown in the left plot of Fig. S8.2-3a. The FOH system sums the overlapping triangles $x(nT)\Delta\left(\frac{t-nT}{2T}\right)$ to produce the linear-interpolation output $y(t)$, as shown in the right plot of Fig. S8.2-3a.
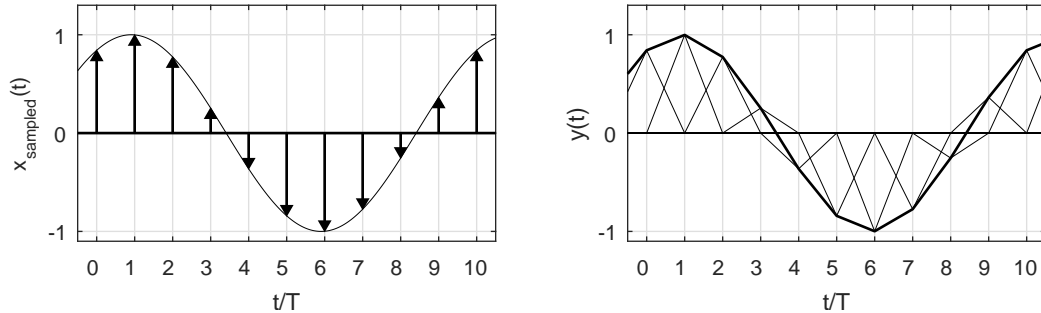
Figure S8.2-3a

(b) From pair 19 of Table 7.1, the frequency response of the FOH system is

$$h_{\text{FOH}}(t) = \Delta\left(\frac{t}{2T}\right) \Longleftrightarrow T\operatorname{sinc}^2\left(\frac{\omega T}{2}\right) = H_{\text{FOH}}(\omega).$$

A ZOH system has impulse response $h_{\text{ZOH}}(t) = \operatorname{rect}\left(\frac{t}{T}\right)$. From pair 17 of Table 7.1, the frequency response of the ZOH system is

$$h_{\text{ZOH}}(t) = \operatorname{rect}\left(\frac{t}{T}\right) \Longleftrightarrow T\operatorname{sinc}\left(\frac{\omega T}{2}\right) = H_{\text{ZOH}}(\omega).$$

Finally from pair 18 of Table 7.1, we know that the frequency response of an ideal reconstruction filter is

$$h_{\text{ideal}}(t) = \operatorname{sinc}(t\pi/T) \Longleftrightarrow T\operatorname{rect}\left(\frac{\omega T}{2\pi}\right) = H_{\text{ideal}}(\omega).$$

As Fig. S8.2-3b makes clear, the FOH magnitude response (solid curve) more closely resembles the ideal reconstruction filter (dash-dot curve) than does the ZOH magnitude response (dashed curve). This is especially true for frequencies beyond $f_{\text{s}}/2$ since the FOH response decays exponentially faster than the ZOH. Neither the FOH or ZOH reconstruction filters do particularly well for higher frequencies in the upper passband of the ideal response, although the ZOH response is slightly better. Both the FOH and ZOH overly attenuate components in the upper passband region.
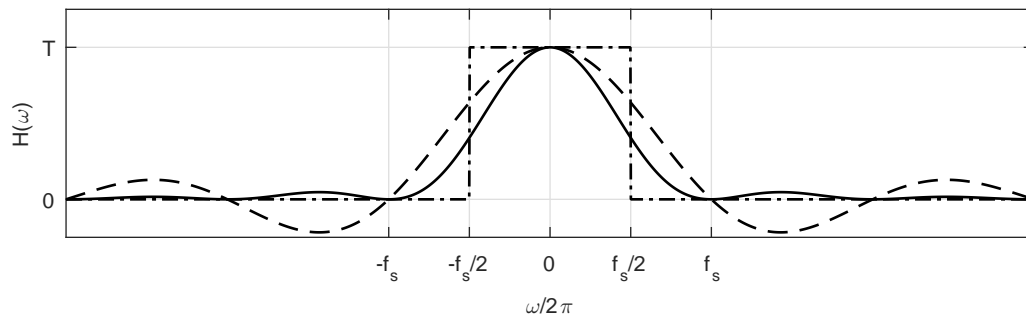
Figure S8.2-3b

(c) A minimum of a T second delay is required to make $h_{\text{FOH}}(t)$ causal (realizable), that is $h_{\text{FOH,causal}}(t) = h_{\text{FOH}}(t - T)$. This delay would not change the filter magnitude response ($|H_{\text{FOH,causal}}(\omega)| = |H_{\text{FOH}}(\omega)|$), but it would add a linear phase component $e^{-j\omega T}$. Of course, delaying the impulse response by T also causes a corresponding delay of T seconds in the reconstructed signal $\hat{x}(t)$. This delay is the necessary price for realizable interpolation.

(d) The impulse response and the frequency response of a causal ZOH circuit are

$$h_{\text{ZOH,causal}}(t) = \text{rect}\left(\frac{t - T/2}{T}\right) \Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)e^{-j\omega T/2} = H_{\text{ZOH,causal}}(\omega).$$

The frequency response of a cascade of two such causal ZOH circuits is given by

$$H_{\text{cascade}}(\omega) = H_{\text{ZOH,causal}}^2(\omega) = T^2 \text{sinc}^2(\omega T/2)e^{-j\omega T}.$$

Earlier, we found that the frequency response of an FOH circuit is given as

$$H_{\text{FOH}}(\omega) = T\text{sinc}^2\left(\frac{\omega T}{2}\right).$$

Comparing, we see that

$$H_{\text{cascade}}(\omega) = TH_{\text{FOH}}(\omega)e^{-j\omega T}.$$

This shows that the frequency response of a cascade of two ZOH circuits is T times the frequency response of the FOH circuit with a time delay of T seconds. The time delay is a desirable feature as it makes the FOH circuit causal, and therefore, realizable. Thus, the cascade of two ZOH acts identical to an FOH circuit except for the (unimportant) amplification by factor T and delay by T seconds.

Solution 8.2-4

This problem considers signal $x(t) = \sin(2\pi t/8)\,(u(t) - u(t-8))$ sampled at a rate $f_s = 1$ Hz to generate signal $x[n]$.

(a) Signals $x(t)$ and $x[n]$ are shown in the upper left plot of Fig. S8.2-4.

(b) Yes. Aliasing has occurred in sampling $x(t)$ to produce $x[n]$. Since $x(t)$ is time-limited, we know that its spectrum $X(\omega)$ is not bandlimited. It is not possible to sample a non-bandlimited signal without aliasing (an infinite sample rate would be required, which is clearly impractical). Another way to understand the aliasing that occurs in sampling $x(t)$ to produce $x[n]$ is to consider the starting and ending times of $x(t)$: it is impossible to exactly locate the start or end times of $x(t)$ using a finite sample rate (there would always be a T-second uncertainty in knowing these times).

(c) The upper right plot of Fig. S8.2-4 shows the output $\hat{x}(t)$ produced when $x[n]$ is applied to the causal ZOH reconstructor of Prob. 8.2-2. The reconstructed signal $\hat{x}(t)$ follows the general trend of $x(t)$, but there are deficiencies as well: there is a clear rightward shift (due to the ZOH being causal), and the stair-step nature of $\hat{x}(t)$ does not match the (generally) smooth nature of $x(t)$.

(d) The lower left plot of Fig. S8.2-4 shows the output $\hat{x}(t)$ produced when $x[n]$ is applied to the (noncausal) FOH reconstructor of Prob. 8.2-3, while the lower right plot of Fig. S8.2-4 shows the output $\hat{x}(t)$ produced when $x[n]$ is applied to the (causal) FOH reconstructor of Prob. 8.2-3. The FOH reconstructor clearly produces a better reconstructor of $x(t)$ than does the ZOH reconstructor considered in part (c). Notice that the causal FOH output is just the (noncausal) FOH output right-shifted by T, the minimum delay to produce a causal reconstructor. Due to the nature of $x(t)$ (which is zero at both its starting and ending points), it is not easy to detect the non-causal nature of the noncausal FOH output (lower left graph of Fig. S8.2-4).
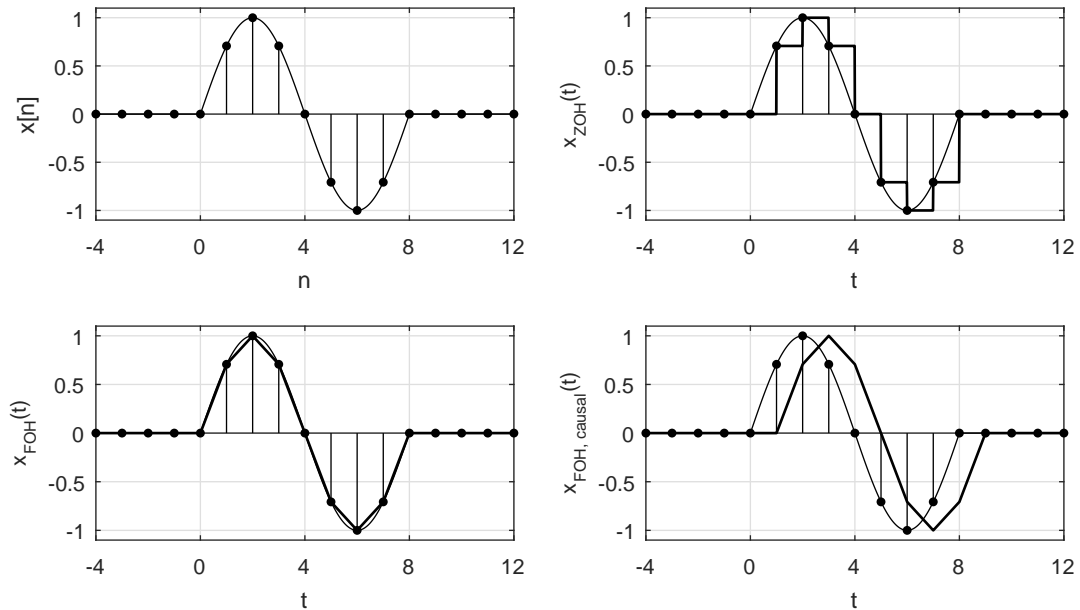
Figure S8.2-4

Solution 8.2-5

This problem considers signal $x(t) = \cos(2\pi t/8)\,(u(t) - u(t-8))$ sampled at a rate $f_s = 1$ Hz to generate signal $x[n]$.

(a) Signals $x(t)$ and $x[n]$ are shown in the upper left plot of Fig. S8.2-5.

(b) Yes. Aliasing has occurred in sampling $x(t)$ to produce $x[n]$. Since $x(t)$ is time-limited, we know that its spectrum $X(\omega)$ is not bandlimited. It is not possible to sample a non-bandlimited signal without aliasing (an infinite sample rate would be required, which is clearly impractical). Another way to understand the aliasing that occurs in sampling $x(t)$ to produce $x[n]$ is to consider the starting and ending times of $x(t)$: it is impossible to exactly locate the start or end times of $x(t)$ using a finite sample rate (there would always be a T-second uncertainty in knowing these times).

(c) The upper right plot of Fig. S8.2-5 shows the output $\hat{x}(t)$ produced when $x[n]$ is applied to the causal ZOH reconstructor of Prob. 8.2-2. The reconstructed signal $\hat{x}(t)$ follows the general trend of $x(t)$, but there are deficiencies as well: there is a clear rightward shift (due to the ZOH being causal), and the stair-step nature of $\hat{x}(t)$ does not match the (generally) smooth nature of $x(t)$.

(d) The lower left plot of Fig. S8.2-5 shows the output $\hat{x}(t)$ produced when $x[n]$ is applied to the (noncausal) FOH reconstructor of Prob. 8.2-3, while the lower right plot of Fig. S8.2-5 shows the output $\hat{x}(t)$ produced when $x[n]$ is applied to the (causal) FOH reconstructor of Prob. 8.2-3. The FOH reconstructor generally produces a better reconstructor of $x(t)$ than does the ZOH reconstructor considered in part (c); this is especially true where $x(t)$ is relatively smooth. The FOH reconstruction has difficulty, however, at the start and end-point discontinuities of $x(t)$. Notice that the causal FOH output is just the (noncausal) FOH output right-shifted by T, the minimum delay to produce a causal reconstructor. Unlike the signal used in Prob. 8.2-4, $x(t)$ in this problem makes it easy to see the non-causal nature of the noncausal FOH output (lower left graph of Fig. S8.2-5).
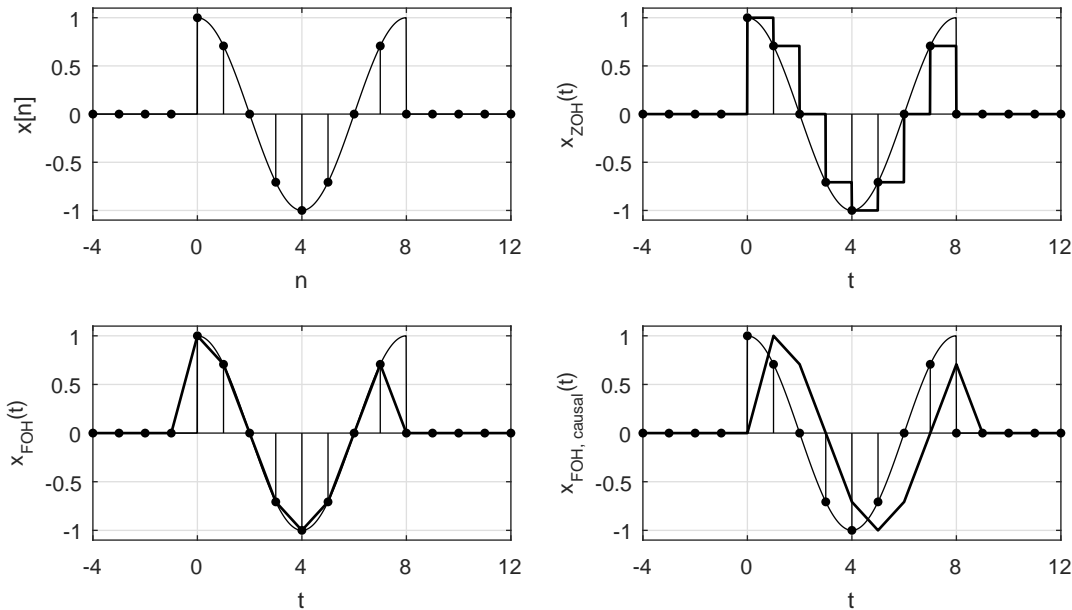
Figure S8.2-5

Solution 8.2-6

Since any practical, physically realizable signal $x(t)$ is time-limited, we know that its spectrum $X(\omega)$ is not bandlimited. It is not possible to sample a non-bandlimited signal without aliasing (an infinite sample rate would be required, which is clearly impractical). Still, most practical signals are effectively bandlimited, which is to say that there is some maximum frequency $f_{\max}$ above which there is negligible signal content. If we sample such a signal at a rate $f_s \geq 2f_{\max}$, the the signal can be sampled with negligible aliasing.

Solution 8.2-7

Referencing Sec. 8.1.1, a practical pulse-sampled signal can be modeled as

$$x_p(t) = p_T(t)x(t) = \sum_{n=-\infty}^{\infty} P_T[n]e^{jn\omega_s t}x(t),$$

where $\sum_{n=-\infty}^{\infty} P_T[n]e^{jn\omega_s t}$ is just the Fourier series of (an arbitrary) periodic pulse train $p_T(t)$. The spectrum of this pulse-sampled signal is

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} P_T[n]X(\omega - n\omega_s).$$

As this equation shows, the only impact of the impulse train on $X_p(\omega)$ is the amplitude (due to Fourier series coefficient $P_T[n]$) and spacing (due to ω_s) of the replicates $X(\omega - n\omega_s)$. As long as $x(t)$ is bandlimited to $B < \frac{f_s}{2}$ Hz (Nyquist sampling or better), the replicates in $X_p(\omega)$ do not overlap *regardless of the shape or duration of pulses in the pulse-train $p_T(t)$*, and the original signal $x(t)$ can be recovered.

To provide an example, consider pulse-train $p_T(t)$ comprised of semi-infinite duration pulses $p(t) = e^{-at}u(t)$. Assuming $a > 0$, which is required for the signal and its Fourier transform to exist, the Fourier transform of $p(t)$ is given as

$$P(\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}\,dt = \left.\frac{e^{t(-a-j\omega)}}{-a - j\omega}\right|_{t=0}^{\infty} = \frac{1}{a + j\omega}.$$

When $p(t)$ is periodically replicated with period T, the Fourier series coefficients are just scaled and sampled values of $P(\omega)$. That is,

$$P_T[n] = \frac{1}{T}P(n2\pi/T) = \frac{1}{T}\left(\frac{1}{a + jn2\pi/T}\right) = \frac{1}{aT + jn2\pi}.$$

Thus, the spectrum of the pulse-sampled signal $x_p(t)$ is

$$X_p(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{aT + jn2\pi}X(\omega - n\omega_s)$$

Since $x(t)$ is bandlimited to $B < \frac{f_s}{2}$ Hz, the replicates $X(\omega - n\omega_s)$ do not overlap. Thus, the original signal $x(t)$ can be recovered by filtering $x_p(t)$ with a ideal lowpass filter with a cutoff frequency of $\frac{f_s}{2}$ Hz and a passband gain of aT. The passband gain aT compensates for the $\frac{1}{aT}$ scale factor of the $n = 0$ term in $X_p(\omega)$, which is the only term to survive lowpass filtering. Notice that the only difference between this reconstruction and reconstruction for ideal impulse sampling is that the lowpass reconstruction filter gain is aT rather than T.

Solution 8.2-8

(a) The signal $x(t) = \text{sinc}^2(5\pi t)$, when sampled by an impulse train, results in the sampled signal $x_\delta(t) = x(t)\delta_T(t)$ (upper left plot of Fig. S8.2-8). The spectrum of $x_\delta(t)$ is

$$X_\delta(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s),$$

where $X(\omega) = 0.2\Delta\left(\frac{\omega}{20\pi}\right)$.
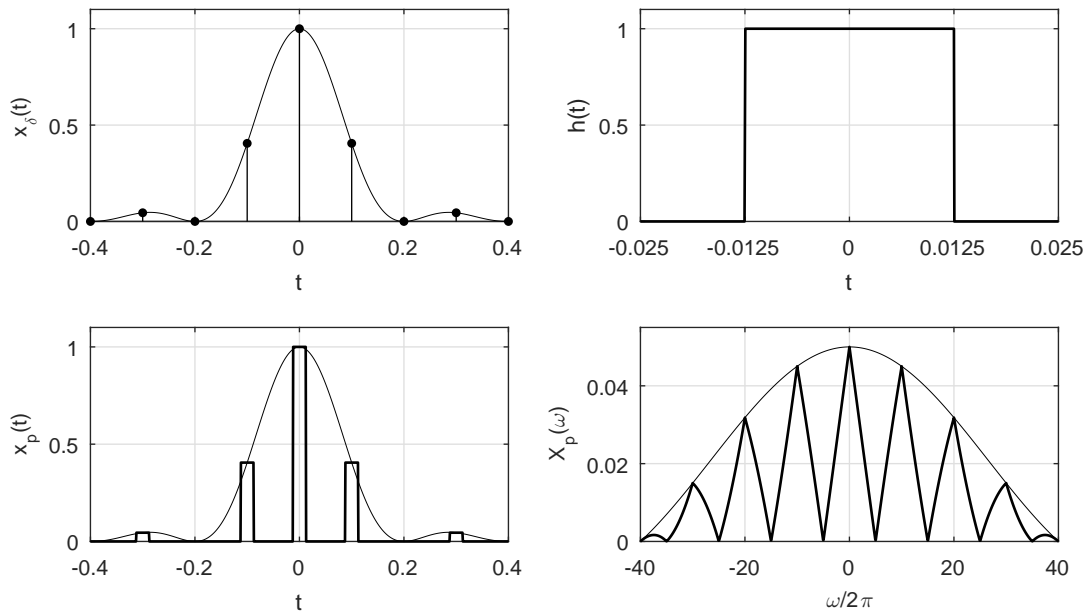
Figure S8.2-8

Now, if $x_\delta(t)$ is transmitted through a filter with impulse response $h(t) = p(t) = \text{rect}\left(\frac{t}{0.025}\right)$ (upper right plot of Fig. S8.2-8), then each impulse in the input will generate a pulse $p(t)$,

resulting in the desired sampled signal $x_p(t)$ (lower left plot of Fig. S8.2-8). Recognizing that the frequency response of $h(t)$ is $H(\omega) = \frac{1}{40}\text{sinc}(\frac{\omega}{80})$, the spectrum of $x_p(t)$ is

$$X_p(\omega) = H(\omega)\left[\frac{1}{T}\sum_{k=-\infty}^{\infty}X(\omega - k\omega_{\text{s}})\right] = \frac{1}{40T}\sum_{k=-\infty}^{\infty}\text{sinc}(\frac{\omega}{80})X(\omega - k\omega_{\text{s}}).$$

The lower right plot of Fig. S8.2-8 shows $|X_p(\omega)|$. Due to the multiplication by $H(\omega)$, each replicate of $X(\omega)$ is distorted (no longer triangular). However, as long as sampling occurs at no less than the Nyquist rate ($\frac{f_s}{2} \geq 5$ Hz), the periodic replicates of $X(\omega)$, while distorted, do not overlap and the original signal $x(t)$ can be recovered.

(b) To recover the signal $x(t)$ from the flat top samples, we reverse the process described in part (a). First, we pass the sampled signal through a filter with frequency response $1/H(\omega)$. This will recover the impulse-sampled signal $x_\delta(t)$. Next, we pass the resulting impulse-sampled signal $x_\delta(t)$ through an ideal lowpass filter with bandwidth $\frac{f_s}{2} = 5$ Hz and gain $T = \frac{1}{10}$ to obtain $x(t)$.

(c) As shown in part (a), the spectrum of signal $x_p(t)$ is given as

$$X_p(\omega) = \frac{1}{40T}\sum_{k=-\infty}^{\infty}\text{sinc}(\frac{\omega}{80})X(\omega - k\omega_{\text{s}}).$$

A plot of this spectrum and the other signal plots in Fig. S8.2-8 are readily generated using MATLAB.

```
>>  snc = @(t) sinc(t/pi); % Conform MATLAB sinc to textbook notation
>>  t = linspace(-.4,.4,1001); T = 0.1; n = -4:4; x = @(t) (snc(5*pi*t)).^2;
>>  subplot(221); stem(n*T,x(n*T),'.k'); line(t,x(t),'linewidth',1/4);
>>  xlabel('t'); ylabel('x_\delta(t)');
>>  axis([-.4 .4 0 1.1]); grid on; box on;
>>  u = @(t) 1.0*(t>0)+0.5*(t==0); rect = @(t) u(t+0.5)-u(t-0.5);
>>  h = @(t) rect(40*t); t = linspace(-2/80,2/80,1001);
>>  subplot(222); plot(t,h(t));
>>  xlabel('t'); ylabel('h(t)'); set(gca,'xtick',-2/80:1/80:2/80);
>>  axis([-2/80 2/80 0 1.1]); grid on; box on;
>>  t = linspace(-.4,.4,1001); xp = zeros(size(t));
>>  for i=n, xp = xp+x(i*T)*h(t-i*T); end
>>  subplot(223); plot(t,xp,'k-');line(t,x(t),'linewidth',1/4);
>>  xlabel('t'); ylabel('x_p(t)');
>>  axis([-.4 .4 0 1.1]); grid on; box on;
>>  omega = 2*pi*linspace(-40,40,1001);
>>  tri = @(omega) (1-2*abs(omega)).*(abs(omega)<=0.5);
>>  X = @(omega) 0.2*tri(omega/(20*pi)); Xp = zeros(size(omega));
>>  for k=-10:10, Xp = Xp+snc(omega/80).*X(omega-k*2*pi/T)/(40*T); end
>>  subplot(224); plot(omega/2/pi,abs(Xp),'k-');
>>  line(omega/2/pi,0.2*snc(omega/80)/(40*T),'linewidth',1/4)
>>  xlabel('\omega/2\pi'); ylabel('X_p(\omega)');
>>  axis([-40 40 0 1.1/20]); grid on; box on;
```

Solution 8.2-9

In this problem, the sampling frequency is $f_{\text{s}} = 20$ Hz. As a general expression, apparent frequency can be computed as

$$f_{\text{a}} = \langle f_0 + f_{\text{s}}/2\rangle_{f_{\text{s}}} - f_{\text{s}}/2,$$

where $<a>_b$ is the operation a modulo b. In all cases, $|f_{\text{a}}| \leq f_{\text{s}}/2 = 10$ Hz.

(a) In this case, $f_0 = 8$ Hz is less than $f_s/2 = 10$ Hz. Hence, this frequency is not aliased and

$$f_a = f_0 = 8 \text{ Hz}.$$

(b) In this case, $f_0 = 12$ Hz and the apparent frequency is computed as

$$f_a = \langle f_0 + f_s/2 \rangle_{f_s} - f_s/2 = \langle 12 + 10 \rangle_{20} - 10 = 2 - 10 = -8 \text{ Hz}.$$

Since f_a is negative, we can also report apparent frequency as $|f_a| = 8$ Hz.

(c) In this case, $f_0 = 20$ Hz and the apparent frequency is computed as

$$f_a = \langle f_0 + f_s/2 \rangle_{f_s} - f_s/2 = \langle 20 + 10 \rangle_{20} - 10 = 10 - 10 = 0 \text{ Hz}.$$

(d) In this case, $f_0 = 21$ Hz and the apparent frequency is computed as

$$f_a = \langle f_0 + f_s/2 \rangle_{f_s} - f_s/2 = \langle 21 + 10 \rangle_{20} - 10 = 11 - 10 = 1 \text{ Hz}.$$

(e) In this case, $f_0 = 22$ Hz and the apparent frequency is computed as

$$f_a = \langle f_0 + f_s/2 \rangle_{f_s} - f_s/2 = \langle 22 + 10 \rangle_{20} - 10 = 12 - 10 = 2 \text{ Hz}.$$

(f) In this case, $f_0 = 32$ Hz and the apparent frequency is computed as

$$f_a = \langle f_0 + f_s/2 \rangle_{f_s} - f_s/2 = \langle 32 + 10 \rangle_{20} - 10 = 2 - 10 = -8 \text{ Hz}.$$

Since f_a is negative, we can also report apparent frequency as $|f_a| = 8$ Hz.

Solution 8.2-10

In this problem, the sampling frequency is $f_s = 60$ Hz and the apparent frequency is $|f_a| = 20$ Hz. To determine the original frequency f_0, we use the general relation between f_a, f_0, and f_s:

$$|f_a| = |\langle f_0 + f_s/2 \rangle_{f_s} - f_s/2|,$$

where $< a >_b$ is the operation a modulo b.

(a) Over $0 \leq f_0 < 30$ Hz, $20 = |\langle f_0 + 30 \rangle_{60} - 30|$ is satisfied only by

$$f_0 = 20 \text{ Hz}.$$

(b) Over $30 < f_0 < 60$ Hz, $20 = |\langle f_0 + 30 \rangle_{60} - 30|$ is satisfied only by

$$f_0 = 40 \text{ Hz}.$$

(c) Over $60 < f_0 < 90$ Hz, $20 = |\langle f_0 + 30 \rangle_{60} - 30|$ is satisfied only by

$$f_0 = 80 \text{ Hz}.$$

(d) Over $90 < f_0 < 120$ Hz, $20 = |\langle f_0 + 30 \rangle_{60} - 30|$ is satisfied only by

$$f_0 = 100 \text{ Hz}.$$

Solution 8.2-11

By inspection, signal $x(t) = 3\cos(6\pi t) + \cos(16\pi t) + 2\cos(20\pi t)$ has spectrum

$$X(\omega) = \pi \left[3\delta(\omega \pm 6\pi)) + \delta(\omega \pm 16\pi)) + 2\delta(\omega \pm 20\pi)) \right].$$

Since the highest frequency is 10 Hz, the Nyquist rate is 20 Hz. Sampling at 25% above this rate yields $f_{\rm s} = 25$ Hz. Hence $T = \frac{1}{25}$ and $X_\delta(\omega)$ consists of $25X(\omega)$ repeating periodically with period 25 Hz (50π rad/s), as shown in Fig. S8.2-11. To reconstruct $x(t)$ from the sampled signal $x_\delta(t)$, we pass $X_\delta(\omega)$ through a lowpass filter with passband gain $\frac{1}{25}$ and having a cutoff frequency anywhere between $(10 + \epsilon)$ Hz to $(15 - \epsilon)$ Hz where ϵ is an arbitrarily small number.

A sampling rate 25% below the Nyquist rate is $f_{\rm s} = 15$ Hz. In this case, the components of frequencies 8 Hz and 10 Hz will alias. The 8 Hz components will appear as $|f_{\rm a}| = |\langle 8 + 7.5 \rangle_{15} - 7.5| = 7$ Hz, and the 10 Hz components will appear as $|f_{\rm a}| = |\langle 10 + 7.5 \rangle_{15} - 7.5| = 5$ Hz. Thus,

the output contains 3, 5, and 7 Hz frequency components when $F_{\rm s} = 15$ Hz.
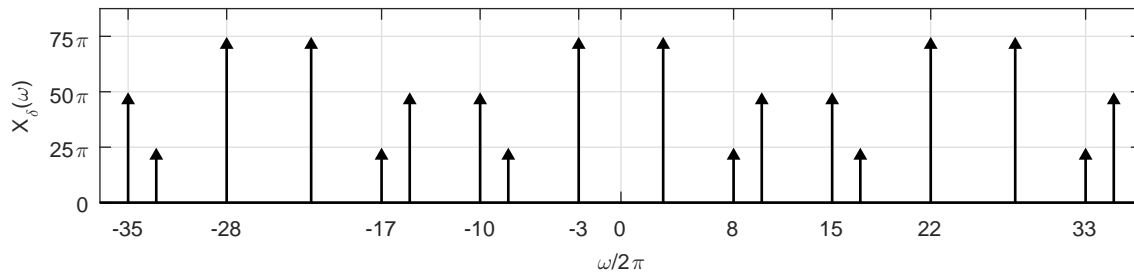
Figure S8.2-11

Solution 8.2-12

This problem considers a complex signal $x(t)$ with spectrum

$$X(\omega) = \begin{cases} \omega & 0 \le \omega \le 2\pi 10 \\ 0 & \text{otherwise} \end{cases}.$$

Here, $x(t)$ is sampled at rate $f_{\rm s} = 24$ Hz to produce signal $x_\delta(t)$ with spectrum $X_\delta(\omega)$.

(a) Signal $x(t)$ is bandlimited to $0 \le f \le 10$ Hz. As shown in Fig. S8.2-12a, signal spectrum $X_\delta(\omega)$ is just $X(\omega)$ scaled by $f_{\rm s}$ and then periodically replicated with interval $f_{\rm s}$.
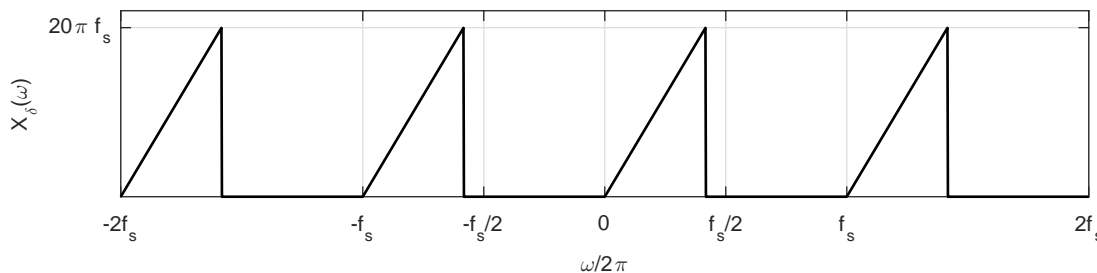
Figure S8.2-12a

(b) Since $f_{\rm s} = 24$ Hz exceeds the Nyquist rate of 20 Hz, we do not expect to see any aliasing. This is easily confirmed with Fig. S8.2-12a: only the original signal spectrum $X(\omega)$ is found in the principle frequency range of $-f_{\rm s}/2 \le f \le f_{\rm s}/2$.

(c) Since the original spectrum $X(\omega)$ is found undistorted in $X_\delta(\omega)$ (see Fig. S8.2-12a), the signal $x(t)$ can be exactly recovered from $x_\delta(t)$. This can be accomplished by using a lowpass filter with cutoff frequency $f_s/2$ and passband gain of $1/f_s$.

Solution 8.2-13

This problem considers a complex signal $x(t)$ with spectrum

$$X(\omega) = \begin{cases} \omega & 0 \le \omega \le 2\pi 10 \\ 0 & \text{otherwise} \end{cases}.$$

Here, $x(t)$ is sampled at rate $f_s = 16$ Hz to produce signal $x_\delta(t)$ with spectrum $X_\delta(\omega)$.

(a) Signal $x(t)$ is bandlimited to $0 \le f \le 10$ Hz. As shown in Fig. S8.2-13a, signal spectrum $X_\delta(\omega)$ is just $X(\omega)$ scaled by f_s and then periodically replicated with interval f_s.
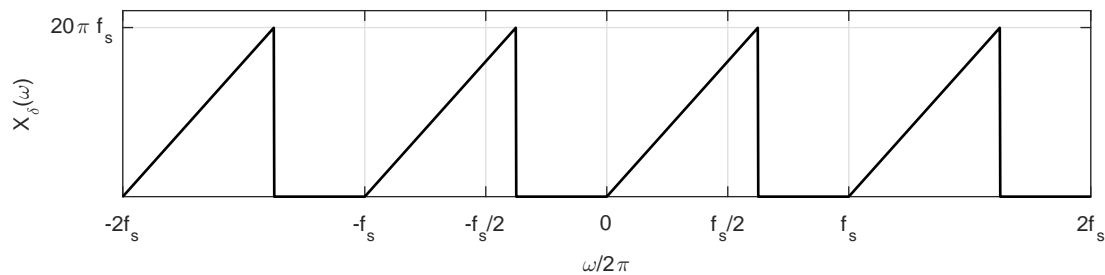
Figure S8.2-13a

(b) Since $f_s = 16$ Hz is lower than the Nyquist rate of 20 Hz, we expect to see aliasing. This is easily confirmed with Fig. S8.2-13a: the original spectrum $X(\omega)$, which is purely positive, does not fit in the positive principle frequency range $0 \le f \le f_s/2$. Consequently, the upper portion of this spectrum aliases to a lower apparent frequency and appears in the negative fundamental frequency band $-f_s/2 \le f \le 0$. That is, components of $X(\omega)$ that are higher than $f_s/2$ alias to a lower apparent frequency in the band $-f_s/2 \le f \le 0$.

(c) Although aliasing occurs in this sampling scenario, the original spectrum $X(\omega)$ remains intact in $X_\delta(\omega)$ (see Fig. S8.2-13a). Thus, the signal $x(t)$ can be exactly recovered from $x_\delta(t)$. This can be accomplished (in this case) by using a complex bandpass filter with passband $0 \le f \le 20\pi$ and passband gain of $1/f_s$.

Solution 8.2-14

This problem considers a complex signal $x(t)$ with spectrum

$$X(\omega) = \begin{cases} \omega & 0 \le \omega \le 2\pi 10 \\ 0 & \text{otherwise} \end{cases}.$$

Here, $x(t)$ is sampled at rate $f_s = 8$ Hz to produce signal $x_\delta(t)$ with spectrum $X_\delta(\omega)$.

(a) Signal $x(t)$ is bandlimited to $0 \le f \le 10$ Hz. As shown in Fig. S8.2-14a, signal spectrum $X_\delta(\omega)$ is just $X(\omega)$ scaled by f_s and then periodically replicated with interval f_s.

(b) Since $f_s = 8$ Hz is lower than the Nyquist rate of 20 Hz, we expect to see aliasing. This is easily confirmed with Fig. S8.2-14a: the original spectrum $X(\omega)$, which is purely positive, does not fit in the positive principle frequency range $0 \le f \le f_s/2$. Consequently, the upper portion of this spectrum aliases to lower frequencies. In fact, not only is there aliasing, but aliased components interfere with content in the fundamental band, causing irrecoverable signal distortion.
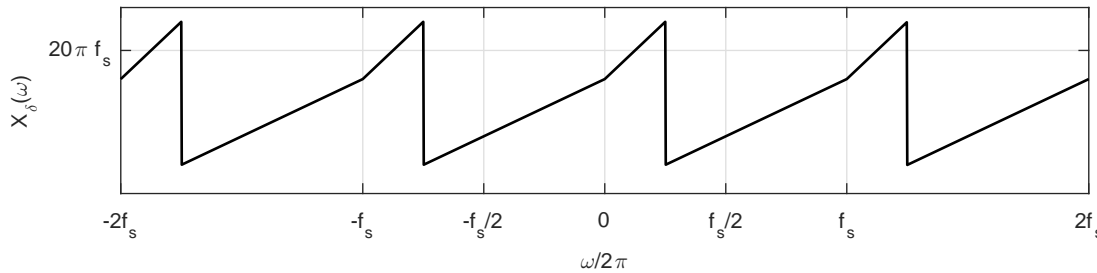
Figure S8.2-14a

(c) Since the aliasing in this case also causes distortion of the original spectrum $X(\omega)$, it is impossible to exactly recover the original signal $x(t)$ from $x_\delta(t)$. Graphically, Fig. S8.2-14a shows that $X_\delta(\omega)$ does not retain the original triangular spectrum $X(\omega)$ but rather a shape-distorted version instead. This distortion cannot be undone, so the original signal is forever lost.

Solution 8.2-15

(a) The output in Eq. (8.6) is the output of an ideal lowpass filter of bandwidth $B = 1/2T$ Hz. Clearly its bandwidth must be $\leq 1/2T$ Hz.

(b) Suppose there is another signal $\hat{x}(t)$ that passes through the samples $x(nT)$ and has bandwidth smaller than that of $x(t)$ obtained through Eq. (8.6). Clearly, both $x(t)$ and $\hat{x}(t)$ have bandwidth $\leq 1/2T$ and the signal $x(t) - \hat{x}(t)$ also has bandwidth $\leq 1/2T$ Hz. Hence, we can reconstruct the signal $x(t) - \hat{x}(t)$ from samples of $x(t) - \hat{x}(t)$ at a rate $1/T$ Hz and using these samples in Eq. (8.6). But because both $x(t)$ and $\hat{x}(t)$ pass through sample $x(nT)$, the samples of $x(t) - \hat{x}(t)$ at a rate $1/T$ Hz are zero for all n. Clearly $x(t) - \hat{x}(t) = 0$ and $\hat{x}(t) = x(t)$.

Solution 8.2-16

For this problem, $T = 1/R$ and the sample values of the bandlimited pulse $p(t)$ are

$$p(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

We can use Eq. (8.6) to reconstruct $p(t)$ from these sample values. There is only one nonzero-valued sample. Hence, we obtain

$$p(t) = p(0)\text{sinc}\left(\tfrac{\pi t}{T}\right) = \text{sinc}\left(\pi R t\right) \iff \tfrac{1}{R}\text{rect}\left(\tfrac{\omega}{2\pi R}\right) = P(\omega).$$

This is the only signal that has bandwidth πR rad/s ($R/2$ Hz) with samples satisfying

$$p(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

Solution 8.2-17

As given in the problem statement, the pulse $p(t)$ has spectrum $P(\omega)$ with odd-like symmetry, such as shown in the left figure of Fig. S8.2-17. When we sample $p(t)$ at a rate R Hz ($T = 1/R$ s), the resulting spectrum consists of $\tfrac{1}{T}P(\omega) = RP(\omega)$ repeated periodically with period R Hz, as shown in the right figure of Fig. S8.2-17. Because of the odd-like symmetry of $P(\omega)$ about the dotted axis, the overlapping (scaled) spectra add to a constant value 1 for all ω. Hence,

$$P_\delta(\omega) = 1 \qquad \text{and} \qquad p_\delta(t) = \delta(t).$$

This shows that samples of the pulse $p(t)$ are, as required,

$$p(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad \text{where} \qquad T = \tfrac{1}{R}.$$
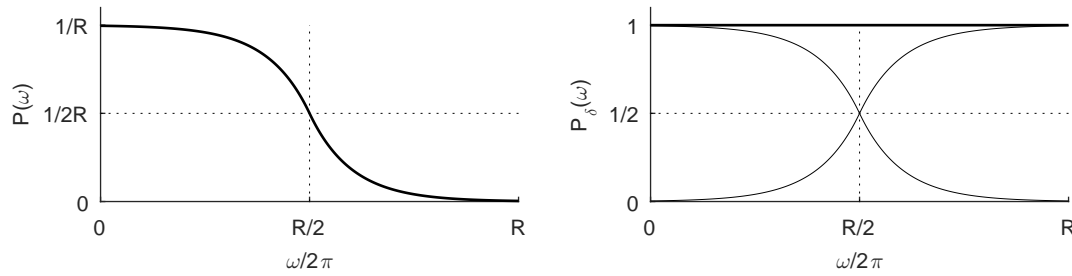
Figure S8.2-17

Solution 8.2-18

Using the Nyquist interval $T = 1/2B$, the Nyquist samples are $x(\pm n/2B)$ for $(n = 0, 1, 2, 3, \ldots)$. We are given that $x(0) = x(1/2B) = 1$ and $x(n/2B) = 0$ for all other cases of n. Hence, from Eq. (8.6)

$$
\begin{aligned}
x(t) &= \text{sinc}(2\pi Bt) + \text{sinc}(2\pi Bt - \pi) \\
&= \frac{\sin(2\pi Bt)}{2\pi Bt} + \frac{\sin(2\pi Bt - \pi)}{2\pi Bt - \pi} \\
&= \frac{\sin(2\pi Bt)}{2\pi Bt} - \frac{\sin(2\pi Bt)}{2\pi Bt - \pi} \\
&= \frac{\sin 2\pi Bt}{2\pi Bt(1 - 2Bt)}.
\end{aligned}
$$

Thus,

$$
x(t) = \frac{\text{sinc}(2\pi Bt)}{1 - 2Bt}.
$$

Solution 8.2-19

To begin, we show that $\int_{-\infty}^{\infty} x(t)\, dt = T \sum_{n=-\infty}^{\infty} x(nT)$ for a signal bandlimited to B Hz and sampled at a rate $f_{\text{s}} > 2B$ Hz. From the ideal interpolation formula [see just before Eq. (8.6)] we know that any bandlimited signal sampled at greater than the Nyquist rate can be reconstructed as

$$
x(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\pi \frac{t - nT}{T}\right).
$$

Integrating both sides yields

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t)\, dt &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\pi \frac{t - nT}{T}\right) dt \\
&= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \text{sinc}\left(\pi \frac{t - nT}{T}\right) dt \\
&= T \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \text{sinc}(\pi t)\, dt
\end{aligned}
$$

Now, from pair 18 of Table 7.1 we know that $\text{sinc}(\pi t) \iff \text{rect}(\omega/2\pi)$. Further, the Fourier transform analysis equation states $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$. Setting $\omega = 0$, we thus see that $X(0) = \int_{-\infty}^{\infty} \text{sinc}(\pi t)\, dt = \text{rect}(0) = 1$. Applying this simplification, we obtain the desired result of

$$
\int_{-\infty}^{\infty} x(t)\, dt = T \sum_{n=-\infty}^{\infty} x(nT)
$$

Next, we show that $\int_{-\infty}^{\infty} |x(t)|^2\, dt = T \sum_{n=-\infty}^{\infty} |x(nT)|^2$ for a signal bandlimited to B Hz and sampled at a rate $f_{\text{s}} > 2B$ Hz. Again using the ideal interpolation formula, we know that any

bandlimited signal sampled at greater than the Nyquist rate can be reconstructed as

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\pi \frac{t-nT}{T}\right).$$

Recognizing that $|x(t)|^2 = x(t)x^*(t)$, we see that

$$|x(t)|^2 = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\pi \frac{t-nT}{T}\right) \sum_{m=-\infty}^{\infty} x^*(mT)\text{sinc}\left(\pi \frac{t-mT}{T}\right).$$

Integrating both sides yields

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}\left(\pi \frac{t-nT}{T}\right) \sum_{m=-\infty}^{\infty} x^*(mT)\text{sinc}\left(\pi \frac{t-mT}{T}\right) dt$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(nT)x^*(mT) \int_{-\infty}^{\infty} \text{sinc}\left(\pi \frac{t-nT}{T}\right) \text{sinc}\left(\pi \frac{t-mT}{T}\right) dt.$$

To handle the inside integral requires some additional work. Recall from the Fourier transform synthesis and anaysis equations that

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} y(t)e^{j\omega t}\, dt = Y(-\omega).$$

Therefore

$$\int_{-\infty}^{\infty} x(t)y(t)\, dt = \int_{-\infty}^{\infty} y(t)\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega\right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\left[\int_{-\infty}^{\infty} y(t)e^{j\omega t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y(-\omega)\, d\omega$$

This is a generalized form of Parseval's theorem. Now, using pair 18 of Table 7.1 and the time-shift property, we know that $\text{sinc}\left(\pi \frac{t-nT}{T}\right) \Longleftrightarrow T\text{rect}(\omega/2B)e^{-j\omega nT}$, where $B = \pi/T$. Thus,

$$\int_{-\infty}^{\infty} \underbrace{\text{sinc}\left(\pi \frac{t-nT}{T}\right)}_{x(t)} \underbrace{\text{sinc}\left(\pi \frac{t-mT}{T}\right)}_{y(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y(-\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} T\text{rect}(\omega/2B)e^{-j\omega nT} T\text{rect}(\omega/2B)e^{j\omega mT}\, d\omega$$

$$= \frac{T^2}{2\pi} \int_{-B}^{B} e^{j\omega(m-n)T}\, d\omega.$$

When $m \neq n$, we are integrating sinusoids over integer numbers of periods and the result is zero. When $m = n$, we are integrating a constant over a width of $2B$. Thus,

$$\int_{-\infty}^{\infty} \text{sinc}\left(\pi \frac{t-nT}{T}\right) \text{sinc}\left(\pi \frac{t-mT}{T}\right) dt = \left\{ \begin{array}{ll} 0 & n \neq m \\ \frac{T^2}{2\pi}2B = T & n = m \end{array} \right. .$$

Since the $m \neq n$ terms are zero, only the $m = n$ terms in our double sum survive, and we obtain our desired result of

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(nT)x^*(mT) \int_{-\infty}^{\infty} \text{sinc}\left(\frac{t-nT}{T}\right) \text{sinc}\left(\frac{t-mT}{T}\right) dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)x^*(nT)T$$

or

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = T \sum_{n=-\infty}^{\infty} |x(nT)|^2.$$

Solution 8.2-20

Assume a signal $x(t)$ that is simultaneously timelimited and bandlimited. Let $X(\omega) = 0$ for $|\omega| > 2\pi B$. Therefore, $X(\omega) = X(\omega)\mathrm{rect}\left(\frac{\omega}{4\pi B'}\right)$ for $B' > B$. Using pair 18 of Table 7.1 and the time-convolution property of Eq. (7.33), we know that

$$x(t) = x(t) * [2B'\mathrm{sinc}(2\pi B't)]$$
$$= 2B'x(t) * \mathrm{sinc}(2\pi B't)$$

Because $x(t)$ is timelimited, $x(t) = 0$ for $|t| > T$. But $x(t)$ is equal to convolution of $x(t)$ with $\mathrm{sinc}(2\pi B't)$ which is not timelimited. It is impossible to obtain a time-limited signal from the convolution of a time-limited signal and a non-timelimited signal. Based on this contradiction, we conclude that a signal cannot be simultaneously timelimited and bandlimited.

Solution 8.3-1

(a) A primary reason that time-sampling is necessary for digital systems relates to memory. To continuously record a waveform digitally would require an infinite amount of memory, even to record the briefest interval of time. Clearly this is impractical. Time sampling helps remedy this memory problem. Further, time sampling need not degrade the underlying waveform: as long as the sampling rate is at least twice the highest frequency of the signal (Nyquist criterion), the original waveform is preserved. Although practical signals are not strictly bandlimited, most are effectively bandlimited, meaning that a suitable sampling rate is possible. If the Nyquist criterion is not met and high-frequency components alias and overlap with low-frequency (or other aliased) components, unrecoverable signal distortion results. Notice that aliasing can be tolerated in special cases when aliasing does not cause signal components to mix or overlap. It is generally preferable to use the lowest practical sampling rates, as this helps save memory and allows the digital system to operate at lower speeds.

(b) Amplitude quantization is necessary for primarily the same reason as is time sampling: finite computer memory. If amplitude quantization was not performed, it would take an infinite amount of memory just to represent a single sample of a signal, which is impractical. Quantization (along with time sampling) neatly solves the memory dilemma. Unfortunately, quantization is a non-linear process that (almost always) results in unrecoverable changes to the signal. Such distortions can be effectively managed by using a suitable number of quantization levels. It is, however, a balancing act. Too few quantization levels saves memory but increases distortion; too many quantization levels can effectively eliminate distortion but can cause excessive memory (and processing) requirements.

Solution 8.3-2

Typical analog-to-digital converters (ADCs) operate over a range of input amplitudes $[-V_{\mathrm{ref}}, V_{\mathrm{ref}}]$. It desirable to condition the input $x(t)$ to an ADC so that its maximum magnitude is close to, but does not exceed, V_{ref} since this produces the most favorable signal-to-quantization noise condition. If the maximum magnitude of $x(t)$ is greater than V_{ref}, the signal is clipped during quantization, which causes substantial and undesirable signal distortion. If the maximum magnitude of $x(t)$ is much smaller than V_{ref}, than the signal-to-quantization noise ratio goes down, meaning the quantized signal is comprised of an increasing – and unnecessary – amount of noise.

Solution 8.3-3

(a) Since the audio signal bandwidth is 15 kHz,

$$\text{the Nyquist rate is 30 kHz.}$$

(b) Since the desired number of levels is $65536 = 2^{16}$,

$$\text{16 binary digits (bits) are needed to encode each sample.}$$

(c) To transmit the audio signal requires a bit rate of

$$30000 \times 16 = 480000 \text{ bits/s.}$$

(d) A CD rate of 44100 Hz with $L = 65536$ levels requires a bit rate of

$$44100 \times 16 = 705600 \text{ bits/s.}$$

Solution 8.3-4

(a) The Nyquist rate for a 4.5 MHz signal is $2 \times 4.5 \times 10^6 = 9$ MHz. Exceeding the Nyquist rate by 20% yields a sampling rate of

$$F_\text{s} = 1.2 \times 9 = 10.8 \text{ MHz.}$$

(b) Since the desired number of levels is $1024 = 2^{10}$,

$$\text{10 binary digits (bits) are needed to encode each sample.}$$

(c) To transmit the TV signal requires a bit rate of

$$10.8 \times 10^6 \times 10 = 108 \times 10^6 \text{ or 108 Mbits/s.}$$

Solution 8.3-5

(a) For $L = 16$, we need a 4-bit binary code because $16 = 2^4$. Assuming a bipolar DAC with level 0 corresponding to the most negative value and level 15 corresponding to the most positive value, the following three codes are consistent with offset binary, two's complement, and one form of gray code.

Level	Offset Binary	Two's Complement	Gray Code
0	0000	1000	0000
1	0001	1001	0001
2	0010	1010	0011
3	0011	1011	0010
4	0100	1100	0110
5	0101	1101	0111
6	0110	1110	0101
7	0111	1111	0100
8	1000	0000	1100
9	1001	0001	1101
10	1010	0010	1111
11	1011	0011	1110
12	1100	0100	1010
13	1101	0101	1011
14	1110	0110	1001
15	1111	0111	1000

For a quaternary code, we use four symbols 0, 1, 2, and 3. For this code, we need only a group of 2 symbols to form 16 combinations $(4 \times 4) = 16$. One possible quaternary code is given below.

Level	Code	Level	Code	Level	Code	Level	Code
0	00	4	10	8	20	12	30
1	01	5	11	9	21	13	31
2	02	6	12	10	22	14	32
3	03	7	13	11	23	15	33

(b) Let a minimum of b_2 binary digits and b_4 quaternary digits be required to represent L level. Now b_2 binary digits can form at most 2^{b_2} distinct combinations. Similarly b_4 quaternary digits can form at most 4^{b_4} distinct combinations. Hence

$$L = 2^{b_2} = 4^{b_4}$$

and

$$b_2 \log 2 = b_4 \log 4 = 2b_4 \log 2.$$

Simplifying, we see that

$$b_2/b_4 = 2.$$

Solution 8.3-6

If V is the peak sample amplitude, then

$$\text{quantization error} \leq \tfrac{0.2}{100}V = \tfrac{V}{500}.$$

Because the maximum quantization error is $\frac{\Delta}{2} = \frac{2V}{2L} = \frac{V}{L}$, it follows that

$$\frac{V}{L} = \frac{V}{500} \quad \Longrightarrow \quad L = 500.$$

Because L should be a power of 2, we choose

$$L = 512 = 2^9.$$

This requires a 9-bit binary code per sample. For each telemetry signal, the Nyquist rate is $2 \times 1000 = 2000$ Hz. Increasing this rate by 20% yields a sampling frequency of $2000 \times 1.2 = 2400$ Hz. Thus, each signal has 2400 samples/s, and each sample is encoded with 9 bits. Therefore, each signal uses $9 \times 2400 = 21.6$ kbits/s. Five such signals are multiplexed. Hence,

$$\text{the required data rate is } 5 \times 21.6 = 108 \text{ kbits/s.}$$

Solution 8.4-1

The spectrum $X(\omega)$ of triangle function $x(t) = \Delta(t/5)$ is sampled at a rate $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ to produce

$$X_{T_0}(\omega) = X(n\omega_0).$$

According to the spectral sampling theorem, the corresponding time-domain function is a T_0-scaled, T_0-replicated version of $x(t)$. That is,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} T_0 x(t - kT_0).$$

(a) When $X(\omega)$ is sampled at 10 samples/Hz, $T_0 = 10$ and signal $x_{10}(t)$ results, as shown in the upper left plot of Fig. S8.4-1. In this case, the replicates of $x(t)$ that form $x_{10}(t)$ are spaced well apart from one another, and the triangular nature of $x(t)$ is fully preserved.

(b) When $X(\omega)$ is sampled at 5 samples/Hz, $T_0 = 5$ and signal $x_5(t)$ results, as shown in the upper right plot of Fig. S8.4-1. In this case, the replicates of $x(t)$ that form $x_5(t)$ are just touching one another, and the triangular nature of $x(t)$ is still preserved.

(c) When $X(\omega)$ is sampled at 4 samples/Hz, $T_0 = 4$ and signal $x_4(t)$ results, as shown in the lower left plot of Fig. S8.4-1. In this case, the replicates of $x(t)$ that form $x_4(t)$ slightly overlap one another, distorting the triangular shape of the waveform.

(d) When $X(\omega)$ is sampled at 2.5 samples/Hz, $T_0 = 2.5$ and signal $x_{2.5}(t)$ results, as shown in the lower right plot of Fig. S8.4-1. In this case, the replicates of $x(t)$ that form $x_{2.5}(t)$ fully overlap one another, completely destroying the triangular shape from $x(t)$.
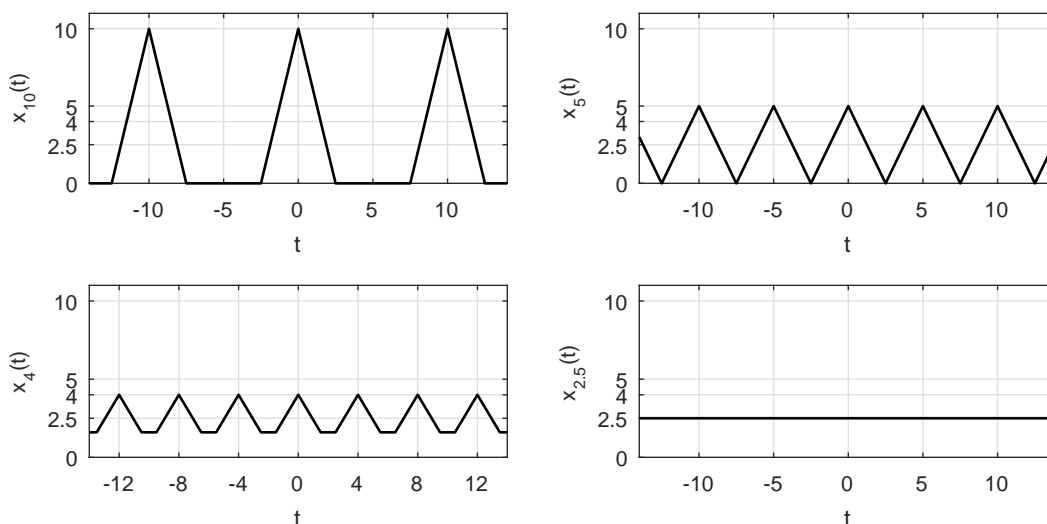
Figure S8.4-1

Solution 8.4-2

In Sec. 8.4, we have shown that when a timelimited signal $x(t)$ is repeated periodically with a period $T_0 > \tau$ (the signal duration), the Fourier series coefficients for the resulting periodic signal $x_{T_0}(t)$ are proportional to the samples of $X(\omega)$, the Fourier transform of $x(t)$ at frequency interval of $f_0 = \frac{1}{T_0}$ Hz. This result is quite general and applied even if $x(t)$ is bandlimited, and therefore, nontimelimited. To show this we convolve $x(t)$ with unit impulse train $\delta_T(t)$. This will result in periodic repetition of $x(t)$ with period T. To begin, notice that

$$y(t) = x(t) * \delta_T(t) \iff \frac{1}{2\pi} X(\omega) \left[\frac{2\pi}{T} \delta_{\omega_s}(\omega) \right] = \frac{1}{T} X(\omega) \delta_{\omega_s}(\omega),$$

where $\omega_s = \frac{2\pi}{T}$. In the present case, $T = \frac{1.25}{B}$ and the fundamental frequency is $\omega = 2\pi/T$. Therefore,

$$Y(\omega) = \frac{1}{T} X(\omega) \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right).$$

Hence $Y(\omega)$ represents the spectrum $\frac{1}{T}X(\omega)$ sampled at intervals of $\frac{1}{T}$ Hz. This means $y(t)$ is a periodic signal with fundamental frequency $f_0 = \frac{1}{T}$ and Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_s t},$$

where

$$D_n = \frac{1}{T}X(n\omega_s) = \frac{1}{T}X\left(\frac{2\pi n}{T}\right).$$

Moreover, $T = \frac{1.25}{B}$. Hence, the fundamental frequency $f_0 = \frac{B}{1.25} = 0.8B$. But $X(\omega)$ is bandlimited to B Hz. This means $Y(\omega)$ contains only the dc and the fundamental component. Frequencies of all the remaining components are beyond $1.6B$, where $X(\omega) = 0$, and hence $Y(\omega) = 0$. The nonzero component amplitudes are $D_0 = \frac{1}{T}X(0)$ and $D_1 = \frac{1}{T}X\left(\frac{2\pi}{T}\right)$. We can write $y(t)$ as a trigonometric Fourier series

$$y(t) = C_0 + C_1 \cos(1.6\pi Bt + \theta_1),$$

where

$$C_0 = D_0 = \frac{1}{T}X(0), \quad C_1 = 2|D_1| = \frac{2}{T}\left|X\left(\frac{2\pi}{T}\right)\right|, \quad \text{and} \quad \theta_1 = \angle D_1 = \angle X\left(\frac{2\pi}{T}\right).$$

Solution 8.5-1

Here,

$$T_0 = \frac{1}{f_o} = \frac{1}{50} = 20\text{ms} \quad \text{and} \quad B = 10000.$$

Hence,

$$f_s \geq 2B = 20000, \quad T = \frac{1}{f_s} = \frac{1}{20000} = 50\mu s, \quad \text{and} \quad N_0 = \frac{T_0}{T} = \frac{20 \times 10^{-3}}{50 \times 10^{-6}} = 400.$$

Since N_0 must be a power of 2, we choose

$$N_0 = 512.$$

Also $T = 50\mu s$, $T_0 = N_0 T = 512 \times 50\mu s = 25.6$ ms, and $f_0 = 1/T_0 = 39.0625$ Hz. Since $x(t)$ is of 10 ms duration, we need zero padding over 15.6 ms. Alternatively, we could also have used

$$T = \frac{20 \times 10^{-3}}{512} = 39.0625 \ \mu s.$$

This gives $T_0 = 20$ ms, $f_0 = 50$ Hz, and $f_s = \frac{1}{T} = 25600$ Hz.
There are also other possibilities of reducing T as well as increasing the frequency resolution.

Solution 8.5-2

For the signal $x(t)$,

$$T_0 \geq \frac{1}{0.25} = 4 \quad \text{and} \quad T \leq \frac{1}{f_s} = \frac{1}{3 \times 2} = \frac{1}{6}.$$

Let us choose $T = 1/8$ and $T_0 = 4$. Therefore, $N_0 = T_0/T = 32$. The signal $x(t)$ repeats every 4 seconds with samples every 1/8 second. The samples are $Tx(nT) = (1/8)x(n/8)$. Thus, the first sample is (at $n = 0$) $1 \times (1/8) = 1/8$. The 32 samples (of a single period) are (starting at $n = 0$)

$$\frac{1}{8}, \frac{7}{64}, \frac{3}{32}, \frac{5}{64}, \frac{1}{16}, \frac{3}{64}, \frac{1}{32}, \frac{1}{64}, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{64}, \frac{1}{32}, \frac{3}{64}, \frac{1}{16}, \frac{5}{64}, \frac{3}{32}, \frac{7}{64}$$

The samples $Tx(nT)$ are shown in Fig. S8.5-2.
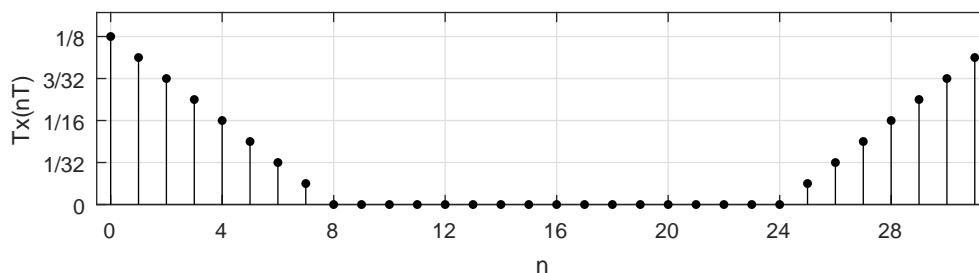
Figure S8.5-2

Solution 8.5-3

(a) When a bandlimited signal is oversampled, its DFT will be 0 (or very small) for high frequencies. Thus, a suitable sample rate f_s can be determined by increasing the DFT size N_0 until high frequency components are sufficiently small, and then computing the sampling rate as

$$f_s = \frac{N_0}{T_0}, \quad \text{where } N_0 \text{ is the smallest DFT size where high frequency components are small.}$$

(b) To test the method of part (a), we consider the signal $x(t) = \Delta(\frac{t-1}{2})$, which has $T_0 = 2$. Using MATLAB, we compute and plot the DFT for various sample sizes N_0. Figure S8.5-3 shows the results for $N_0 = 16$, 32, and 64. For $N_0 = 16$, we see that the high frequency components are not yet very small. For $N_0 = 32$, the high frequency components are becoming small, and at $N_0 = 64$, the DFT is small for a fairly broad range of frequencies. Thus,

$$N_0 = 32 \text{ produces a good result, suggesting a sampling rate of } f_s = 16 \text{ Hz.}$$

This result aligns well with the known spectral characteristics of $x(t) = \Delta(\frac{t-1}{2})$.
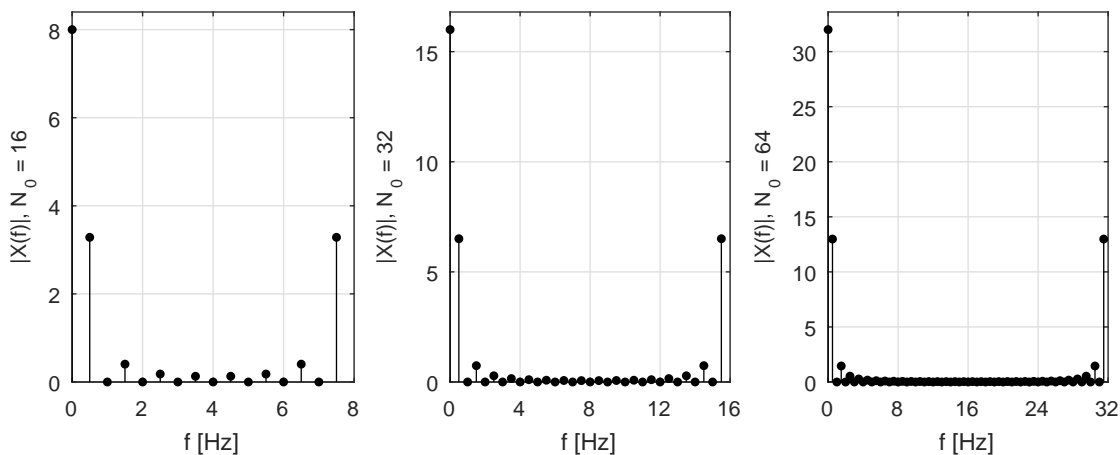
Figure S8.5-3

Solution 8.5-4

In this problem,

$$x(t) = e^{-t}u(t) \qquad \text{and} \qquad X(\omega) = \frac{1}{j\omega + 1}.$$

(a) We take the folding frequency $f_s/2$ to be the frequency where $|X(\omega)|$ is 1% of its peak value, which happens to be 1 (at $\omega = 0$). Hence,

$$|X(\omega)| \approx \frac{1}{\omega} = 0.01 \Rightarrow \omega = 2\pi B = 100.$$

This yields $B = 50/\pi$, and $T \le 1/2B = \pi/100$. Let us reduce T to 0.03125, resulting in 32 samples per second. The time constant of e^{-t} is 1. For T_0, a reasonable choice is 5 to 6 time constants or more. Value of $T_0 = 5$ or 6 results in $N_0 = 160$ or 192, neither of which is a power of 2. Hence, we choose $T_0 = 8$, resulting in $N_0 = 32 \times 8 = 256$, which is a power of 2.

(b) Here,

$$|X(\omega)| = \frac{1}{\sqrt{\omega^2 + 1}} \simeq \frac{1}{\omega}, \qquad \omega \gg 1.$$

We take the folding frequency $f_s/2$ to be the 99% energy frequency as explained in Ex. 7.20). From the results in Ex. 7.20), we have (with $a = 1$)

$$\frac{0.99\pi}{2} = \tan^{-1}\frac{W}{a} \quad \Rightarrow W = 63.66a = 63.66 \text{ rad/sec}.$$

This yields $B = \frac{W}{2\pi} = 10.13$ Hz. Also $T \le 1/2B = 0.04936$. This results in the sampling rate $\frac{1}{T} = 20.26$ Hz. Also $T_0 = 8$ as explained in part (a). This yields $N_0 = 20.26 \times 8 = 162.08$, which is not a power of 2. Hence, we choose the next higher value, that is $N_0 = 256$, which yields $T = 0.03125$ and $T_0 = 8$, the same as in part (a).

Solution 8.5-5

(a) Here,

$$x(t) = \frac{2}{t^2 + 1}.$$

Application of the duality property to pair 3 of Table 7.1 yields

$$\frac{2}{t^2 + 1} \Longleftrightarrow 2\pi e^{-|\omega|}.$$

Next, we observe that the peak value of $|X(\omega)| = 2\pi e^{-|\omega|}$ is 2π (occurring at $\omega = 0$). Also, $2\pi e^{-|\omega|}$ becomes $0.01 \times 2\pi$ (1% of the peak value) at $\omega = \ln 100 = 4.605$. Hence, $B = 4.605/2\pi = 0.733$ Hz, and $T \le 1/2B = 0.682$. Also,

$$x(0) = 2 \quad \text{and} \quad x(t) \simeq \frac{2}{t^2} \quad t \gg 1.$$

Next, we choose T_0 (the duration of $x(t)$) to be the instant where $x(t)$ is 1% of $x(0)$:

$$x(T_0) = \frac{2}{T_0^2 + 1} = \frac{2}{100} \Longrightarrow T_0 \approx 10.$$

This results in $N_0 = T_0/T = 10/0.682 = 14.66$. Rounding up to a power of two yields $N_0 = 16$, for which $T = 0.625$ and $T_0 = 10$.

(b) The energy of $x(t) = \frac{2}{t^2+1}$ is

$$E_x = \frac{2}{2\pi} \int_0^\infty (2\pi)^2 e^{-2\omega} \, d\omega = 2\pi.$$

The energy within the band from $\omega = 0$ to W is given by

$$E_W = \frac{8\pi^2}{2\pi} \int_0^W e^{-2\omega}\, d\omega = 2\pi(1 - e^{-2W}).$$

But $E_W = 0.99E_x = 0.99 \times 2\pi$. Hence,

$$0.99(2\pi) = 2\pi(1 - e^{-2W}) \quad \Rightarrow W = 2.303.$$

Hence, $B = W/2\pi = 0.366$ Hz. Thus, $T \leq 1/2B = 1.366$. Also, $T_0 = 10$ as found in part (a). Hence, $N_0 = T_0/T = 7.32$. We select $N_0 = 8$ (a power of 2), resulting in $N_0 = 8$ and $T = 1.25$.

Solution 8.5-6

The widths of $x(t)$ and $g(t)$ are 1 and 2 respectively. Hence the width of the convolved signal is $1 + 2 = 3$. This means we need to zero-pad $x(t)$ for 2 s and $g(t)$ for 1 s, making $T_0 = 3$ for both signals. Since $T = 0.125$,

$$N_0 = \frac{3}{0.125} = 24.$$

Preferably, N_0 should be a power of 2. Choose $N_0 = 32$. This permits us to adjust T_0 to 4. Hence the final values are $T = 0.125$ and $T_0 = 4$. The scaled samples of $x(t)$ and $g(t)$ are shown in Fig. S8.5-6.
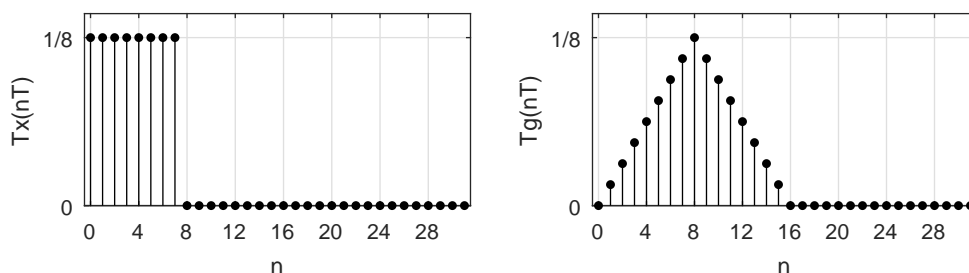
Figure S8.5-6

Solution 8.5-7

(a) Since $X_a(r) = j - \pi$ is a constant, it satisfies the periodicity requirement $X_a(r) = X_a(r + N)$ for integer $N \geq 1$, which makes $X_a(r)$ a valid DFT. Since $X_a(r)$ does not possess conjugate symmetry, the time-domain signal $x_a[n]$ is not real. Thus,

$$X_a(r) \text{ is a valid DFT for integer } N \geq 1, \text{ and } x_a[n] \text{ is not real.}$$

(b) For $X_b(r) = \sin(k/10)$, there exists no integer N such that $X_b(r) = X_b(r + N)$, and $X_b(r)$ is necessarily aperiodic. Thus,
$$X_b(r) \text{ is not a valid DFT.}$$

(c) In this case, $X_c(r) = \sin(\pi k/10)$ satisfies the periodicity requirement $X_c(r) = X_c(r + N)$ for $N = 20, 40, \ldots$, so $X_c(r)$ is a valid DFT. Since $X_c(r)$ does not possess conjugate symmetry, the time-domain signal $x_c[n]$ is not real. Thus,

$$X_c(r) \text{ is a valid DFT for } N = [20, 40, \ldots], \text{ and } x_c[n] \text{ is not real.}$$

Notice, since $X_c(r)$ is conjugate antisymmetric, the time-domain signal $x_c[n]$ is purely imaginary.

(d) To begin, we us express $X_{\mathrm{d}}(r) = \left(\frac{1+j}{\sqrt{2}}\right)^r$ as $X_{\mathrm{d}}(r) = e^{jk\pi/4}$. Clearly, $X_{\mathrm{d}}(r)$ satisfies the periodicity requirement $X_{\mathrm{d}}(r) = X_{\mathrm{d}}(r + N)$ for $N = 8, 16, \ldots$, so $X_{\mathrm{d}}(r)$ is a valid DFT. Further, $X_{\mathrm{d}}(r)$ is conjugate symmetric ($X_{\mathrm{d}}(r) = X_{\mathrm{d}}^*(-r)$), so the time-domain signal $x_{\mathrm{d}}[n]$ is real. Thus,

$$X_{\mathrm{d}}(r) \text{ is a valid DFT for } N = [8, 16, \ldots], \text{ and } x_{\mathrm{d}}[n] \text{ is real.}$$

(e) In this case, $X_{\mathrm{e}}(r) = \langle k + \pi \rangle_{10}$ satisfies the periodicity requirement $X_{\mathrm{e}}(r) = X_{\mathrm{e}}(r + N)$ for $N = 10, 20, \ldots$, so $X_{\mathrm{e}}(r)$ is a valid DFT. Since $X_{\mathrm{e}}(r)$ is not conjugate symmetric, the time-domain signal $x_{\mathrm{e}}[n]$ is not real. Thus,

$$X_{\mathrm{e}}(r) \text{ is a valid DFT for } N = [10, 20, \ldots], \text{ and } x_{\mathrm{e}}[n] \text{ is not real.}$$

Solution 8.7-1

Given signal $x[n]$ with DFT $X_r = \sum_{n=0}^{N_0-1} x[n] e^{-jr\Omega_0 n}$, the time-shifting property tells us that $y[n] = x[n - n_0]$ has DFT $Y_r = e^{-jr\Omega_0 n_0} X_r$. Therefore, if MATLAB computes the DFT X_r for a signal $x[n]$ assuming that it starts at 0, then the DFT of the signal shifted to start at n_0 is found by scaling X_r by $e^{-jr\Omega_0 n_0}$.

In MATLAB, this is easy to accomplish. Assuming constants `n_0` and `Omega_0` are defined and DFT `X` is already computed, the corrected DFT `X_shift` is computed by:

```
>>  X_shift = exp(-j*([0:length(X)-1]')*Omega_0*n_0).*X(:);
```

Solution 8.7-2

Ideally, $x_1[n] = e^{j2\pi n 30/100} + e^{j2\pi n 33/100}$ is characterized by two spikes of equal height located at $f_r = 0.30$ and $f_r = 0.33$. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $x_1[n]$.

(a)
```
>>  n = (0:9); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
>>  X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>>  stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([-0.5 0.5 0 20]);
```


Figure S8.7-2a

For this DFT, only ten samples of $x_1[n]$ are used. As a result, the DFT has only 10 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Fig. S8.7-2a shows, there is insufficient frequency resolution to separately identify the two closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.33$.

(b)
```
>>  n = (0:9); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
>>  x1 = [x1,zeros(1,490)];
```

```
>>  X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([-0.5 0.5 0 20]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([0.2 0.4 0 20]);
```


Figure S8.7-2b

For this DFT, only ten samples of $x_1[n]$ are used but the sequence is zero-padded to a length of 500. Although the DFT has 500 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$, there is insufficient information about $x_1[n]$ (only 10 samples) to resolve the closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.33$. Using the picket fence analogy, zero-padding increases the number of "pickets" in our DFT fence, but it does not change what lies behind the fence. Still, Fig. S8.7-2b does show a concentration of signal energy centered at $f_r = 0.315$, the average of the two exponential frequencies.

(c)
```
>>  n = (0:99); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
>>  X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([-0.5 0.5 0 110]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([0.2 0.4 0 110]);
```

For this DFT, 100 samples of $x_1[n]$ are used. As a result, the DFT has 100 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. The set of DFT bins also happens to include both exponential frequencies $f_r = 0.30$ and $f_r = 0.33$. As Fig. S8.7-2c shows, the two exponentials are each easily identified. It is rare, however, for the windowed data record $x_1[n]$ to contain an integer number of periods, as occurs in this case, so Fig. S8.7-2c paints a somewhat optimistic picture of the performance of this 100-point DFT.

(d)
```
>>  n = (0:99); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
```
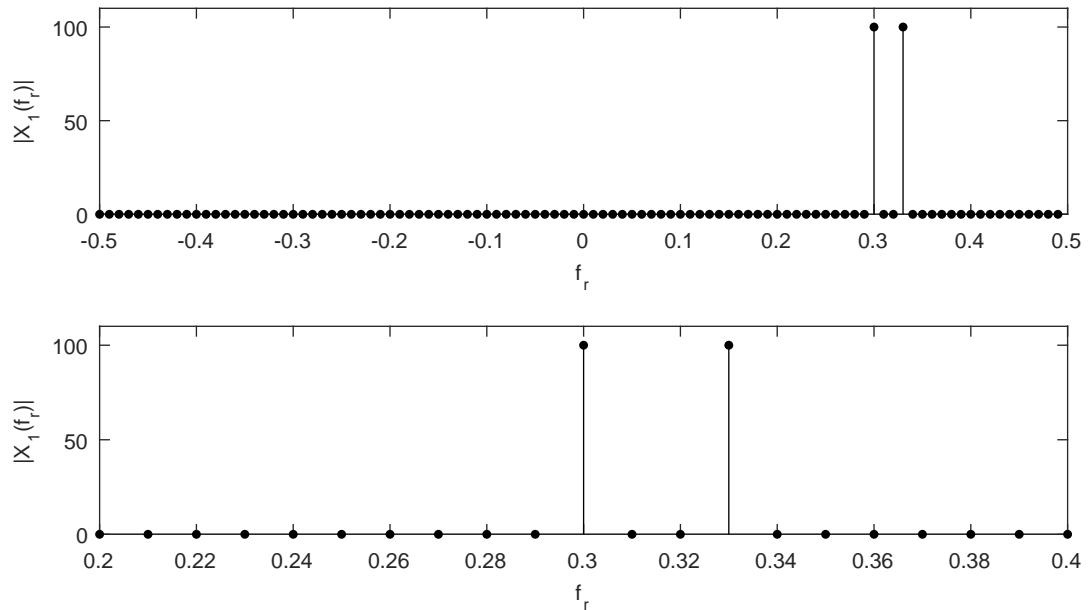
Figure S8.7-2c

```
>>  x1 = [x1,zeros(1,400)];
>>  X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([-0.5 0.5 0 110]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>>  xlabel('f_r'); ylabel('|X_1(f_r)|');
>>  axis([0.2 0.4 0 110]);
```


Figure S8.7-2d

For this DFT, 100 samples of $x_1[n]$ are used, and the sequence is zero-padded to a length of

500. The DFT has 500 frequency bins uniformly spaced in the frequency interval $[-0.5, 0.5)$, and these bins include both exponential frequencies $f_r = 0.30$ and $f_r = 0.33$. As shown in Fig. S8.7-2d, the two exponentials can be separately identified, but there is some added clutter throughout the frequency spectrum. This clutter is the result of applying a finite-length window to the signal $x_1[n]$. Although Fig. S8.7-2d may appear less accurate than Fig. S8.7-2c, both contain the same information about $x_1[n]$. Using the picket fence analogy, Fig. S8.7-2d uses more pickets than does Fig. S8.7-2c, but the background behind both is the same. In many respects, Fig. S8.7-2d paints a more honest picture of the data than does Fig. S8.7-2c; recall that Fig. S8.7-2c looks uncommonly good since data record $x_1[n]$ includes an integer number of periods (a rare occurrence).

Solution 8.7-3

Ideally, $x_2[n] = e^{j2\pi n30/100} + e^{j2\pi n31.5/100}$ is characterized by two spikes of equal height located at $f_r = 0.30$ and $f_r = 0.315$. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $x_2[n]$.

(a)
```
>>  n = (0:9); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>>  X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>>  stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([-0.5 0.5 0 20]);
```
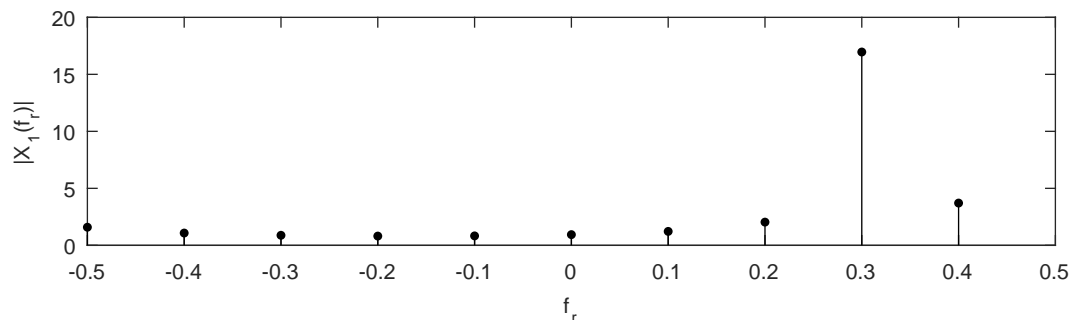

Figure S8.7-3a

For this DFT, only ten samples of $x_2[n]$ are used. As a result, the DFT has only 10 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Fig. S8.7-3a shows, there is insufficient frequency resolution to separately identify the two closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.315$.

(b)
```
>>  n = (0:9); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>>  x2 = [x2,zeros(1,490)];
>>  X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([-0.5 0.5 0 20]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([0.2 0.4 0 20]);
```
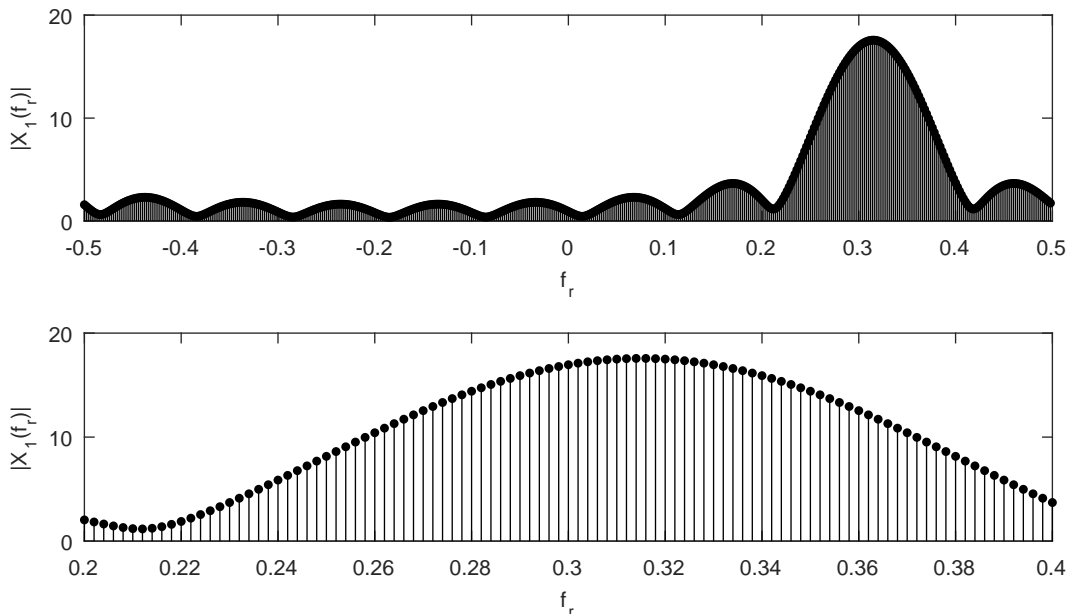
For this DFT, only ten samples of $x_2[n]$ are used but the sequence is zero-padded to a length of 500. Although the DFT has 500 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$, there is insufficient information about $x_2[n]$ (only 10 samples) to resolve the closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.315$. Using the picket fence analogy, zero-padding
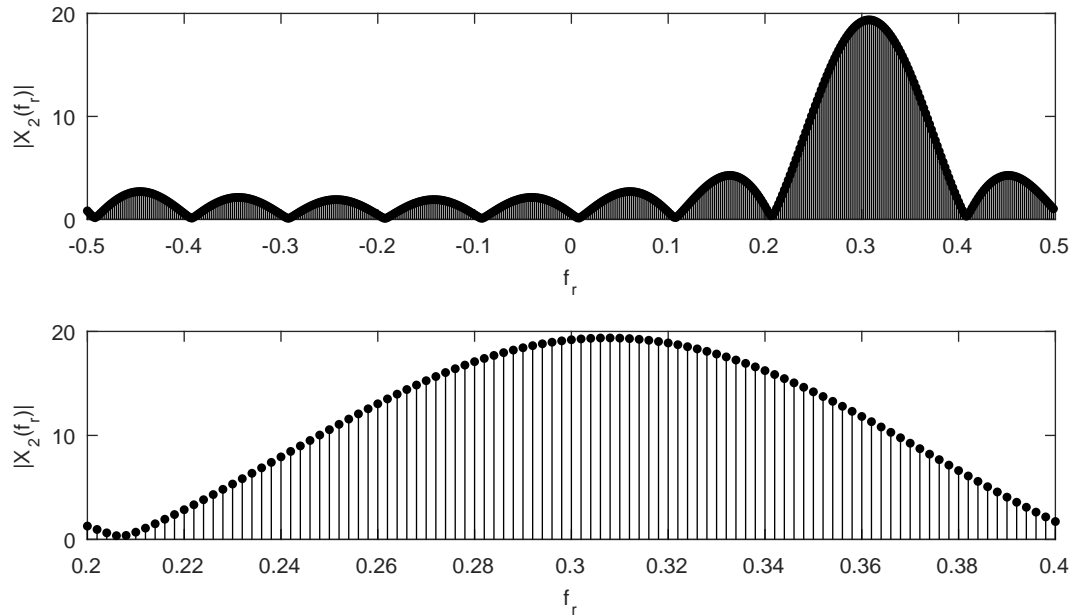
Figure S8.7-3b

increases the number of "pickets" in our DFT fence, but it does not change what lies behind the fence. Still, Fig. S8.7-3b does show a concentration of signal energy centered around $f_r = 0.308$, the average of the two exponential frequencies.

(c)
```
>>  n = (0:99); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>>  X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([-0.5 0.5 0 110]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([0.2 0.4 0 110]);
```

For this DFT, 100 samples of $x_2[n]$ are used. As a result, the DFT has 100 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Fig. S8.7-3c shows, the two exponentials are not easily distinguished; even the number of dominant frequency components is difficult to identify. There difficulties partially occur because the exponentials are closely spaced and the data window is insufficiently large. Features are also obscured since the frequency $f_r = 0.315$ does not lie directly on a DFT frequency bin; therefore the effects of frequency leakage and smearing are pronounced.

(d)
```
>>  n = (0:99); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>>  x2 = [x2,zeros(1,400)];
>>  X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([-0.5 0.5 0 110]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>>  xlabel('f_r'); ylabel('|X_2(f_r)|');
>>  axis([0.2 0.4 0 110]);
```

For this DFT, 100 samples of $x_2[n]$ are used, and the sequence is zero-padded to a length of 500. The DFT has 500 frequency bins uniformly spaced in the frequency interval $[-0.5, 0.5)$. As shown in Fig. S8.7-3d, two dominant frequency components can be separately identified, but
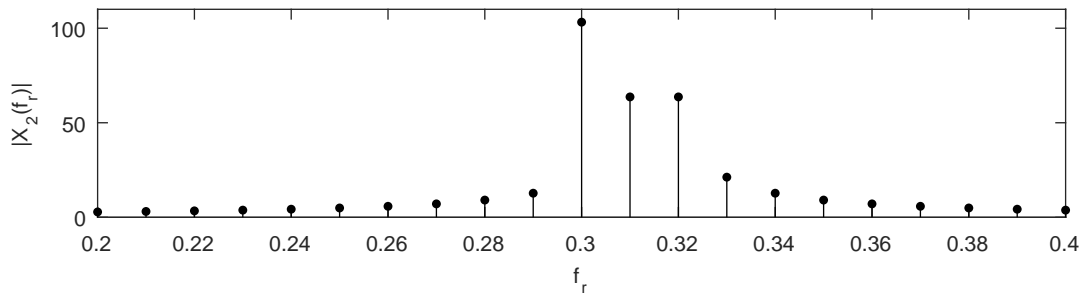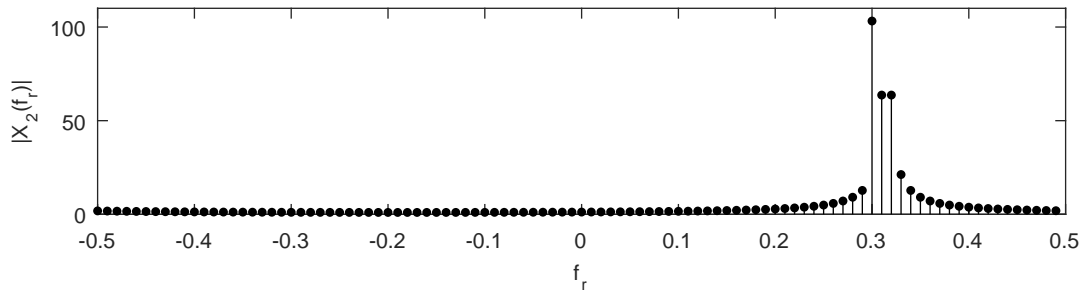
Figure S8.7-3c
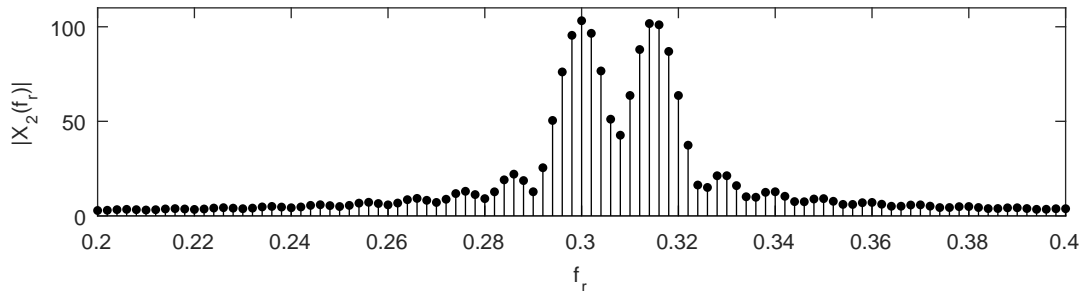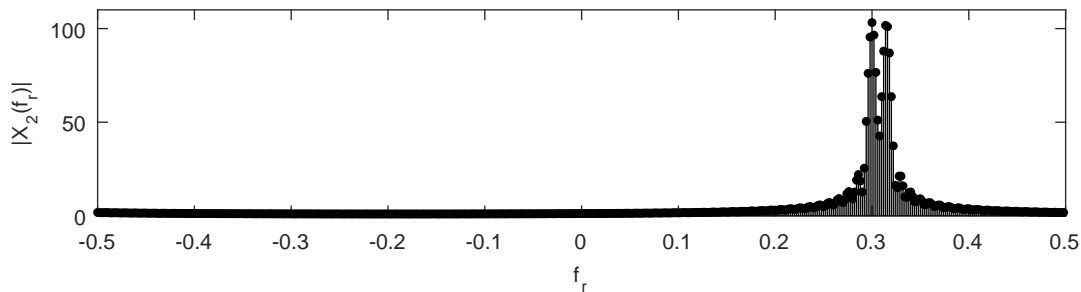
Figure S8.7-3d

there is some added "clutter" throughout the frequency spectrum. This "clutter" is the result of applying a finite-length window to the signal $x_2[n]$. Comparing Figs. S8.7-3c and S8.7-3d, it is clear that zero-padding assists in separating and locating the two dominant frequencies. That is, Fig. S8.7-3d displays two discernable modes at the correct frequencies $f_r = 0.30$ and $f_r = 0.315$ while Fig. S8.7-3c cannot distinguish these two features.

Solution 8.7-4

Ideally, $y_1[n] = 1 + e^{j2\pi n30/100} + 0.5 * e^{j2\pi n43/100}$ is characterized by three spikes located at $f_r = 0$, $f_r = 0.3$, and $f_r = 0.43$. The spikes at $f_r = 0$ and $f_r = 0.3$ should have equal height and the spike

at $f_r = 0.43$ should have have a height that is half as high as the other two. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $x_2[n]$.
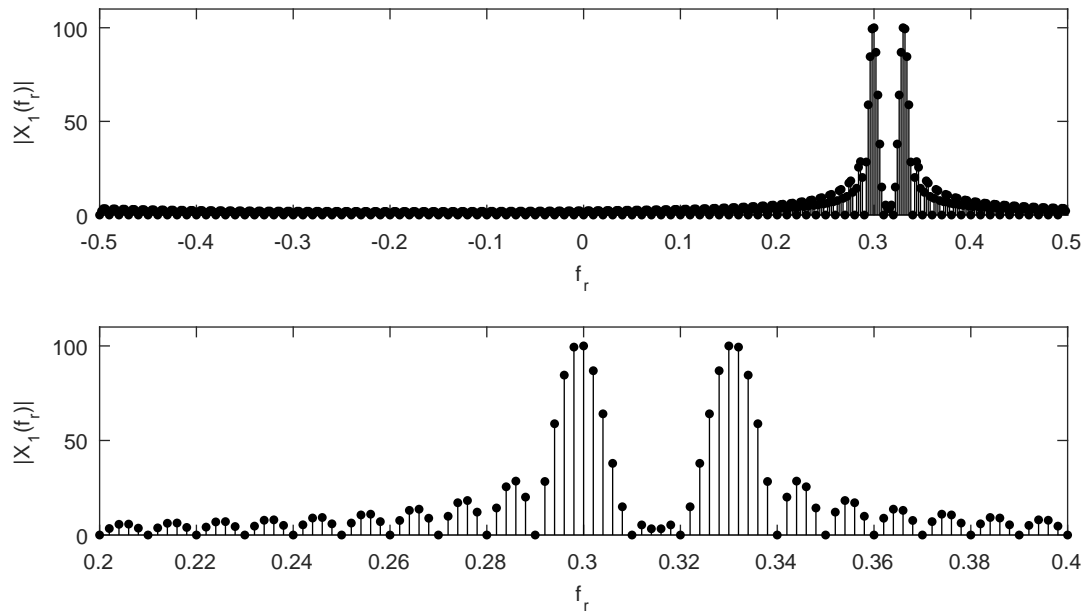
(a)
```
>>  n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>>  Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>>  stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>>  xlabel('f_r'); ylabel('|Y_1(f_r)|');
>>  axis([-0.5 0.5 0 25]);
```
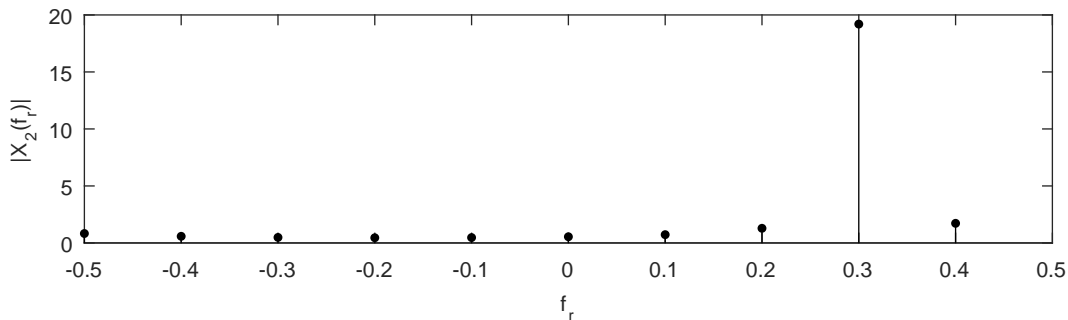

Figure S8.7-4a

For this DFT, only 20 samples of $y_1[n]$ are used. As a result, the DFT has only 20 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Fig. S8.7-4a shows, strong content is seen at $f_r = 0$ and $f_r = 0.3$, but there is insufficient detail to identify the component at $f_r = 0.43$. As a result, it is not really possible to determine the relative strength of the two non-DC components.

(b)
```
>>  n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>>  y1 = [y1,zeros(1,480)];
>>  Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>>  xlabel('f_r'); ylabel('|Y_1(f_r)|');
>>  axis([-0.5 0.5 0 25]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>>  xlabel('f_r'); ylabel('|Y_1(f_r)|');
>>  axis([0.25 0.45 0 25]);
```

As shown in Fig. S8.7-4b, the picture is improved by zero-padding the signal from part (a). In this case, each of the three signal components can be identified near their correct frequencies $f_r = 0$, $f_r = 0.3$, and $f_r = 0.43$. Interestingly, however, the peak amplitude near $f_r = 0.3$ is around 20.2 while the peak amplitude near $f_r = 0.43$ is around 11.5; the ratio of signal amplitudes appears to be $\frac{20.2}{11.5} = 1.7565$, which does not equal the true ratio of 2. The primary reason for this distortion is frequency leakage that results from applying a rectangular window to the signal $y_1[n]$.

(c)
```
>>  n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>>  y1 = y1.*window(@hanning,length(y1))';
>>  Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>>  stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>>  xlabel('f_r'); ylabel('|Y_1(f_r)|');
>>  axis([-0.5 0.5 0 12]);
```

Compared to a rectangular window, a Hanning window has a broader main lobe, which tends to broaden a signal's spectral features. This broadening, or frequency smearing as it is sometimes called, is evident when Fig. S8.7-4c(a) is compared to Fig. S8.7-4a; the DFT of the Hanning-windowed signal has broader features than the DFT of the rectangular-windowed signal. As
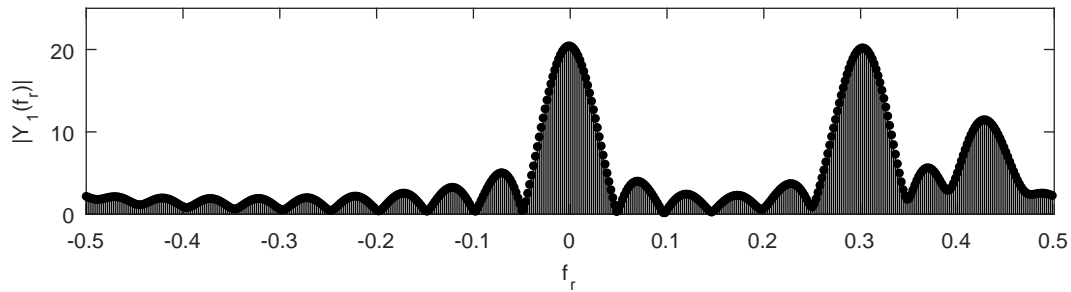
Figure S8.7-4b
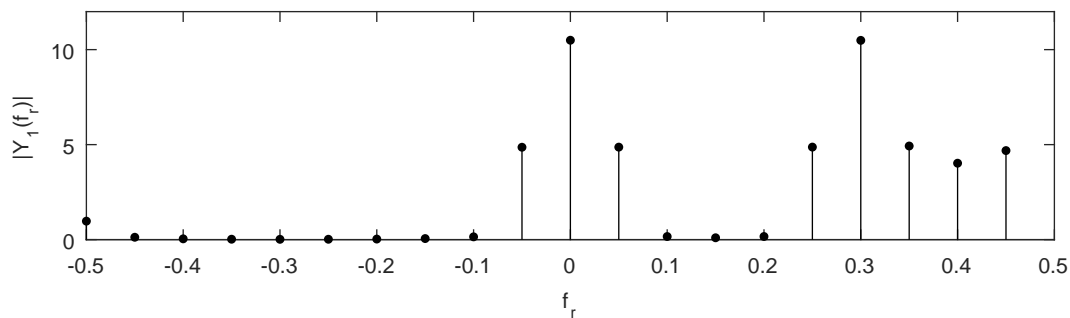
Figure S8.7-4c(a)

with Fig. S8.7-4a, the component at $f_r = 0.43$ is not discernable in Fig. S8.7-4c(a). Finally, notice that the peak amplitudes in Fig. S8.7-4c(a) are lower than the peak amplitudes in Fig. S8.7-4a. This is primarily because the Hanning-window attenuates the edges of the original signal, resulting in a loss of signal energy (and thus smaller DFT coefficients).

```
>>  n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>>  y1 = [y1.*window(@hanning,length(y1))',zeros(1,480)];
>>  Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>>  xlabel('f_r'); ylabel('|Y_1(f_r)|');
>>  axis([-0.5 0.5 0 12]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>>  xlabel('f_r'); ylabel('|Y_1(f_r)|');
>>  axis([0.25 0.45 0 12]);
```

By zero-padding the Hanning-windowed signal $y_1[n]$, the picture is again improved. Figure S8.7-4c(b) shows that each of the three components of $y_1[n]$ are located at the correct frequencies $f_r = 0$, $f_r = 0.3$, and $f_r = 0.43$. Additionally, the peak amplitude at $f_r = 0.3$ is around 10.5 and the peak amplitude at $f_r = 0.43$ is around 5.24; the ratio of signal amplitudes is $\frac{10.5}{5.24} = 2.0038$, which is very close to the true ratio of 2. The computed ratio more accurate
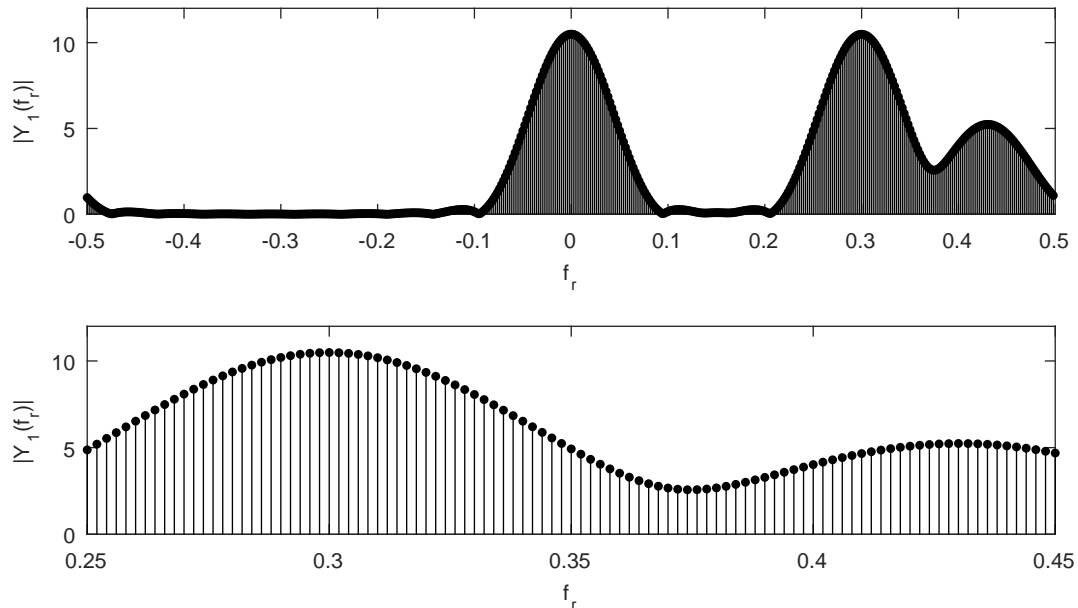
Figure S8.7-4c(b)

than that computed in Fig. S8.7-4b; the reason for this improvement is that the Hanning window has lower side lobes than the rectangular window, and is thus less susceptible to leakage from distantly spaced components.

In this particular case, the Hanning window improves the analysis of the signal. The lower side lobes of the Hanning window result in reduced leakage. Although the Hanning's broad main lobe results in increased smearing, signal components are spaced sufficiently far apart that each component can still be distinguished.

Solution 8.7-5

Ideally, $y_2[n] = 1 + e^{j2\pi n30/100} + 0.5 * e^{j2\pi n38/100}$ is characterized by three spikes located at $f_r = 0$, $f_r = 0.3$, and $f_r = 0.38$. The spikes at $f_r = 0$ and $f_r = 0.3$ should have equal height and the spike at $f_r = 0.38$ should have have a height that is half as high as the other two. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $y_2[n]$.

(a)
```
>>  n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>>  Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>>  stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>>  xlabel('f_r'); ylabel('|Y_2(f_r)|');
>>  axis([-0.5 0.5 0 25]);
```

For this DFT, only 20 samples of $y_2[n]$ are used. As a result, the DFT has only 20 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Fig. S8.7-5a shows, strong content is seen at $f_r = 0$ and $f_r = 0.3$, but there is insufficient detail to identify the component at $f_r = 0.38$. As a result, it is not really possible to determine the relative strength of the two non-DC components.

(b)
```
>>  n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>>  y2 = [y2,zeros(1,480)];
>>  Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>>  xlabel('f_r'); ylabel('|Y_2(f_r)|');
```
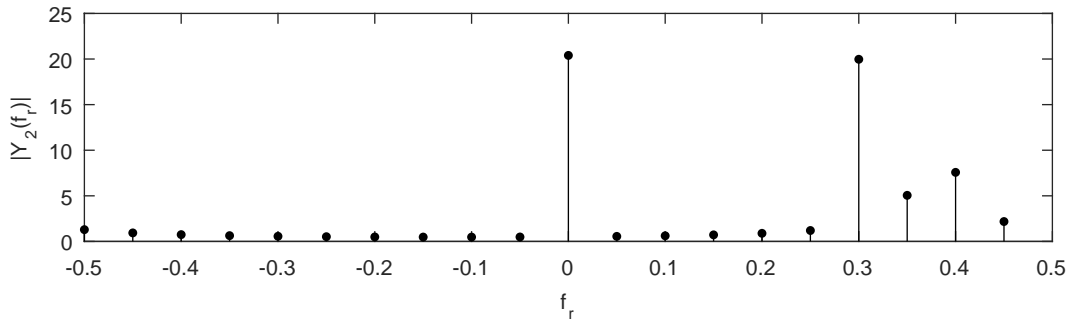
Figure S8.7-5a

```
>>   axis([-0.5 0.5 0 25]);
>>   subplot(212),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>>   xlabel('f_r'); ylabel('|Y_2(f_r)|');
>>   axis([0.25 0.45 0 25]);
```
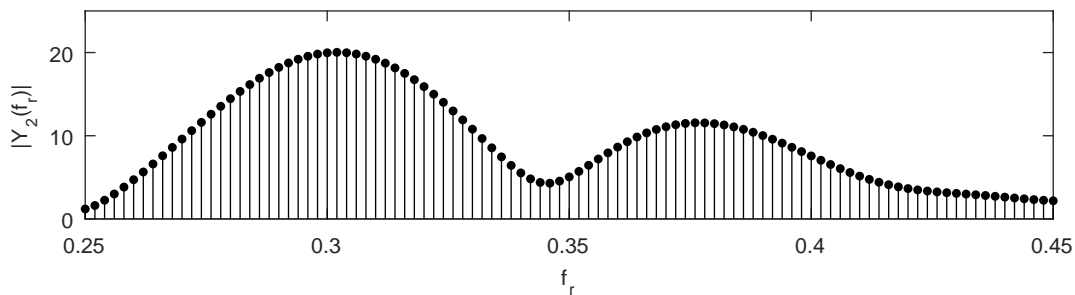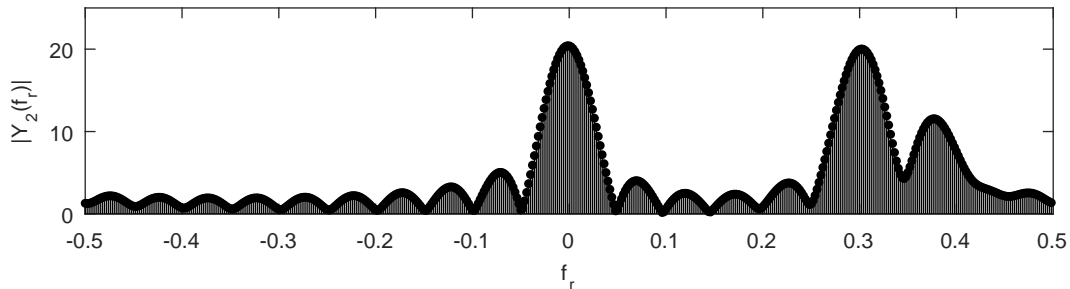

Figure S8.7-5b

As shown in Fig. S8.7-5b, the picture is greatly improved by zero-padding the signal part (a). In this case, each of the three signal components can be identified near their correct frequencies $f_r = 0$, $f_r = 0.3$, and $f_r = 0.38$. Interestingly, however, the peak amplitude near $f_r = 0.3$ is around 20.0 while the peak amplitude near $f_r = 0.38$ is around 11.5; the ratio of signal amplitudes appears to be $\frac{20.0}{11.5} = 1.7391$, which does not equal the true ratio of 2. The primary reason for this distortion is frequency leakage that results from applying a rectangular window to the signal $y_2[n]$.

(c)
```
>>   n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>>   y2 = y2.*window(@hanning,length(y2))';
>>   Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>>   stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>>   xlabel('f_r'); ylabel('|Y_2(f_r)|');
>>   axis([-0.5 0.5 0 12]);
```
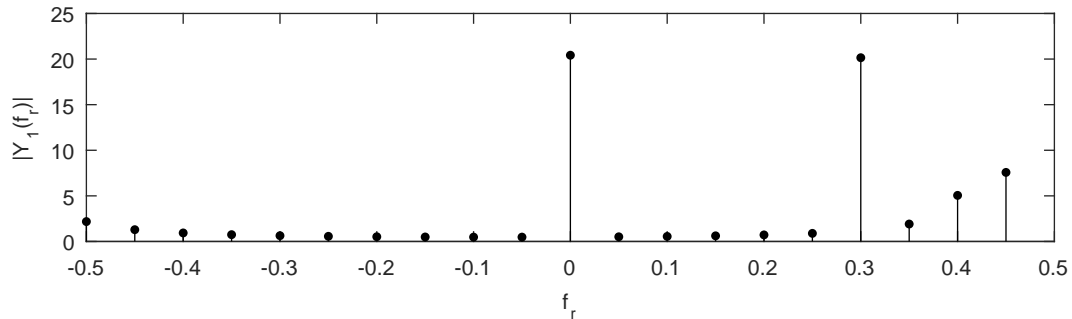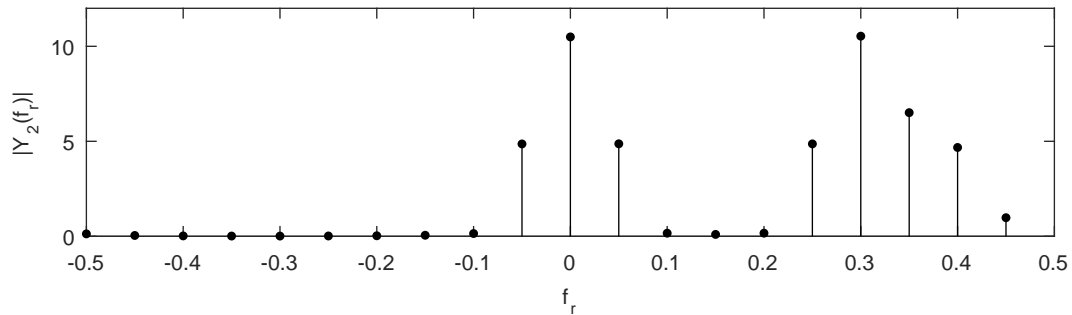
Figure S8.7-5c(a)

Compared to a rectangular window, a Hanning window has a broader main lobe, which tends to broaden a signal's spectral features. This broadening, or frequency smearing as it is sometimes called, is evident when Fig. S8.7-5c(a) is compared to Fig. S8.7-5a; the DFT of the Hanning-windowed signal has broader features than the DFT of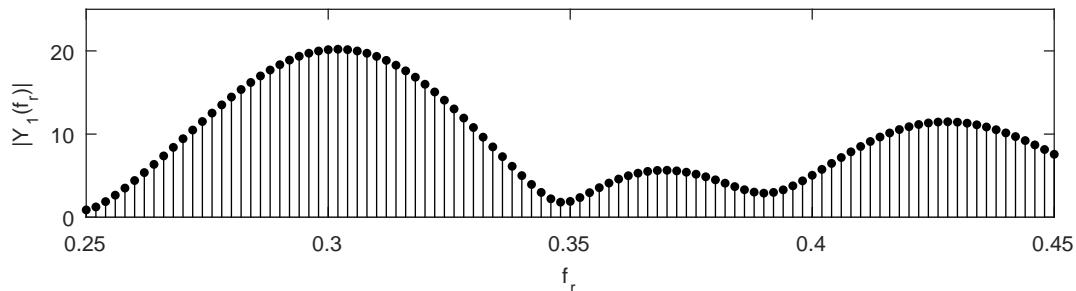 the rectangular-windowed signal. As with Fig. S8.7-5a, the component at $f_r = 0.38$ is not discernable in Fig. S8.7-5c(a). Finally, notice that the peak amplitudes in Fig. S8.7-5c(a) are lower than the peak amplitudes in Fig. S8.7-5a. This is primarily because the Hanning-window attenuates the edges of the original signal, resulting in a loss of signal energy (and thus smaller DFT coefficients).

```
>>  n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>>  y2 = [y2.*window(@hanning,length(y2))',zeros(1,480)];
>>  Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>>  subplot(211),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>>  xlabel('f_r'); ylabel('|Y_2(f_r)|');
>>  axis([-0.5 0.5 0 12]);
>>  subplot(212),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>>  xlabel('f_r'); ylabel('|Y_2(f_r)|');
>>  axis([0.25 0.45 0 12]);
```
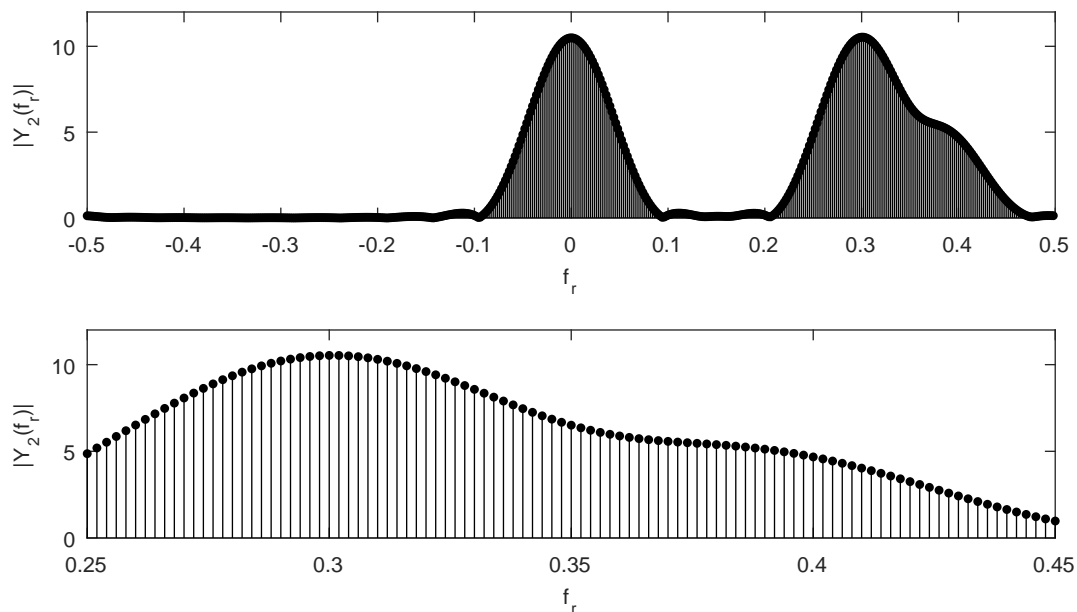

Figure S8.7-5c(b)

As shown in Fig. S8.7-5c(b), zero-padding does not improve the picture of the Hanning-

windowed signal $y_2[n]$. While the components near $f_r = 0$ and $f_r = 0.3$ can be identified, the component at $f_r = 0.38$ appears mostly lost! As such, the relative strengths of the components at $f_r = 0.3$ and $f_r = 0.38$ cannot be determined.

In this particular case, the Hanning window does not improve the analysis of the signal. While the lower side lobes of the Hanning window may reduce leakage, the Hanning's broad main lobe smears the components $f_r = 0.3$ and $f_r = 0.38$ to the point that the component $f_r = 0.38$ is completely obscured.

Solution 8.7-6

(a) MATLAB is used to plot the four samples corresponding to one period of the periodic signal $x[n] = \cos(n\pi/2)$.

```
>>  N = 4; n = (0:N-1); x = cos(n*pi/2);
>>  stem(n,x,'k.'); xlabel('n'); ylabel('x[n]');
>>  axis([-.5 3.5 -1.1 1.1]);
```
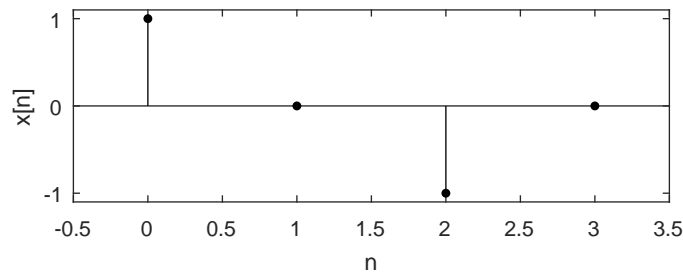

Figure S8.7-6a

Since the signal is so sparsely sampled, it doesn't much resemble a sinusoid.

(b)
```
>>  N = 4; X = fft(x); r=(0:N-1); fr = r/N;
>>  stem(fr-0.5,fftshift(abs(X)),'k.'); xlabel('f_r'); ylabel('|X(f_r)|');
>>  axis([-.5 0.5 -0.1 2.1]);
```
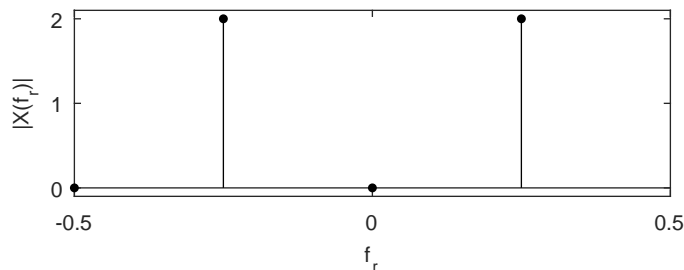

Figure S8.7-6b

The DFT shown in Fig. S8.7-6b seems sensible. A pair of spikes appears that are consistent with the original sinusoid.

(c) Inserting zeros in the middle of the DFT X_r has the effect of increasing the sampling rate of $x[n]$. Zeros need to be placed in the middle to maintain the necessary symmetry of the DFT. Thought of another way, adding zeros to the middle of the DFT effectively specifies zero signal content for the newly added range of higher frequencies. The original signal content at lower frequencies is left unchanged.

```
>> Y = [X(1:3),zeros(1,100-length(X)),X(4)];
>> stem([0:99],real(ifft(Y)),'k.');
>> xlabel('n'); ylabel('y[n]');
```
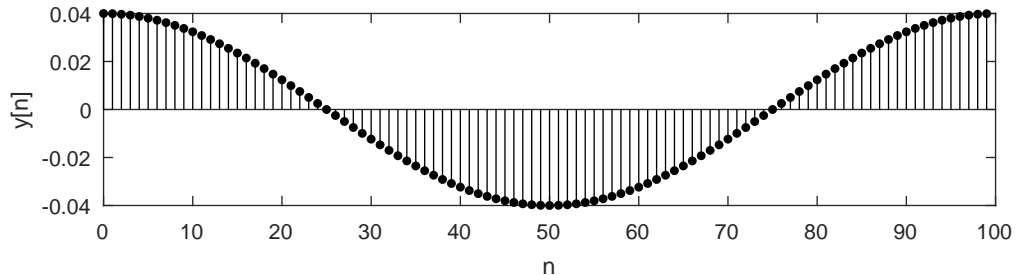

Figure S8.7-6c

As seen in Fig. S8.7-6c, the signal $y[n]$ looks much more sinusoidal than $x[n]$. Both signals are plotted for one full period, but $y[n]$ has 25 times as many samples as $x[n]$. Notice also that the magnitude of $y[n]$ is $1/25$ as great as $x[n]$.

Zero-padding in the frequency domain achieves a similar effect as zero-padding in the time domain. Using the picket fence analogy, zero-padding in frequency increases the number of pickets in the time-domain (increases the sampling rate), but it does not, other than a scale factor, change what is behind the pickets (in this case, one period of a sinusoid).

(d) As seen in the previous part, increasing the size of an N-point DFT by a factor K causes a reduction of the time-domain signal's amplitude by a factor $1/K$. To correct this reduction, scale the zero-padded DFT by the factor K. For example, a 4-point DFT zero-padded to a length of 100 would need to be scaled by $K = 100/4 = 25$.

(e)
```
>> temp = fft([1 1 1 1 -1 -1 -1 -1]);
>> S = (100/length(temp))*[temp(1:5),zeros(1,100-length(temp)),temp(6:8)];
>> stem([0:99],real(ifft(S)),'k.');
>> xlabel('n'); ylabel('s[n]');
```
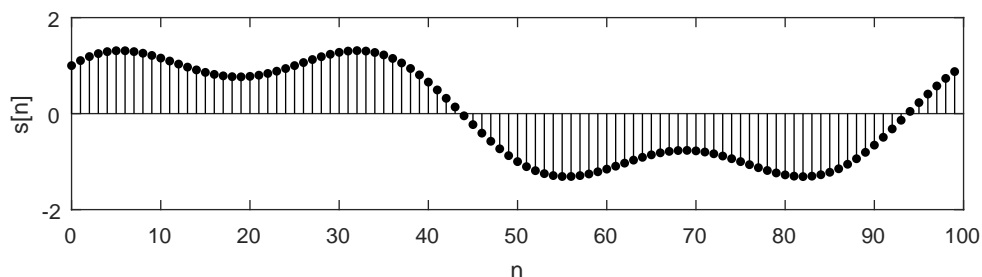

Figure S8.7-6e

As shown in Fig. S8.7-6e, the reconstructed signal $s[n]$ has some appearance of a square wave, but lacks the sharp edges typical of a square wave. In fact, $s[n]$ might be best called a band-limited square wave. Although zero-padding in the frequency domain increases the sampling rate in the time-domain, zero-padding cannot add the high-frequency harmonics needed to achieve a better square-wave approximation.

Solution 8.7-7

(a) Following an approach similar to the text, we use MATLAB to plot the transfer characteristics for a 3-bit truncating asymmetric converter operating over $(-10, 10)$ (see Fig. S8.7-7a).

```
>>  x = (-15:.0001:15); xmax = 10; B = 3; xq = x;
>>  xq(abs(xq)>xmax)=xmax*sign(xq(abs(xq)>xmax));    % Limit amplitude to xmax
>>  xq = xmax/(2^(B-1))*floor(xq*2^(B-1)/xmax);      % Quantize
>>  xq(xq>=xmax)=xmax*(1-2^(1-B));                   % Ensure 2^B levels
>>  plot(x,xq,'k'); axis([-15 15 -10.5 10.5]); grid on;
>>  line([-10 10],[-10 10],'linestyle',':','linewidth',1/4);
>>  xlabel('Quantizer input'); ylabel('Quantizer output');
```
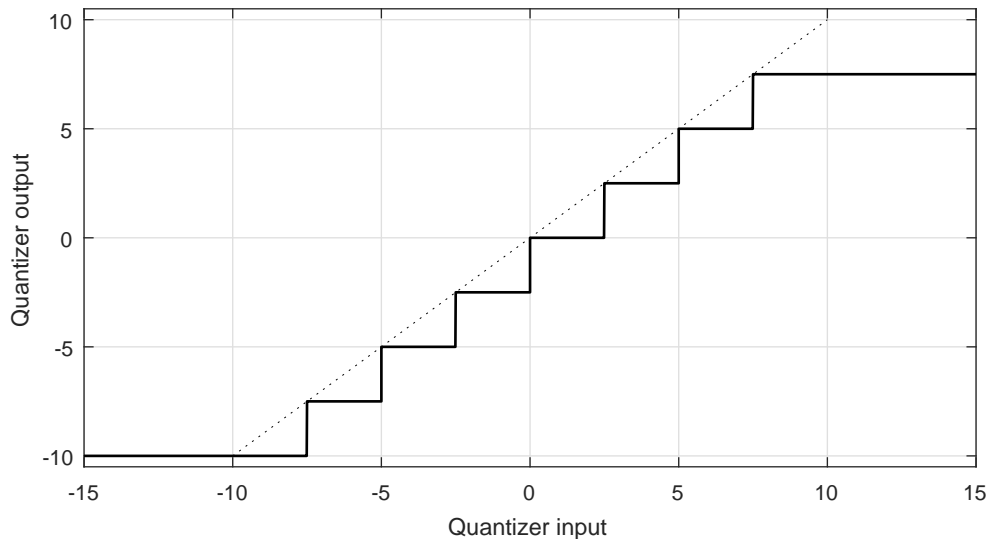

Figure S8.7-7a

(b) Next, we apply 3-bit truncating asymmetric quantization to a 1 Hz cosine sampled at $f_\mathrm{s} = 50$ Hz over 1 second. Figure S8.7-7b shows the original signal $x(t)$, the quantized signal $x_\mathrm{q}(t)$, and the magnitude spectra of both. The (3-bit) truncating asymmetric quantization shown in Fig. S8.7-7b operates similarly to the (2-bit) asymmetric rounding quantization shown in Fig. 8.33. Although not easy to see from the plots, truncating asymmetric quantization has a level transition at 0 and thus (undesirably) operates as a low-noise amplifier.

```
>>  fs = 50; T = 1/fs; N0 = 50; n = 0:N0-1;
>>  x = cos(2*pi*n*T); xmax = 1; B = 3; xq = x;
>>  xq(abs(xq)>xmax)=xmax*sign(xq(abs(xq)>xmax));    % Limit amplitude to xmax
>>  xq = xmax/(2^(B-1))*floor(xq*2^(B-1)/xmax);      % Quantize
>>  xq(xq>=xmax)=xmax*(1-2^(1-B));                   % Ensure 2^B levels
>>  subplot(221); stem(n,x,'k.'); xlabel('n'); ylabel('x[n]');
>>  axis([-.5 N0-.5 -1.1*xmax 1.1*xmax]); grid on;
>>  subplot(222); stem(n-25,fftshift(abs(fft(x))),'k.');
>>  xlabel('f'); ylabel('|X(f)|');
>>  axis([-N0/2-.5 N0/2-.5 0 1.1*N0/2]); grid on;
>>  subplot(223); stem(n,xq,'k.'); xlabel('n'); ylabel('x_q[n]');
>>  axis([-.5 N0-.5 -1.1*xmax 1.1*xmax]); grid on;
>>  subplot(224); stem(n-25,fftshift(abs(fft(xq))),'k.');
>>  xlabel('f'); ylabel('|X_q(f)|');
>>  axis([-N0/2-.5 N0/2-.5 0 1.1*N0/2]); grid on;
```

Figure S8.7-7b

Solution 8.7-8

(a) Following an approach similar to the text, we use MATLAB to plot the transfer characteristics for a 3-bit truncating symmetric converter operating over $(-10, 10)$ (see Fig. S8.7-8a).

```
>>  x = (-15:.0001:15); xmax = 10; B = 3; xq = x;
>>  xq(abs(xq)>xmax)=xmax*sign(xq(abs(xq)>xmax));   % Limit amplitude to xmax
>>  xq = xmax/(2^(B-1))*(floor(xq*2^(B-1)/xmax-1/2)+1/2); % Quantize
>>  xq(xq<=-xmax)=-xmax*(1-2^(-B));                 % Ensure 2^B levels
>>  plot(x,xq,'k'); axis([-15 15 -10.5 10.5]); grid on;
>>  line([-10 10],[-10 10],'linestyle',':','linewidth',1/4);
>>  xlabel('Quantizer input'); ylabel('Quantizer output');
```

(b) Next, we apply 3-bit truncating symmetric quantization to a 1 Hz cosine sampled at $f_s = 50$ Hz over 1 second. Figure S8.7-8b shows the original signal $x(t)$, the quantized signal $x_q(t)$, and the magnitude spectra of both. The (3-bit) truncating symmetric quantization shown in Fig. S8.7-8b operates similarly to the (2-bit) asymmetric rounding quantization shown in Fig. 8.33. Still, asymmetric converters are often preferred over symmetric converters since they include the desirable value of 0 as a quantization level.

```
>>  fs = 50; T = 1/fs; N0 = 50; n = 0:N0-1;
>>  x = cos(2*pi*n*T); xmax = 1; B = 3; xq = x;
>>  xq(abs(xq)>xmax)=xmax*sign(xq(abs(xq)>xmax));   % Limit amplitude to xmax
>>  xq = xmax/(2^(B-1))*(floor(xq*2^(B-1)/xmax-1/2)+1/2); % Quantize
>>  xq(xq<=-xmax)=-xmax*(1-2^(-B));                 % Ensure 2^B levels
>>  subplot(221); stem(n,x,'k.'); xlabel('n'); ylabel('x[n]');
>>  axis([-.5 N0-.5 -1.1*xmax 1.1*xmax]); grid on;
>>  subplot(222); stem(n-25,fftshift(abs(fft(x))),'k.');
>>  xlabel('f'); ylabel('|X(f)|');
>>  axis([-N0/2-.5 N0/2-.5 0 1.1*N0/2]); grid on;
>>  subplot(223); stem(n,xq,'k.'); xlabel('n'); ylabel('x_q[n]');
>>  axis([-.5 N0-.5 -1.1*xmax 1.1*xmax]); grid on;
```
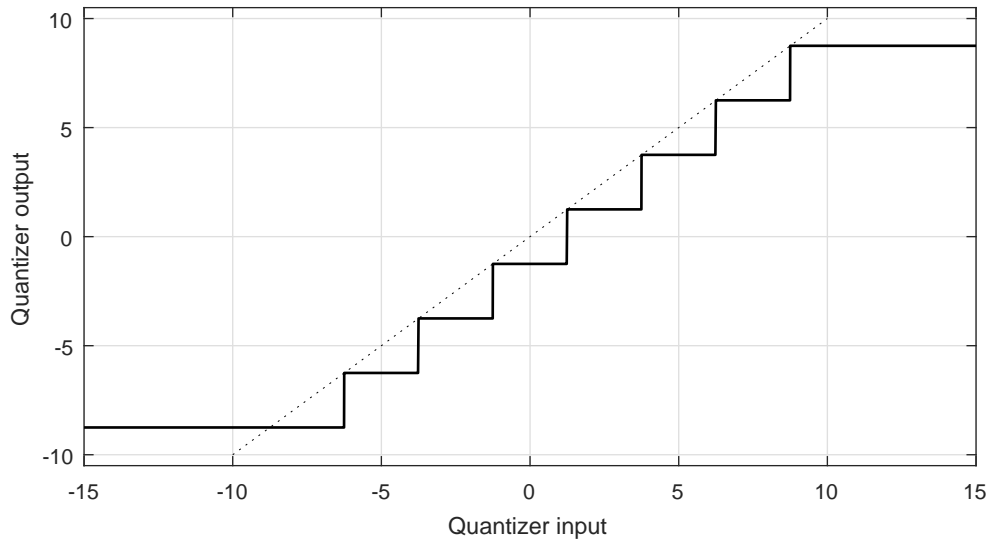
Figure S8.7-8a

```
>> subplot(224); stem(n-25,fftshift(abs(fft(xq)))),'k.');
>> xlabel('f'); ylabel('|X_q(f)|');
>> axis([-N0/2-.5 N0/2-.5 0 1.1*N0/2]); grid on;
```
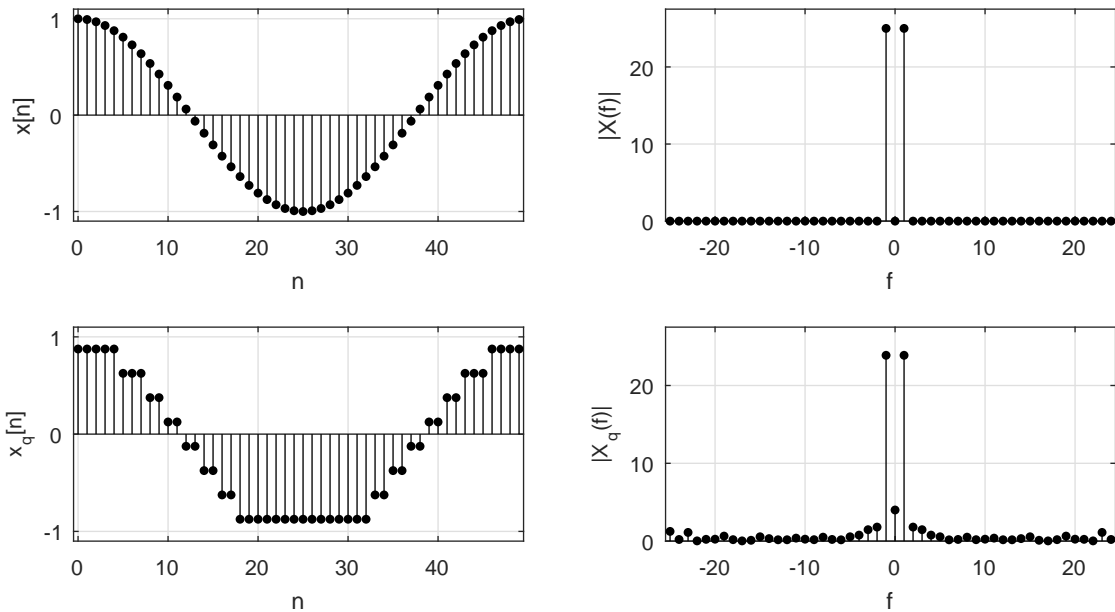

Figure S8.7-8b

Chapter 9 Solutions

Solution 9.1-1

Here,

$$x[n] = 4\cos 2.4\pi n + 2\sin 3.2\pi n = 4\cos 0.4\pi n + 2\sin 1.2\pi n$$

$$= 2[e^{j0.4\pi n} + e^{-j0.4\pi n}] + \frac{1}{j}[e^{j1.2\pi n} - e^{-j1.2\pi n}]$$

$$= 2e^{j0.4\pi n} + 2e^{-j0.4\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{-j(1.2\pi n - \pi/2)}.$$

By inspection, $\Omega_0 = 0.4\pi$ and $N_0 = \frac{2\pi}{\Omega_0} = 5$. Note also that

$$e^{-j0.4\pi n} = e^{j1.6\pi n} \quad \text{and} \quad e^{-j1.2\pi n} = e^{j0.8\pi n}.$$

Therefore,

$$x[n] = 2e^{j0.4\pi n} + 2e^{j1.6\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{j(0.8\pi n + \pi/2)}.$$

We see $x[n]$ is comprised of the first, second, third and fourth harmonics with coefficients

$$\mathcal{D}_1 = \mathcal{D}_2 = 2, \quad \mathcal{D}_3 = -j, \quad \text{and} \quad \mathcal{D}_4 = j.$$

The magnitude and phase spectra are shown in Fig. S9.1-1.
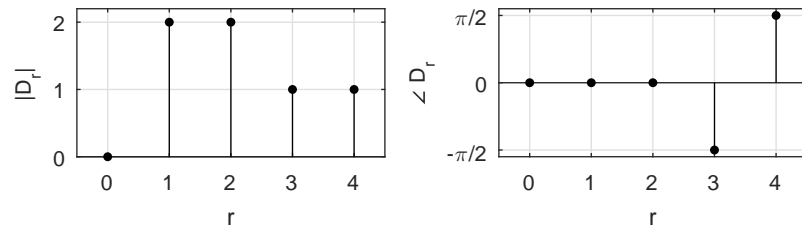
Figure S9.1-1

Solution 9.1-2

Here,

$$x[n] = \cos 2.2\pi n \cos 3.3\pi n = \frac{1}{2}[\cos 5.5\pi n + \cos 1.1\pi n] = \frac{1}{2}[\cos 1.5\pi n + \cos 1.1\pi n]$$

$$= \frac{1}{2}[e^{j1.5\pi n} + e^{-j1.5\pi n} + e^{j1.1\pi n} + e^{-j1.1\pi n}] = \frac{1}{2}[e^{j1.5\pi n} + e^{j0.5\pi n} + e^{j1.1\pi n} + e^{j0.9\pi n}].$$

The fundamental frequency is $\Omega_0 = 0.1$, and $N_0 = \frac{2\pi}{\Omega_0} = 20$. Thus, this signal is comprised of the 5th, 9th, 11th and 15th harmonics with coefficients

$$\mathcal{D}_5 = \mathcal{D}_9 = \mathcal{D}_{11} = \mathcal{D}_{15} = \frac{1}{2}.$$

All the form coefficients are real (phases zero). The spectrum is shown in Fig. S9.1-2.
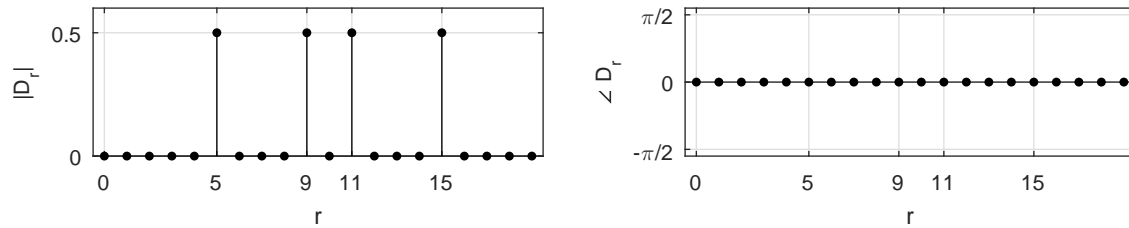
Figure S9.1-2

Solution 9.1-3

Here,

$$x[n] = 2\cos 3.2\pi(n-3) = 2\cos(3.2\pi n - 9.6\pi) = 2\cos(1.2\pi n - 1.6\pi)$$
$$= e^{j(1.2\pi n - 1.6\pi)} + e^{-j(1.2\pi n - 1.6\pi)} = e^{j(1.2\pi n - 1.6\pi)} + e^{j(0.8\pi n + 1.6\pi)}.$$

The fundamental frequency is $\Omega_0 = 0.4\pi$, and $N_0 = \frac{2\pi}{\Omega_0} = 5$. By inspection, we see that $x[n]$ is comprised of only the 2nd and 3rd harmonics with coefficients

$$\mathcal{D}_2 = e^{j1.6\pi} = e^{-j0.4\pi} \quad \text{and} \quad \mathcal{D}_3 = e^{-j1.6\pi} = e^{j0.4\pi}.$$

The magnitude and phase spectra are shown in Fig. S9.1-3.
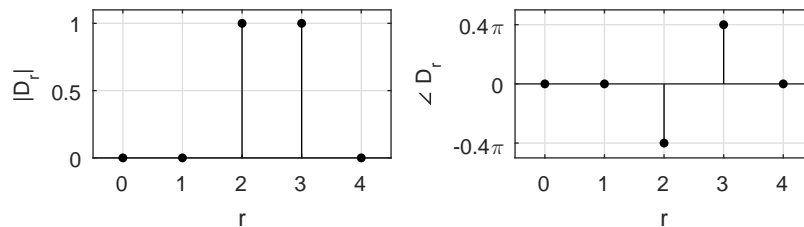
Figure S9.1-3

Solution 9.1-4

In this case, $N_0 = 7$ and $\Omega_0 = \frac{2\pi}{7}$. Now, the DTFS coefficients $\mathcal{D}_r$ are just the DFT scaled by $\frac{1}{N_0}$. We use MATLAB to perform the (tedious) calculations.

```
>>  x = [0, 1, -2, 3, -4, 5, 6]; N0 = 7; Omega0 = 2*pi/N0;
>>  r = (0:N0-1); n = (0:N0-1); Dr = fft(x)/N0
    Dr =   -0.4286+0.0000i   -0.4120-0.2408i   -0.3163-0.6270i    0.9425-2.1906i
            0.9425+2.1906i   -0.3163+0.6270i   -0.4120+0.2408i
```

Thus, over $0 \le r \le 6$, the DTFS coefficients are

$$\mathcal{D}_r = [\, -0.4286 + 0.0000j,\ -0.4120 - 0.2408j,\ -0.3163 - 0.6270j,\ 0.9425 - 2.1906j$$
$$0.9425 + 2.1906j,\ -0.3163 + 0.6270j,\ -0.4120 + 0.2408j].$$

The magnitude and phase spectra, computed using MATLAB, are shown in Fig. S9.1-4.

```
>>  subplot(121); stem(r,abs(Dr),'.k'); xlabel('r'); ylabel('|D_r|');
>>  axis([-.5 6.5 0 1.7]); grid on;
>>  subplot(122); stem(r,angle(Dr),'.k'); xlabel('r'); ylabel('\angle D_r');
>>  axis([-.5 6.5 -1.1*pi 1.1*pi]); grid on; set(gca,'ytick',-pi:pi/2:pi);
```
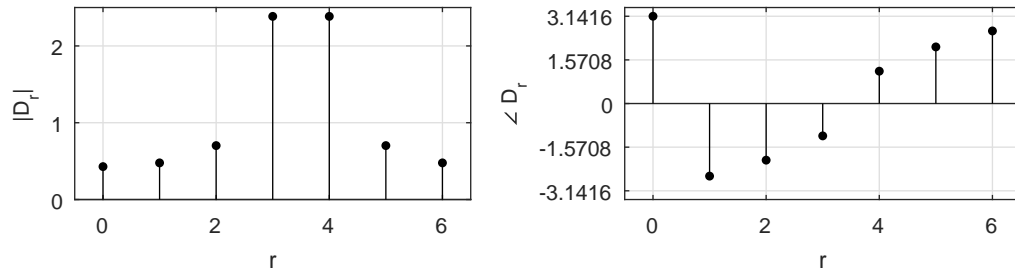
Figure S9.1-4

Since the DTFS coefficients are just a scaled version of the DFT, the DTFS shares the properties of the DFT. Thus, if the 7-periodic signal $x[n]$ is time reversed, the DTFS spectrum is (modulo-N) reversed. Over $0 \le r \le 6$, the DTFS coefficients of $x[-n]$ are therefore

$$[-0.4286 + 0.0000j, \; -0.4120 + 0.2408j, \; -0.3163 + 0.6270j, \; 0.9425 + 2.1906j$$
$$0.9425 - 2.1906j, \; -0.3163 - 0.6270j, \; -0.4120 - 0.2408j].$$

We verify this result using MATLAB.

```
>> fft([x(1),fliplr(x(2:end))])/N0
   ans =  -0.4286+0.0000i  -0.4120+0.2408i  -0.3163+0.6270i   0.9425+2.1906i
          0.9425-2.1906i  -0.3163-0.6270i  -0.4120-0.2408i
```

Solution 9.1-5

To compute coefficients $\mathcal{D}_r$, we use Eq. (9.7) where the summation is performed over any interval N_0. We choose this interval to be $-N_0/2, (N_0/2) - 1$ (for even N_0). Therefore

$$\mathcal{D}_r = \frac{1}{N_0} \sum_{n=-N_0/2}^{(N_0/2)-1} x[n] e^{-jr\Omega_0 n}.$$

In the present case $N_0 = 6$, $\Omega_0 = \frac{2\pi}{N_0} = \frac{\pi}{3}$, and

$$\mathcal{D}_r = \frac{1}{6} \sum_{n=-3}^{2} x[n] e^{-jr\frac{\pi}{3}n}.$$

We have $x[0] = 3$, $x[\pm 1] = 2$, $x[\pm 2] = 1$, and $x[\pm 3] = 0$. Therefore

$$\mathcal{D}_r = \frac{1}{6}[3 + 2(e^{j\frac{\pi}{3}r} + e^{-j\frac{\pi}{3}r}) + (e^{j\frac{2\pi}{3}r} + e^{-j\frac{2\pi}{3}r})]$$
$$= \frac{1}{6}[3 + 4\cos(\frac{\pi}{3}r) + 2\cos(\frac{2\pi}{3}r)]$$

Over $0 \le r \le N_0 - 1$, we obtain

$$\mathcal{D}_0 = \frac{3}{2}, \quad \mathcal{D}_1 = \frac{2}{3}, \quad \mathcal{D}_2 = 0, \quad \mathcal{D}_3 = \frac{1}{6}, \quad \mathcal{D}_4 = 0, \quad \text{and} \quad \mathcal{D}_5 = \frac{2}{3}.$$

Solution 9.1-6

In this case, $N_0 = 12$ and $\Omega_0 = \frac{\pi}{6}$.

$$x[0] = 0, \quad x[1] = 1, \quad x[-1] = -1, \quad x[2] = 2, \quad x[-2] = -2,$$
$$x[3] = 3, \quad x[-3] = -3, \quad \text{and} \quad x[\pm 4] = x[\pm 5] = x[\pm 6] = 0.$$

Therefore

$$
\begin{aligned}
\mathcal{D}_r &= \frac{1}{12} \sum_{n=-6}^{5} x[n] e^{-jr\frac{\pi}{6}n} \\
&= \frac{1}{12}[e^{-j\frac{\pi}{6}r} - e^{j\frac{\pi}{6}r} + 2(e^{-j\frac{2\pi}{6}r} - e^{j\frac{2\pi}{6}r}) + 3(e^{-j\frac{3\pi}{6}r} - e^{j\frac{3\pi}{6}r})] \\
&= \frac{-j}{12}[2\sin(\frac{\pi}{6}r) + 4\sin(\frac{\pi}{3}r) + 6\sin(\frac{\pi}{2}r)]
\end{aligned}
$$

Solution 9.1-7

Here, the period is N_0, and $\Omega_0 = 2\pi/N_0$. Using Eq. (9.4), we obtain

$$
\mathcal{D}_r = \frac{1}{N_0} \sum_{n=0}^{N_0-1} a^n e^{-jr\Omega_0 n} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} (ae^{-jr\Omega_0})^n.
$$

This is a geometric progression, whose sum is found from Sec. B.8.3 as

$$
\mathcal{D}_r = \frac{1}{N_0} \left(\frac{a^{N_0} e^{-jr\Omega_0 N_0} - 1}{ae^{-jr\Omega_0} - 1} \right) = \frac{a^{N_0} - 1}{N_0(ae^{-jr\Omega_0} - 1)} \quad \text{because} \quad e^{-jr\Omega_0 N_0} = e^{-jr2\pi} = 1.
$$

Therefore,

$$
\begin{aligned}
\frac{a^{N_0}}{N_0(ae^{-jr\Omega_0} - 1)} &= \frac{a^{N_0}}{N_0(a\cos r\Omega_0 - ja\sin\Omega_0 - 1)} \\
&= \underbrace{\frac{a^{N_0}}{N_0(\sqrt{a^2 - 2a\cos r\Omega_0 + 1}}}_{|\mathcal{D}_r|} \angle \underbrace{\{-\tan^{-1}\frac{-a\sin r\Omega_0}{a\cos r\Omega_0 - 1}.\}}_{\angle \mathcal{D}_r}
\end{aligned}
$$

Solution 9.1-8

Because $|x[n]|^2 = x[n]x^*[n]$, using Eq. (9.3), we obtain

$$
P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} \left| \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 n} \right|^2 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} \left[\sum_{r=0}^{N_0-1} D_r e^{j\Omega_0 n} \sum_{m=0}^{N_0-1} D_m^* e^{-jm\Omega_0 n} \right].
$$

Interchanging the order of summation yields

$$
P_x = \frac{1}{N_0} \sum_{r=0}^{N_0-1} \sum_{m=0}^{N_0-1} D_r D_m^* \left[\sum_{n=0}^{N_0-1} e^{j(r-m)\Omega_0 n} \right].
$$

From Eq. (8.15), the sum inside the parenthesis is N_0 when $r = m$, and is zero otherwise. Hence,

$$
P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{r=0}^{N_0-1} |D_r|^2.
$$

Solution 9.1-9

(a) Yes, the sum of aperiodic discrete-time sequences can be periodic. For example, consider two signals $x_1[n] = \sin(n)u[n]$ and $x_2[n] = \sin(n)u[-n-1]$. The sum of these two aperiodic signals is the periodic function $x_1[n] + x_2[n] = \sin(n)$.

(b) No, it is not possible for a sum of periodic discrete-time sequences to be aperiodic. Consider arbitrary periodic signals $x_1[n]$ and $x_2[n]$ with periods N_1 and N_2, respectively. Let $y[n] = x_1[n] + x_2[n]$. Notice that $y[n + N_1 N_2] = x_1[n + N_1 N_2] + x_2[n + N_1 N_2]$. By periodicity, $x_1[n+kN_1] = x_1[n]$ and $x_2[n+kN_2] = x_2[n]$ for any k. Thus, $y[n+N_1 N_2] = x_1[n]+x_2[n] = y[n]$. That is, the sum of two periodic signals must also be periodic.

Solution 9.2-1

Using Eq. (9.18), the DTFT of $x[n]$ is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)| e^{j\angle X(\Omega)} e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |X(\Omega)| \cos[\Omega n + \angle X(\Omega)] \, d\Omega + j \int_{-\pi}^{\pi} |X(\Omega)| \sin[\Omega n + \angle X(\Omega)] \, d\Omega \right).$$

Since $|X(\Omega)|$ is an even function of Ω and $\angle X(\Omega)$ is an odd function of Ω, the integrand in the second integral is an odd function of Ω, and the integral thus evaluates to zero. Moreover the integrand in the first integral is an even function of Ω, and therefore

$$x[n] = \frac{1}{\pi} \int_0^{\pi} |X(\Omega)| \cos[\Omega n + \angle X(\Omega)] \, d\Omega.$$

Solution 9.2-2

(a) Because $x[n] = x_e[n] + x_o[n]$ and $e^{-j\Omega n} = \cos(\Omega n) - j\sin(\Omega n)$, the DTFT of $x[n]$ is

$$X(\Omega) = \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n]) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n]) \cos(\Omega n) - j \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n]) \sin(\Omega n).$$

Because $x_e[n] \sin(\Omega n)$ and $x_o[n] \cos(\Omega n)$ are odd functions of n, the sums involving these terms evaluate to zero. Thus,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x_e[n] \cos(\Omega n) - j \sum_{n=-\infty}^{\infty} x_o[n] \sin(\Omega n).$$

Now, if $x[n]$ is real, then $x_e[n]$ and $x_o[n]$ are also real. Thus,

$$\operatorname{Re}\{X(\Omega)\} = \sum_{n=-\infty}^{\infty} x_e[n] \cos(\Omega n) \quad \text{and} \quad j \operatorname{Im}\{X(\Omega)\} = -j \sum_{n=-\infty}^{\infty} x_o[n] \sin(\Omega n).$$

Next, we determine the DTFT of $x_e[n]$ as

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty} x_e[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_e[n] \cos(\Omega n) - j \sum_{n=-\infty}^{\infty} x_e[n] \sin(\Omega n).$$

Because $x_e[n]\sin(\Omega n)$ is odd, the second integral is zero, and

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty} x_e[n]\cos(\Omega n) = \operatorname{Re}\{X(\Omega)\}.$$

Similarly, the DTFT of $x_o[n]$ is

$$X_o(\Omega) = \sum_{n=-\infty}^{\infty} x_o[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_o[n]\cos(\Omega n) - j\sum_{n=-\infty}^{\infty} x_o[n]\sin(\Omega n).$$

Because $x_o[n]\cos(\Omega n)$ is odd, the first integral is zero, and

$$X_o(\Omega) = -j\sum_{n=-\infty}^{\infty} x_o[n]\sin(\Omega n) = j\operatorname{Im}\{X(\Omega)\}.$$

Taking everything together and assuming $x[n]$ is real, we obtain the desired results of

$$X_e(\Omega) = \operatorname{Re}\{X(\Omega)\} \qquad \text{and} \qquad X_o(\Omega) = j\operatorname{Im}\{X(\Omega)\}.$$

(b) We shall prove the result for a general exponential $x[n] = \gamma^n u[n]$ with real parameter γ. Using entry 2 from Table 9.1, we see that the DTFT of $x[n]$ is given as

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}\left(\frac{e^{-j\Omega} - \gamma}{e^{-j\Omega} - \gamma}\right)$$

$$= \frac{1 - \gamma e^{j\Omega}}{\gamma^2 - 2\gamma\cos(\Omega) + 1} = \frac{1 - \gamma\cos(\Omega)}{\gamma^2 - 2\gamma\cos(\Omega) + 1} + j\frac{-\gamma\sin(\Omega)}{\gamma^2 - 2\gamma\cos(\Omega) + 1}.$$

The even and odd components of $x[n] = \gamma^n u[n]$ are

$$x_e[n] = 0.5(\gamma^n u[n] + \gamma^{-n}u[-n]) \qquad \text{and} \qquad x_o[n] = 0.5(\gamma^n u[n] - \gamma^{-n}u[-n]).$$

We know that $\gamma^n u[n] \Longleftrightarrow \frac{1}{1-\gamma e^{j\Omega}}$. Moreover,

$$\gamma^{-n}u[-n] = \left(\frac{1}{\gamma}\right)^n u[-(n+1)] + \delta[n].$$

Hence, using entries 1 and 3 from Table 9.1, we see that

$$\gamma^{-n}u[-n] \Longleftrightarrow \frac{1}{\frac{1}{\gamma}e^{-j\Omega} - 1} + 1 = \frac{1}{1 - \gamma e^{j\Omega}}.$$

As long as γ is real, we therefore see that

$$X_e(\Omega) = 0.5\left(\frac{1}{1 - \gamma e^{-j\Omega}} + \frac{1}{1 - \gamma e^{j\Omega}}\right) = \frac{1 - \gamma\cos(\Omega)}{\gamma^2 - 2\gamma\cos(\Omega) + 1} = \operatorname{Re}\{X(\Omega)\}$$

and

$$X_o(\Omega) = 0.5\left(\frac{1}{1 - \gamma e^{-j\Omega}} - \frac{1}{1 - \gamma e^{j\Omega}}\right) = \frac{-j\gamma\sin(\Omega)}{\gamma^2 - 2\gamma\cos(\Omega) + 1} = j\operatorname{Im}\{X(\Omega)\}.$$

These direct calculations confirm our earlier part (a) results.

Solution 9.2-3

For the following signals, we assume $|\gamma| < 1$ and find the DTFT directly using Eq. (9.19).

(a) Applying Eq. (9.19) to $x_a[n] = \delta[n]$ yields

$$X_a(\Omega) = \sum_{n=-\infty}^{\infty} x_a[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = 1.$$

Figure S9.2-3a shows $x_a[n]$, $|X_a(\Omega)|$, and $\angle X_a(\Omega)$.
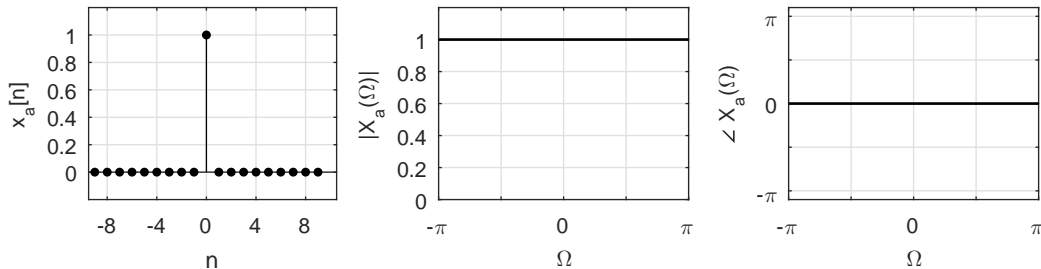
Figure S9.2-3a

(b) Applying Eq. (9.19) to $x_b[n] = \delta[n-k]$ yields

$$X_b(\Omega) = \sum_{n=-\infty}^{\infty} x_b[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n-k]e^{-j\Omega n} = e^{-j\Omega k}.$$

Figure S9.2-3b shows $x_b[n]$, $|X_b(\Omega)|$, and $\angle X_b(\Omega)$. Notice, $x_b[n]$ is just a shifted version of $x_a[n]$. The magnitude spectrum $|X_b(\Omega)|$ equals the magnitude spectrum $|X_a(\Omega)|$, and the phase spectrum $\angle X_b(\Omega)$ equals the phase spectrum $\angle X_a(\Omega)$ plus the linear phase component $-\Omega k$.
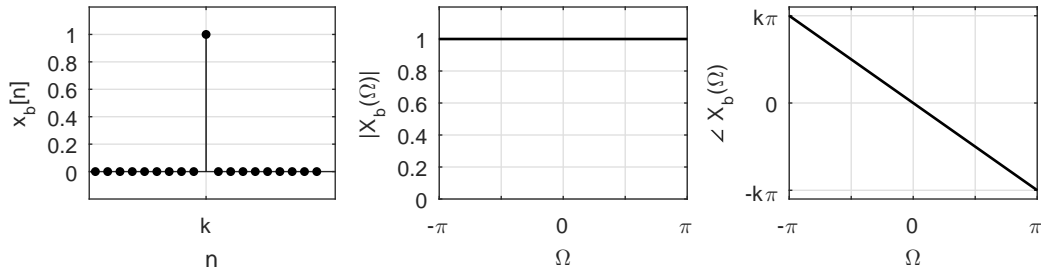
Figure S9.2-3b

(c) Applying Eq. (9.19) to $x_c[n] = \gamma^n u[n-1]$ yields

$$X_c(\Omega) = \sum_{n=-\infty}^{\infty} x_c[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \gamma^n u[n-1]e^{-j\Omega n}$$

$$= \sum_{n=1}^{\infty} \left(\gamma e^{-j\Omega}\right)^n = \frac{\gamma e^{-j\Omega}}{1 - \gamma e^{-j\Omega}}.$$

Figure S9.2-3c shows $x_c[n]$, $|X_c(\Omega)|$, and $\angle X_c(\Omega)$ for $\gamma = 0.8$. Comparing $X_c(\Omega)$ to the spectrum for $\gamma^n u[n]$ found in Ex. 9.3, we see that $X_c(\Omega)$ is just $\gamma e^{-j\Omega}$ times that signal's
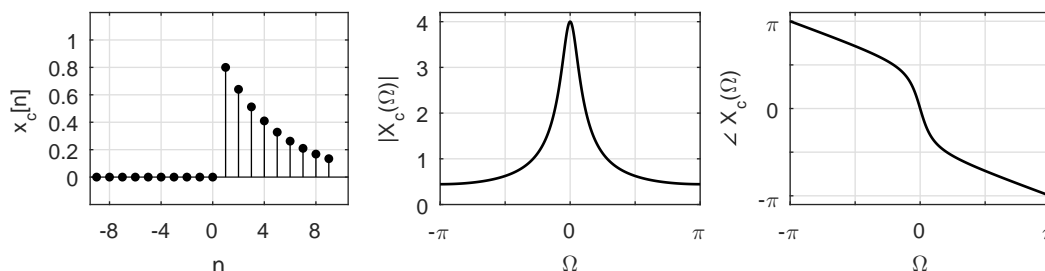
Figure S9.2-3c

spectrum of $1/(1 - \gamma e^{-j\Omega})$. Thus, the magnitude spectrum $|X_c(\Omega)|$ is just $|\gamma|$ times the magnitude spectrum shown in Fig. 9.5b, and the phase spectrum $\angle X_c(\Omega)$ is just the phase spectrum of Fig. 9.5c plus the linear phase component $-\Omega + \angle\gamma$.

(d) Applying Eq. (9.19) to $x_d[n] = \gamma^n u[n+1]$ yields

$$X_d(\Omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \gamma^n u[n+1]e^{-j\Omega n} = \sum_{n=-1}^{\infty} \left(\gamma e^{-j\Omega}\right)^n$$

$$= \frac{(\gamma e^{-j\Omega})^{-1}}{1 - \gamma e^{-j\Omega}} = \frac{\frac{1}{\gamma}e^{j\Omega}}{1 - \gamma e^{-j\Omega}}.$$

Figure S9.2-3d shows $x_d[n]$, $|X_d(\Omega)|$, and $\angle X_d(\Omega)$ for $\gamma = 0.8$. Comparing $X_d(\Omega)$ to the spectrum for $\gamma^n u[n]$ found in Ex. 9.3, we see that $X_d(\Omega)$ is just $\frac{1}{\gamma}e^{j\Omega}$ times that signal's spectrum of $1/(1 - \gamma e^{-j\Omega})$. Thus, the magnitude spectrum $|X_d(\Omega)|$ is just $|\frac{1}{\gamma}|$ times the magnitude spectrum shown in Fig. 9.5b, and the phase spectrum $\angle X_d(\Omega)$ is just the phase spectrum of Fig. 9.5c plus the linear phase component $\Omega - \angle\gamma$.
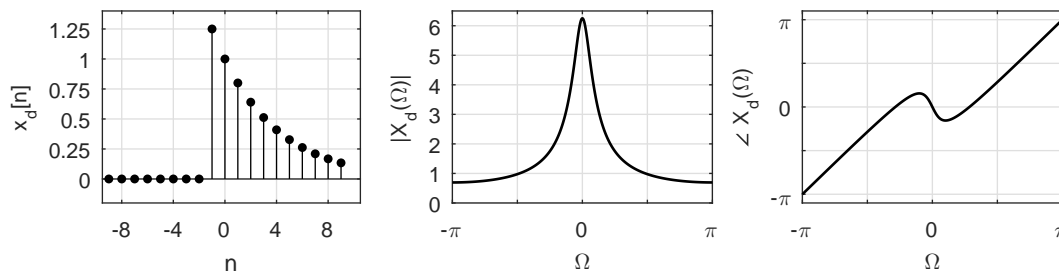
Figure S9.2-3d

(e) Applying Eq. (9.19) to $x_e[n] = (-\gamma)^n u[n]$ yields

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty} x_e[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} (-\gamma)^n u[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(-\gamma e^{-j\Omega}\right)^n$$

$$= \frac{1}{1 + \gamma e^{-j\Omega}} = \frac{1}{1 - \gamma e^{-j(\Omega+\pi)}}.$$

Figure S9.2-3e shows $x_e[n]$, $|X_e(\Omega)|$, and $\angle X_e(\Omega)$ for $\gamma = 0.8$. Comparing $X_e(\Omega)$ to the spectrum for $\gamma^n u[n]$ found in Ex. 9.3, we see that $X_e(\Omega)$ is just that signal's spectrum of $1/(1 - \gamma e^{-j\Omega})$ with $\Omega + \pi$ substituted for Ω. Thus, the magnitude spectrum $|X_e(\Omega)|$ is just the magnitude spectrum shown in Fig. 9.5b shifted in frequency by π, and the phase spectrum $\angle X_e(\Omega)$ is just the phase spectrum of Fig. 9.5c also shifted in frequency by π.
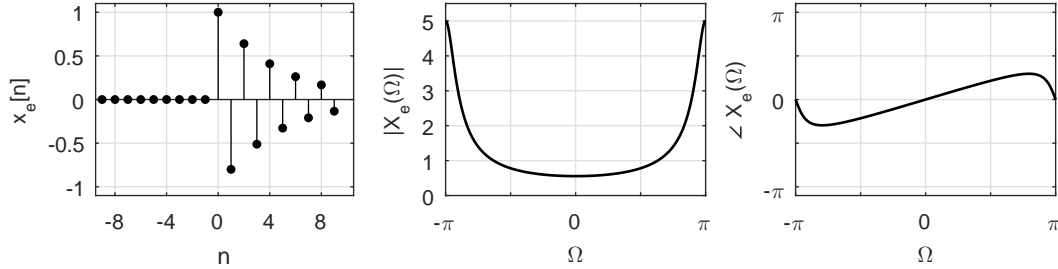
Figure S9.2-3e

(f) Applying Eq. (9.19) to $x_\mathrm{f}[n] = \gamma^{|n|}$ yields

$$X_\mathrm{f}(\Omega) = \sum_{n=-\infty}^{\infty} x_\mathrm{f}[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \gamma^{|n|}e^{-j\Omega n} = \sum_{n=-\infty}^{-1} \gamma^{-n}e^{-j\Omega n} + \sum_{n=0}^{\infty} \gamma^{n}e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{\gamma}e^{-j\Omega}\right)^n + \sum_{n=0}^{\infty} \left(\gamma e^{-j\Omega}\right)^n = \frac{-1}{1 - \frac{1}{\gamma}e^{-j\Omega}} + \frac{1}{1 - \gamma e^{-j\Omega}}$$

$$= \frac{-\gamma}{\gamma - e^{-j\Omega}}\left(\frac{\gamma - e^{j\Omega}}{\gamma - e^{j\Omega}}\right) + \frac{1}{1 - \gamma e^{-j\Omega}}\left(\frac{1 - \gamma e^{j\Omega}}{1 - \gamma e^{j\Omega}}\right)$$

$$= \frac{-\gamma^2 + \gamma e^{j\Omega}}{1 - 2\gamma\cos(\Omega) + \gamma^2} + \frac{1 - \gamma e^{j\Omega}}{1 - 2\gamma\cos(\Omega) + \gamma^2}$$

$$= \frac{1 - \gamma^2}{1 - 2\gamma\cos(\Omega) + \gamma^2}.$$

Figure S9.2-3f shows $x_\mathrm{f}[n]$, $|X_\mathrm{f}(\Omega)|$, and $\angle X_\mathrm{f}(\Omega)$ for $\gamma = 0.8$.

Figure S9.2-3f

Solution 9.2-4

(a) Using Eq. (9.18), the IDTFT of $X_\mathrm{a}(\Omega)$ is

$$x_\mathrm{a}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_\mathrm{a}(\Omega)e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jk\Omega}e^{jn\Omega}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n+k)\Omega}\,d\Omega = \left.\frac{e^{j(n+k)\Omega}}{2\pi j(n+k)}\right|_{\Omega=-\pi}^{\pi}$$

$$= \frac{\sin[\pi(n+k)]}{\pi(n+k)} = \mathrm{sinc}\,[\pi(n+k)] = \delta[n+k].$$

The final step follows by observing that both n and k are integers and $\sin[(n+k)\pi] = 0$ for all $n \neq -k$. For $n = -k$, $\mathrm{sinc}\,[\pi(n+k)] = 1$.

(b) Using Eq. (9.18), the IDTFT of $X_b(\Omega)$ is

$$x_b[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_b(\Omega) e^{j\Omega n}\, d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(k\Omega) e^{jn\Omega}\, d\Omega$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(e^{j(n+k)\Omega} + e^{j(n-k)\Omega} \right) d\Omega$$

Applying the results obtained from part (a), we see that

$$x_b[n] = \frac{1}{2} \left(\mathrm{sinc}\,[\pi(n+k)] + \mathrm{sinc}\,[\pi(nik)] \right) = \frac{1}{2} \left(\delta[n+k] + \delta[n-k] \right).$$

(c) Using Eq. (9.18), the IDTFT of $X_c(\Omega)$ is

$$x_c[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_c(\Omega) e^{j\Omega n}\, d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(\Omega/2) e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left(1 + \cos(\Omega) \right) e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{j\Omega n}\, d\Omega + \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos(\Omega) e^{j\Omega n}\, d\Omega.$$

Using the results from parts (a) and (b), we obtain

$$x_c[n] = \frac{1}{2}\delta[n] + \frac{1}{4} \left(\delta[n+1] + \delta[n-1] \right).$$

(d) In this problem, we assume that $0 < \Omega_c < \pi$. Using Eq. (9.18), the IDTFT of $X_d(\Omega)$ is

$$x_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\Omega) e^{j\Omega n}\, d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta\left(\frac{\Omega}{2\Omega_c} \right) e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{0} \left(1 + \frac{\Omega}{\Omega_c} \right) e^{j\Omega n}\, d\Omega + \frac{1}{2\pi} \int_{0}^{\Omega_c} \left(1 - \frac{\Omega}{\Omega_c} \right) e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} \left[\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\, d\Omega + \frac{1}{\Omega_c} \int_{-\Omega_c}^{0} \Omega e^{j\Omega n}\, d\Omega - \frac{1}{\Omega_c} \int_{0}^{\Omega_c} \Omega e^{j\Omega n}\, d\Omega \right]$$

$$= \frac{1}{2\pi} \left[\left. \frac{e^{j\Omega n}}{jn} \right|_{\Omega=-\Omega_c}^{\Omega_c} + \frac{1}{\Omega_c} \left[\left. \frac{e^{j\Omega n}}{(jn)^2}(j\Omega n - 1) \right|_{\Omega=-\Omega_c}^{0} - \left. \frac{e^{j\Omega n}}{(jn)^2}(j\Omega n - 1) \right|_{\Omega=0}^{\Omega_c} \right] \right]$$

$$= \frac{1}{2\pi} \left[\frac{2\sin(\Omega_c n)}{n} + \frac{1}{\Omega_c} \left(\frac{1}{n^2} + \frac{e^{-j\Omega_c n}}{n^2}(-j\Omega_c n - 1) \right) - \frac{1}{\Omega_c} \left(\frac{e^{j\Omega_c n}}{-n^2}(j\Omega_c n - 1) - \frac{1}{n^2} \right) \right]$$

$$= \frac{1}{2\pi} \left[\frac{2\sin(\Omega_c n)}{n} + \frac{2}{\Omega_c n^2} + \frac{1}{\Omega_c n^2} \left(-e^{-j\Omega_c n}(j\Omega_c n + 1) + e^{j\Omega_c n}(j\Omega_c n - 1) \right) \right]$$

$$= \frac{1}{2\pi\Omega_c n^2} \left[2\Omega_c n \sin(\Omega_c n) + 2 - 2\Omega_c n \sin(\Omega_c n) - 2\cos(\Omega_c n) \right]$$

$$= \frac{2}{2\pi\Omega_c n^2} \left[1 - \cos(\Omega_c n) \right] = \frac{4}{2\pi\Omega_c n^2} \sin^2(\Omega_c n/2)$$

$$= \frac{\Omega_c}{2\pi} \mathrm{sinc}^2\left(\frac{\Omega_c n}{2} \right).$$

(e) Using Eq. (9.18), the IDTFT of $X_e(\Omega)$ is

$$x_e[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_e(\Omega) e^{j\Omega n}\, d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\Omega - \Omega_0) e^{j\Omega n}\, d\Omega$$

$$= e^{j\Omega_0 n}.$$

(f) Using Eq. (9.18), the IDTFT of $X_f(\Omega)$ is

$$x_f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_f(\Omega)e^{j\Omega n}\, d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi\left(\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right)e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) = \cos(\Omega_0 n).$$

Solution 9.2-5

(a) The DTFT of $x[n]$ is determined as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-5}^{5} x[n]e^{-j\Omega n}$$

$$= \frac{1}{7}e^{j6\Omega} + \frac{2}{7}e^{j5\Omega} + \frac{3}{7}e^{j4\Omega} + \frac{4}{7}e^{j3\Omega} + \frac{5}{7}e^{j2\Omega} + \frac{6}{7}e^{j\Omega} + 1$$

$$+ \frac{6}{7}e^{-j\Omega} + \frac{5}{7}e^{-j2\Omega} + \frac{4}{7}e^{-j3\Omega} + \frac{3}{7}e^{-j4\Omega} + \frac{2}{7}e^{-j5\Omega} + \frac{1}{7}e^{-j6\Omega}$$

$$= \frac{2}{7}\cos(6\Omega) + \frac{4}{7}\cos(5\Omega) + \frac{6}{7}\cos(4\Omega) + \frac{8}{7}\cos(3\Omega) + \frac{10}{7}\cos(2\Omega) + \frac{12}{7}\cos(\Omega) + 1.$$

Figure S9.2-5a shows the spectrum $X(\Omega)$ over the traditional interval $-\pi \leq \Omega \leq \pi$.

```
>>   X = @(Om) 2/7*cos(6*Om)+4/7*cos(5*Om)+6/7*cos(4*Om)+8/7*cos(3*Om)+...
>>       10/7*cos(2*Om)+12/7*cos(Om)+1;
>>   Omega = -pi:2*pi/2000:pi;
>>   plot(Omega,X(Omega),'k'); xlabel('\Omega'); ylabel('X(\Omega)');
>>   axis([-pi pi 0 7.5]); grid on; set(gca,'xtick',-pi:pi/2:pi);
>>   set(gca,'xticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'});
```
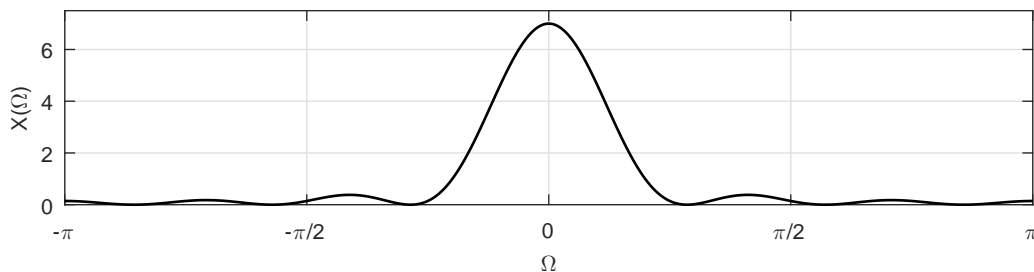

Figure S9.2-5a

(b) Using Ex. 9.5 as a guide, we next use MATLAB and the FFT to validate the DTFT calculations and plot of part (a). As shown in Fig. S9.2-5b, the FFT samples align exactly with the analytical DTFT, thereby confirming the result of part (a).

```
>>   X = @(Om) 2/7*cos(6*Om)+4/7*cos(5*Om)+6/7*cos(4*Om)+8/7*cos(3*Om)+...
>>       10/7*cos(2*Om)+12/7*cos(Om)+1;
>>   Omega = 0:2*pi/2000:2*pi;
>>   N_0 = 64; x = [7 6 5 4 3 2 1 zeros(1,N_0-13) 1 2 3 4 5 6]/7;
>>   Xr = fft(x); Omega_0 = 2*pi/N_0; r = 0:N_0-1;
>>   plot(Omega,abs(X(Omega)),'k-',Omega_0*r,abs(Xr),'k.');
>>   axis([0 2*pi 0 7.5]); xlabel('\Omega'); ylabel('|X(\Omega)|');
```
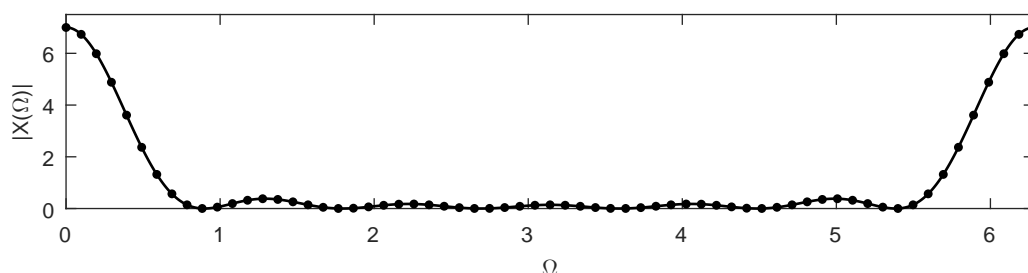
Figure S9.2-5b

Solution 9.2-6

As shown in Fig. S9.2-6, the spectrum $X(\Omega) = \text{rect}\left(\frac{\Omega - \pi/4}{\pi}\right)$ is just a π-width rectrangle function that is shifted to the right by $\pi/4$. Furthermore, the shifted rectangle remains within the fundamental band. Using Eq. (9.18), we compute the IDTFT as

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n} \, d\Omega = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \frac{e^{j\Omega n}}{jn}\Big|_{\Omega = -\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{1}{2\pi}\left(\frac{e^{-j\pi n/4}}{jn} - \frac{e^{j3\pi n/4}}{jn}\right)$$

$$= \frac{1}{\pi n}\left(\frac{e^{j\pi n/2} - e^{-j\pi n/2}}{2j} e^{j\pi n/4}\right) = \frac{1}{2}\left(\frac{\sin(\pi n/2)}{\pi n/2}\right) e^{j\pi n/4}$$

$$= 0.5\text{sinc}(\pi n/2)e^{j\pi n/4}.$$

This is the result that needed to be shown.
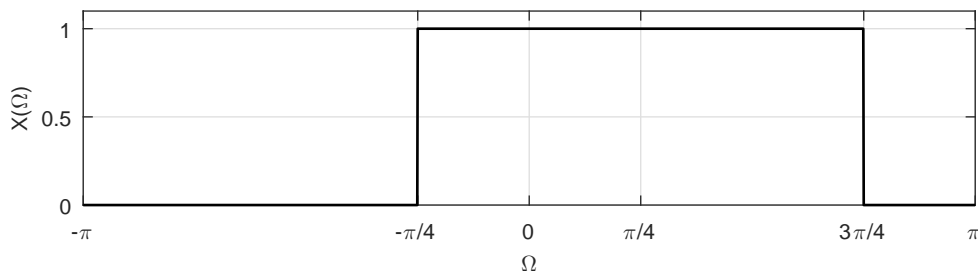
Figure S9.2-6

Solution 9.2-7

(a) For $x_a[n] = a^n(u[n] - u[n - (N_0 + 1)])$ and $0 < a < 1$, we use Eq. (9.19) to obtain

$$X_a(\Omega) = \sum_{n=0}^{N_0} a^n e^{-j\Omega n} = \sum_{n=0}^{N_0} \left(ae^{-j\Omega}\right)^n = \frac{1 - a^{N_0+1}e^{-j(N_0+1)\Omega}}{1 - ae^{-j\Omega}}.$$

(b) For $x_b[n] = a^n(u[n] - u[n - (N_0 + 1)])$ and $a > 1$, we use Eq. (9.19) to obtain

$$X_b(\Omega) = \sum_{n=0}^{N_0} a^n e^{-j\Omega n} = \sum_{n=0}^{N_0} \left(ae^{-j\Omega}\right)^n = \frac{1 - a^{N_0+1}e^{-j(N_0+1)\Omega}}{1 - ae^{-j\Omega}}.$$

Notice that $X_a(\Omega)$ and $X_b(\Omega)$ have exactly the same form. The only difference is that $0 < a < 1$ for $X_a(\Omega)$ and $a > 1$ for $X_b(\Omega)$.

Solution 9.2-8

(a) Using Eq. (9.19), the DTFT of $x_a[n]$ is

$$X_a(\Omega) = \sum_{n=-\infty}^{\infty} x_a[n]e^{-j\Omega n} = \sum_{n=0}^{6} 2e^{-j\Omega n} + \sum_{n=7}^{12} e^{-j\Omega n}$$

$$= 2\frac{1 - e^{-j7\Omega}}{1 - e^{-j\Omega}} + \frac{e^{-j7\Omega} - e^{-j13\Omega}}{1 - e^{-j\Omega}}$$

$$= \frac{2 - e^{-j7\Omega} - e^{-j13\Omega}}{1 - e^{-j\Omega}}.$$

(b) Using Eq. (9.19), the DTFT of $x_b[n]$ is

$$X_b(\Omega) = \sum_{n=-\infty}^{\infty} x_b[n]e^{-j\Omega n} = \sum_{n=-(N_0-1)}^{-1} \left(\frac{-n}{N_0 - 1}\right) e^{-j\Omega n} + \sum_{n=0}^{N_0-1} \left(\frac{n}{N_0 - 1}\right) e^{-j\Omega n}$$

$$= \frac{1}{N_0 - 1}\left(\sum_{n'=1}^{N_0-1} n' e^{j\Omega n'} + \sum_{n=0}^{N_0} n e^{-j\Omega n}\right) = \frac{1}{N_0 - 1}\left(\sum_{n=0}^{N_0} n e^{j\Omega n} + \sum_{n=0}^{N_0} n e^{-j\Omega n}\right).$$

Using the fourth sum of Sec. B.8.3, we obtain

$$X_b(\Omega) = \frac{1}{N_0 - 1}\left(\frac{e^{j\Omega} + [(N_0 - 1)(e^{j\Omega} - 1) - 1]e^{j\Omega N_0}}{(e^{j\Omega} - 1)^2}\right.$$

$$\left. + \frac{e^{-j\Omega} + [(N_0 - 1)(e^{-j\Omega} - 1) - 1]e^{-j\Omega N_0}}{(e^{-j\Omega} - 1)^2}\right).$$
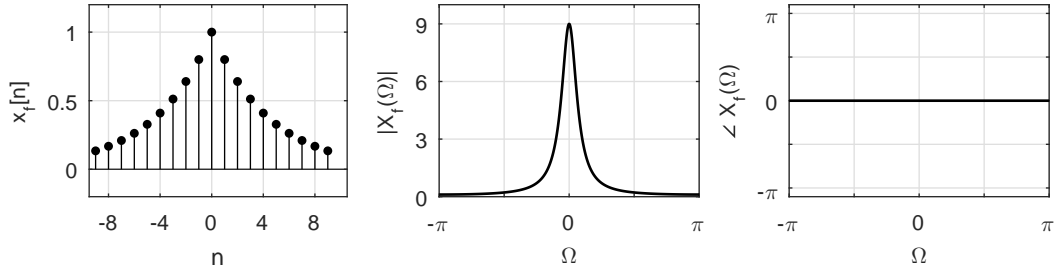
Solution 9.2-9

(a) Using Eq. (9.18), the IDTFT of $X_a(\Omega)$ is

$$x_a[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_a(\Omega)e^{j\Omega n}\, d\Omega = \int_{-0.75\pi}^{0.75\pi} \Omega^2 e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\frac{e^{j\Omega n}}{(jn)^3}\left[-\Omega^2 n^2 - 2j\Omega n + 2\right]\Big|_{-0.75\pi}^{0.75\pi}$$

$$= \frac{(0.5625\pi^2 n^2 - 2)\sin(0.75\pi n) + 1.5\pi n \cos(0.75\pi n)}{\pi n^3}.$$

(b) Computation of the IDTFT of $X_b(\Omega)$ can be simplified by observing that $X_b(\Omega)$ can be expressed as a sum of two gate functions $\mathrm{rect}(\Omega/2)$ and $\mathrm{rect}(\Omega/4)$. Using this fact and Eq. (9.18) we obtain

$$x_b[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_b(\Omega)e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\mathrm{rect}(\frac{\Omega}{2}) + \mathrm{rect}(\frac{\Omega}{4})]e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\left(\int_{-2}^{2} e^{j\Omega n}\, d\Omega + \int_{-1}^{1} e^{j\Omega n}\, d\Omega\right)$$

$$= \frac{\sin(2n) + \sin(n)}{\pi n} = \frac{1}{\pi}\mathrm{sinc}\,(n) + \frac{2}{\pi}\mathrm{sinc}\,(2n).$$

Solution 9.2-10

(a) Using Eq. (9.18), the IDTFT of $X_a(\Omega)$ is

$$x_a[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_a(\Omega) e^{j\Omega n} \, d\Omega = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos(\Omega) e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2} \left(e^{j\Omega} + e^{-j\Omega} \right) e^{j\Omega n} \, d\Omega = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} \left(e^{j\Omega(n+1)} + e^{j\Omega(n-1)} \right) d\Omega$$

$$= \left. \left(\frac{1}{4\pi j(n+1)} e^{j\Omega(n+1)} + \frac{1}{4\pi j(n-1)} e^{j\Omega(n-1)} \right) \right|_{\Omega=-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi} \left(\frac{\sin(\frac{\pi(n+1)}{2})}{n+1} + \frac{\sin(\frac{\pi(n-1)}{2})}{n-1} \right)$$

$$= \frac{1}{2\pi} \left(\frac{\cos(\frac{\pi n}{2})}{n+1} - \frac{\cos(\frac{\pi n}{2})}{n-1} \right) = \frac{1}{2\pi} \frac{-2\cos(\frac{\pi n}{2})}{n^2 - 1}$$

$$= \frac{\cos(\frac{\pi n}{2})}{\pi(1 - n^2)}.$$

(b) Since it is not specified in the problem, designate parameter P as the peak height of $X_b(\Omega)$. Using Eq. (9.18), the IDTFT of $X_b(\Omega)$ is

$$x_b[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_b(\Omega) e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_b(\Omega) \cos(\Omega n) \, d\Omega + \frac{j}{2\pi} \int_{-\pi}^{\pi} X_b(\Omega) \sin(\Omega n) \, d\Omega$$

Because $X_b(\Omega)$ is even function, $X_b(\Omega) \sin(\Omega n)$ is an odd function of Ω and the corresponding integral evaluates to zero. The integrand of the remaining integral is an even function so that

$$x_b[n] = \frac{1}{\pi} \int_0^{\pi} X_b(\Omega) \cos(\Omega n) \, d\Omega = \frac{1}{\pi} \int_0^{\pi/3} \frac{4P}{\pi} \Omega \cos(\Omega n) \, d\Omega$$

$$= \left. \frac{4P}{\pi^2} \frac{\cos(n\Omega) + n\Omega \sin(n\Omega)}{n^2} \right|_{\Omega=0}^{\pi/4}$$

$$= \frac{4P}{\pi^2 n^2} \left[\cos(\pi n/4) + \frac{\pi n}{4} \sin(\pi n/4) - 1 \right].$$

Solution 9.2-11

(a) Using Eq. (9.19), the DTFT of $x_a[n]$ is

$$X_a(\Omega) = \sum_{n=-\infty}^{\infty} x_a[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2] \right) e^{-j\Omega n}$$

$$= 3 + 2(e^{-j\Omega} + e^{j\Omega}) + (e^{-j2\Omega} + e^{j2\Omega})$$

$$= 3 + 4\cos(\Omega) + 2\cos(2\Omega).$$

(b) Using Eq. (9.19), the DTFT of $x_{\mathrm{b}}[n]$ is

$$X_{\mathrm{b}}(\Omega) = \sum_{n=-\infty}^{\infty} x_{\mathrm{b}}[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4] \right) e^{-j\Omega n}$$

$$= e^{-j\Omega} + 2e^{-j2\Omega} + 3e^{-j3\Omega} + 2e^{-j4\Omega} + e^{-j5\Omega}$$

$$= e^{-j3\Omega}[3 + 2(e^{j\Omega} + e^{-j\Omega}) + (e^{j2\Omega} + e^{-j2\Omega})]$$

$$= e^{-j3\Omega}[3 + 4\cos(\Omega) + 2\cos(2\Omega)].$$

(c) Using Eq. (9.19), the DTFT of $x_{\mathrm{c}}[n]$ is

$$X_{\mathrm{c}}(\Omega) = \sum_{n=-\infty}^{\infty} x_{\mathrm{c}}[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(-9\delta[n+3] - 6\delta[n+2] - 3\delta[n+1] + 3\delta[n-1] + 6\delta[n-2] + 9\delta[n-3] \right) e^{-j\Omega n}$$

$$= -9e^{j3\Omega} - 6e^{j2\Omega} - 3e^{j\Omega} + 3e^{-j\Omega} + 6e^{-j2\Omega} + 9e^{-j6\Omega}$$

$$= -3(e^{j\Omega} - e^{-j\Omega}) - 6(e^{j2\Omega} - e^{-j2\Omega}) - 9(e^{j3\Omega} - e^{-j3\Omega})$$

$$= -6j[\sin(\Omega) + 2\sin(2\Omega) + 3\sin(3\Omega)].$$

(d) Using Eq. (9.19), the DTFT of $x_{\mathrm{d}}[n]$ is

$$X_{\mathrm{d}}(\Omega) = \sum_{n=-\infty}^{\infty} x_{\mathrm{d}}[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(4\delta[n+2] + 2\delta[n+1] + 2\delta[n-1] + 4\delta[n-2] \right) e^{-j\Omega n}$$

$$= 4e^{j2\Omega} + 2e^{j\Omega} + 2e^{-j\Omega} + 4e^{-j2\Omega}$$

$$= 4\cos(\Omega) + 8\cos(2\Omega).$$

Solution 9.2-12

(a) Here, the IDTFT is

$$x[n] = \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} e^{-j\Omega n_0} e^{j\Omega n} \, d\Omega = \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} e^{j\Omega(n-n_0)} \, d\Omega$$

$$= \frac{1}{(2\pi)j(n-n_0)} e^{j\Omega(n-n_0)} \Big|_{-\Omega_0}^{\Omega_0} = \frac{\sin \Omega_0(n-n_0)}{\pi(n-n_0)} = \frac{\Omega_0}{\pi} \operatorname{sinc}[\Omega_0(n-n_0)].$$

(b) In this case, the IDTFT is

$$x[n] = \frac{1}{2\pi} \left[\int_{-\Omega_0}^{0} j e^{j\Omega n} \, d\Omega + \int_{0}^{\Omega_0} -j e^{j\Omega n} \, d\Omega \right]$$

$$= \frac{1}{2\pi n} e^{j\Omega n} \Big|_{-\Omega_0}^{0} - \frac{1}{2\pi n} e^{j\Omega n} \Big|_{0}^{\Omega_0} = \frac{1 - \cos \Omega_0 n}{\pi n}.$$

Although the magnitude spectra for parts (a) and (b) are identical, the different phase spectra ensure that the two time-domain signals are dramatically different from one another.

Solution 9.2-13

(a) Let us consider the sum

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n - Lk].$$

When $n \neq mL$ where m is an integer, then for any integer values of k, $n - Lk$ cannot be zero and $\delta[n - Lk] = 0$ for all k and the sum on the left-hand side is zero for all $n \neq mL$ (m integer). When $n = mL$ (m integer), then $\delta[mL - Lk] = 1$ for $k = m$ and is zero for all $k \neq m$. Hence, the sum on the left-hand side has only one term $x[m]$ or, in terms of n, the one term is $x[n/L]$. Putting these two conditions together, we see that

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n - Lk] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$$

$$= x_e[n]$$

(b) Here,

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\delta[n - Lk] \right) e^{-j\Omega n}.$$

Interchanging the order of the summation yields

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty} x[k] \left(\sum_{k=-\infty}^{\infty} \delta[n - Lk]e^{-j\Omega n} \right)$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega Lk} = X(L\Omega).$$

(c) The signal $z[n]$ is just the signal $x[n] = 1$ expanded by factor $L = 3$. Using pair 11 of Table 9.1 and the result of part (b), we obtain

$$Z(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(3\Omega - 2\pi k) = \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{3}).$$

Solution 9.2-14

(a) We shall consider spectra within the band $|\Omega| \leq \pi$ only.

Pair 8: $\frac{\Omega_c}{\pi}\text{sinc}(\Omega_c n) \Longleftrightarrow \text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$. This is identical to pair 18 in Table 7.1 with W replaced by Ω_c and t replaced by n.

Pair 9: In the same way, we see that pair 9 is identical to pair 20 in Table 7.1.

Pair 11: is identical to pair 7 in Table 7.1.

Pair 12: is identical to pair 8 in Table 7.1.

Pair 13: is identical to pair 9 in Table 7.1.

Pair 14: is identical to pair 10 in Table 7.1.

(b) This method cannot be used for pairs 2, 3, 4, 5, 6, 7, 10, 15 and 16 because in all these cases $X(\Omega)$ is not bandlimited.

Solution 9.2-15

(a) The spectrum $X_{\mathrm{a}}(\Omega) = \Omega + \pi$ is not a valid DTFT because it is not 2π-periodic.

(b) The spectrum $X_{\mathrm{b}}(\Omega) = j + \pi$ is a valid DTFT because it is a constant, which satisfies the requirement of being 2π-periodic.

(c) The spectrum $X_{\mathrm{c}}(\Omega) = \sin(10\Omega)$ is a valid DTFT because it is a $\frac{2\pi}{10}$-periodic, hence also a 2π-periodic, function of Ω.

(d) The spectrum $X_{\mathrm{d}}(\Omega) = \sin(\Omega/10)$ is not a valid DTFT because although it is 20π-periodic, it is not 2π-periodic.

(e) The spectrum $X_{\mathrm{e}}(\Omega) = \delta(\Omega)$ is not a valid DTFT because it is not 2π-periodic.

Solution 9.3-1

We determine the DTFTs of the each of the following signals using pairs 2 and 5 (Table 9.1) and the time-shifting property of Eq. (9.31).

(a) The DTFT of $x_{\mathrm{a}}[n] = u[n] - u[n-9]$ is

$$X_{\mathrm{a}}(\Omega) = \left[\frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega) \right] - \left[\frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega) \right] e^{-j9\Omega}.$$

We know that $\delta(\Omega) e^{-j9\Omega} = \delta(\Omega)$ because $e^{-j9\Omega} = 1$ at $\Omega = 0$. Therefore,

$$\begin{aligned} X_{\mathrm{a}}(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega} - 1} \left[1 - e^{-j9\Omega} \right] \\ &= \frac{e^{j\Omega} e^{-j4.5\Omega} \left[e^{j4.5\Omega} - e^{-j4.5\Omega} \right]}{e^{j\Omega/2} \left[e^{j\Omega/2} - e^{-j\Omega/2} \right]} \\ &= e^{-j4\Omega} \frac{\sin(4.5\Omega)}{\sin(0.5\Omega)}. \end{aligned}$$

(b) The DTFT of $x_{\mathrm{b}}[n] = a^{n-m} u[n-m]$ is

$$X_{\mathrm{b}}(\Omega) = e^{-jm\Omega} \left(\frac{e^{j\Omega}}{e^{j\Omega} - a} \right) = \frac{e^{j(1-m)\Omega}}{e^{j\Omega} - a}.$$

(c) To begin, we rewrite the signal $x_{\mathrm{c}}[n]$ as

$$x_{\mathrm{c}}[n] = a^{n-3} (u[n] - u[n-10]) == a^{-3} a^n u[n] - a^7 a^{n-10} u[n-10].$$

Thus, the DTFT of $x_{\mathrm{c}}[n] = a^{n-3} (u[n] - u[n-10])$ is

$$\begin{aligned} X_{\mathrm{c}}(\Omega) &= a^{-3} \frac{e^{j\Omega}}{e^{j\Omega} - a} - a^7 \frac{e^{j\Omega}}{e^{j\Omega} - a} e^{-j10\Omega} \\ &= \frac{e^{j\Omega} \left(a^{-3} - a^7 e^{-j10\Omega} \right)}{e^{j\Omega} - a}. \end{aligned}$$

(d) Because $x_{\mathrm{d}}[n] = a^{n-m} u[n] = a^{-m} a^n u[n]$, the DTFT is

$$X_{\mathrm{d}}(\Omega) = a^{-m} \frac{e^{j\Omega}}{e^{j\Omega} - a}.$$

(e) Because $x_e[n] = a^n u[n-m] = a^m a^{n-m} u[n-m]$, the DTFT is

$$X_e(\Omega) = a^m \frac{e^{j\Omega}}{e^{j\Omega} - a} e^{-jm\Omega}$$

$$= a^m \frac{e^{j(1-m)\Omega}}{e^{j\Omega} - a}.$$

(f) The DTFT of $x_f[n] = (n-m)a^{n-m} u[n-m]$ is

$$X_f(\Omega) = \frac{a e^{j\Omega} e^{-jm\Omega}}{(e^{j\Omega} - a)^2} = \frac{a e^{j(1-m)\Omega}}{(e^{j\Omega} - a)^2}.$$

(g) Because $x_g[n] = (n-m)a^n u[n] = na^n u[n] - ma^n u[n]$, the DTFT is

$$X_g(\Omega) = \frac{a e^{j\Omega}}{(e^{j\Omega} - a)^2} - \frac{m e^{j\Omega}}{e^{j\Omega} - a}$$

$$= \frac{e^{j\Omega}(a - m e^{j\Omega} + ma)}{(e^{j\Omega} - a)^2}.$$

(h) To begin, we rewrite the signal $x_h[n]$ as

$$x_h[n] = na^{n-m} u[n-m] = (n-m)a^{n-m} u[n-m] + ma^{n-m} u[n-m].$$

Thus, the DTFT of $x_h[n] = (na^{n-m} u[n-m]$ is

$$X_h(\Omega) = \frac{a e^{j\Omega} e^{-jm\Omega}}{(e^{j\Omega} - a)^2} + \frac{m e^{j\Omega} e^{-jm\Omega}}{e^{j\Omega} - a}$$

$$= \frac{e^{j(1-m)\Omega}}{(e^{j\Omega} - a)^2} \left(a + m e^{j\Omega} - ma \right).$$

Solution 9.3-2

We represent the DTFTs of signals $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ in terms of $X(\Omega)$. In the first case, notice that $x_1[n] = x[n-4] + x[-n-4] - 4\delta[n]$. Thus,

$$X_1(\Omega) = X(\Omega)e^{-j4\Omega} + X(-\Omega)e^{j4\Omega} - 4.$$

In the second case, notice that $x_2[n] = x[n] + x[-n]$. Thus,

$$X_2(\Omega) = X(\Omega) + X(-\Omega).$$

In the third case, notice that $x_3[n] = x[n-2] + x[-n-2]$. Thus,

$$X_3(\Omega) = X(\Omega)e^{-j2\Omega} + X(-\Omega)e^{j2\Omega}.$$

In the fourth case, notice that $x_4[n] = x_3[n] + x[n-7] + x[-n-7]$. Thus,

$$X_4 = X(\Omega)e^{-j2\Omega} + X(-\Omega)e^{j2\Omega} + X(\Omega)e^{-j7\Omega} + X(-\Omega)e^{j7\Omega}.$$

In all these expression, we define

$$X(\Omega) = \frac{4 e^{j6\Omega} - 5 e^{j5\Omega} + e^{j\Omega}}{(e^{j\Omega} - 1)^2}.$$

Solution 9.3-3
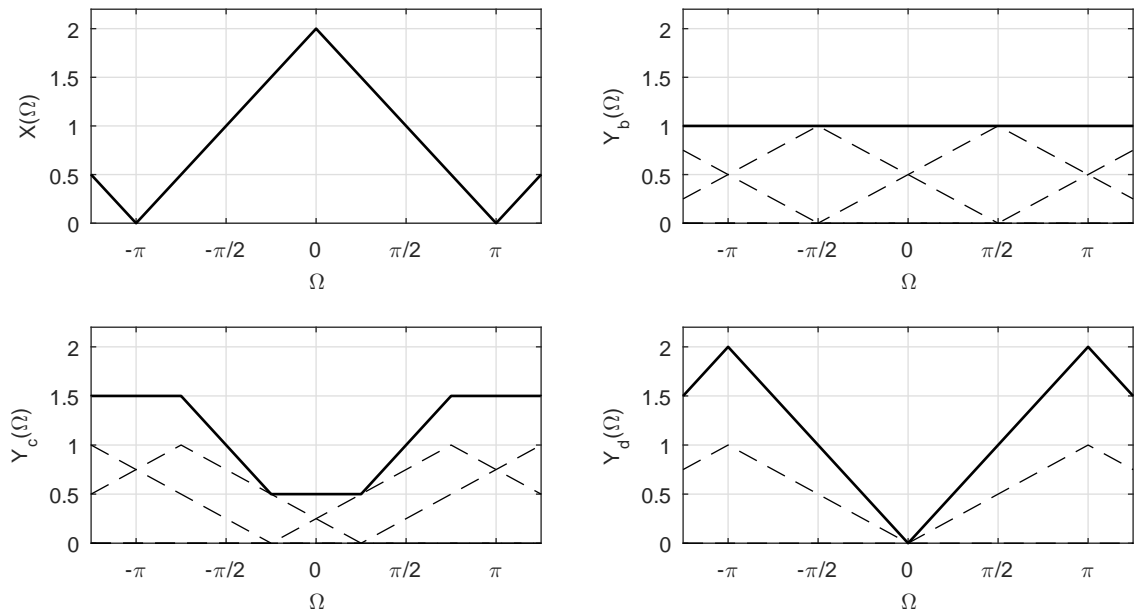
Figure S9.3-3

(a) From pair 9 of Table 9.1, we know that

$$x[n] = \text{sinc}^2(\pi n/2) \Longleftrightarrow 2 \sum_{k=-\infty}^{\infty} \Delta\left(\frac{\Omega - 2\pi k}{2\pi}\right) = X(\Omega).$$

The spectrum $X(\Omega)$ is shown in Fig. S9.3-3.

(b) Using the modulation property of Eq. (9.33), we see that

$$y_b[n] = x[n]\cos(\pi n/2) \Longleftrightarrow \frac{1}{2}\left[X(\Omega - \frac{\pi}{2}) + X(\Omega + \frac{\pi}{2})\right] = Y_b(\Omega).$$

The spectrum $Y_b(\Omega)$, shown in Fig. S9.3-3, completely loses the information in $X(\Omega)$. The overlap in the shifted spectra results in $Y_b(\Omega)$ being a constant value of 1. Thus, $y_b[n] = \delta[n]$. This is easily confirmed in MATLAB.

```
>>  snc = @(t) sinc(t/pi); x = @(n) (snc(pi*n/2)).^2; n = -5:5; ya = x(n).*cos(pi/2*n)
    ya = 0.0000  0.0000  0.0000  0.0000  0.0000  1.0000  0.0000  0.0000  0.0000  0.0000  0.0000
```

(c) Using the modulation property of Eq. (9.33), we see that

$$y_c[n] = x[n]\cos(3\pi n/4) \Longleftrightarrow \frac{1}{2}\left[X(\Omega - \frac{3\pi}{4}) + X(\Omega + \frac{3\pi}{4})\right] = Y_c(\Omega).$$

The spectrum $Y_c(\Omega)$ is shown in Fig. S9.3-3. Again, the overlap in the shifted spectra results in a loss of (most of) $X(\Omega)$ in $Y_c(\Omega)$.

(d) Using the modulation property of Eq. (9.33), we see that

$$y_d[n] = x[n]\cos(\pi n) \Longleftrightarrow \frac{1}{2}[X(\Omega - \pi) + X(\Omega + \pi)] = Y_d(\Omega).$$

The spectrum $Y_d(\Omega)$ is shown in Fig. S9.3-3. In this case, there is no distortion of the original spectrum $X(\Omega)$. In fact, we see that $Y_d(\Omega)$ is just $X(\Omega)$ shifted by π, a result known as spectral inversion. This is more readily seen by noting that $y_d[n] = x[n]\cos(\pi n)$ can be equivalently written as $y_d[n] = x[n](-1)^n = x[n]e^{j\pi n}$. Using the shifting property, we see that $Y_d(\Omega) = X(\Omega - \pi)$.

Solution 9.3-4

From pair 7 of Table 9.1, we know that

$$Y(\Omega) = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega} \iff u[n] - u[n-5] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] = y[n].$$

Furthermore, using pair 1 of Table 9.1, we know that

$$X(\Omega) = \sum_{k=0}^{4} a_k e^{-jk\Omega} \iff a_0\delta[n] + a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4] = x[n].$$

Both $x[n]$ and $y[n]$ are zero outside the range $0 \leq n \leq 4$, and $y[n]$ is unity over the range $0 \leq n \leq 4$. Thus, we see that

$$x[n]y[n] = x[n].$$

Using the frequency convolution property, the DTFT of this expression yields the desired result of

$$\frac{1}{2\pi}Y(\Omega)\circledast X(\Omega) = X(\Omega) \qquad \text{or} \qquad Y(\Omega)\circledast X(\Omega) = 2\pi X(\Omega),$$

where $Y(\Omega) = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)} e^{-j2\Omega}$ and $X(\Omega) = \sum_{n=0}^{4} c_n e^{-jn\Omega}$.

Solution 9.3-5

We determine the DTFTs of the each of the following signals using only pair 2 from Table 9.1 and properties of the DTFT. In each case, we assume that $|a| < 1$ and $\Omega_0 < \pi$.

(a) Applying the modulation property of Eq. (9.33) to pair 2 from Table 9.1, the DTFT of $x_a[n] = a^n \cos(\Omega_0 n) u[n]$ is

$$X_a(\Omega) = \frac{1}{2} \left[\frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)} - a} + \frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)} - a} \right]$$

$$= \frac{1}{2} \left(\frac{e^{j(\Omega-\Omega_0)} \left(e^{j(\Omega+\Omega_0)} - a\right) + e^{j(\Omega+\Omega_0)} \left(e^{j(\Omega-\Omega_0)} - a\right)}{e^{j2\Omega} - ae^{j\Omega} \left(e^{-j\Omega_0} + e^{j\Omega_0}\right) + a^2} \right)$$

$$= \frac{1}{2} \left(\frac{2e^{j2\Omega} - ae^{j(\Omega-\Omega_0)} - ae^{j(\Omega+\Omega_0)}}{e^{j2\Omega} - 2ae^{j\Omega} \cos(\Omega_0) + a^2} \right)$$

$$= \frac{e^{j\Omega} \left(e^{j\Omega} - a\cos(\Omega_0)\right)}{e^{j2\Omega} - 2ae^{j\Omega} \cos(\Omega_0) + a^2}.$$

(b) Applying the differentiation property of Eq. (9.30) to pair 2 from Table 9.1 yields

$$na^n u[n] \iff j \left(j\frac{e^{j\Omega}}{e^{j\Omega} - a} - je^{j\Omega}\frac{e^{j\Omega}}{(e^{j\Omega} - a)^2} \right) = \frac{ae^{j\Omega}}{(e^{j\Omega} - a)^2}.$$

Next, we apply the differentiation property of Eq. (9.30) to this result to obtain the DTFT of $x_b[n] = n^2 a^n u[n]$ as

$$X_b(\Omega) = j \left(j\frac{ae^{j\Omega}}{(e^{j\Omega} - a)^2} - 2je^{j\Omega}\frac{ae^{j\Omega}}{(e^{j\Omega} - a)^3} \right)$$

$$= \frac{ae^{j\Omega}(e^{j\Omega} + a)}{(e^{j\Omega} - a)^3}.$$

(c) To begin, we note that we can represent the signal $x_c[n]$ as

$$x_c[n] = (n-k)a^{2n}\, u[n-m] = a^{2m}(n-m)a^{2(n-m)}u[n-m] + a^{2m}(m-k)a^{2(n-m)}u[n-m].$$

Working to transform the first term, we apply the differentiation property of Eq. (9.30) to pair 2 from Table 9.1 to obtain

$$na^{2n}u[n] \Longleftrightarrow j\left(j\frac{e^{j\Omega}}{e^{j\Omega}-a^2} - je^{j\Omega}\frac{e^{j\Omega}}{(e^{j\Omega}-a^2)^2}\right) = \frac{a^2 e^{j\Omega}}{(e^{j\Omega}-a^2)^2}.$$

Next, we apply the shifting property of Eq. (9.31) to this result and also pair 2 from Table 9.1 to obtain

$$X_c(\Omega) = a^{2m}e^{-jm\Omega}\frac{a^2 e^{j\Omega}}{(e^{j\Omega}-a^2)^2} + a^{2m}(m-k)e^{-jm\Omega}\frac{e^{j\Omega}}{e^{j\Omega}-a^2}$$

$$= \frac{a^{2m}e^{j\Omega(1-m)}}{e^{j\Omega}-a^2}\left(\frac{a^2}{e^{j\Omega}-a^2} + m - k\right).$$

Solution 9.3-6

In this problem, we derive pairs 11, 12, 13, 14, 15 and 16 in Table 9.1 using only pair 10 and properties of the DTFT.

(a) To derive pair 11, we first note that

$$1 = u[n] + u[-(n+1)].$$

From pair 10, we know that

$$u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k).$$

Using the time-reversal and time-shifting properties, we know that

$$\text{DTFT}\{u[-(n+1)]\} = e^{j\Omega}\left(\frac{e^{-j\Omega}}{e^{-j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)\right)$$

$$= \frac{1}{e^{-j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}e^{j\Omega}\delta(\Omega-2\pi k)$$

$$= -\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k).$$

The last step follows since $e^{j\Omega}\delta(\Omega-2\pi k) = \delta(\Omega-2\pi k)$. Using these results, we see that

$$\text{DTFT}\{1\} = \left(\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)\right) - \left(\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)\right)$$

$$= 2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k).$$

(b) To derive pair 12, we simply apply the frequency-shifting property of Eq. (9.32) to the result of part (a). Thus,

$$e^{j\Omega_0 n} \Longleftrightarrow 2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k).$$

(c) To derive pair 13, we first note that

$$\cos(\Omega_0 n) = \frac{e^{j\Omega_0 n}}{2} + \frac{e^{-j\Omega_0 n}}{2}.$$

As done in part (b), we next apply the frequency-shifting property of Eq. (9.32) to the result of part (a) to obtain

$$\text{DTFT}\{\cos(\Omega_0 n)\} = \left(\frac{2\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) \right) + \left(\frac{2\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k) \right)$$

$$= \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k).$$

(d) To derive pair 14, we first note that

$$\sin(\Omega_0 n) = \frac{e^{j\Omega_0 n}}{2j} - \frac{e^{-j\Omega_0 n}}{2j}.$$

As done in part (b), we next apply the frequency-shifting property of Eq. (9.32) to the result of part (a) to obtain

$$\text{DTFT}\{\sin(\Omega_0 n)\} = \left(\frac{2\pi}{2j} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) \right) - \left(\frac{2\pi}{2j} \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k) \right)$$

$$= \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)$$

$$= j\pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k).$$

(e) To derive pair 15, we first note that

$$\cos(\Omega_0 n) u[n] = \frac{e^{j\Omega_0 n}}{2} u[n] + \frac{e^{-j\Omega_0 n}}{2} u[n].$$

We next apply the frequency-shifting property of Eq. (9.32) to pair 10 to obtain

$$\text{DTFT}\{\cos(\Omega_0 n) u[n]\} = \frac{1}{2} \left(\frac{e^{j(\Omega - \Omega_0)}}{e^{j(\Omega - \Omega_0)} - 1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) \right)$$

$$+ \frac{1}{2} \left(\frac{e^{j(\Omega + \Omega_0)}}{e^{j(\Omega + \Omega_0)} - 1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k) \right)$$

$$= \frac{\frac{1}{2} \left(e^{j2\Omega} - e^{j(\Omega - \Omega_0)} + e^{j2\Omega} - e^{j(\Omega + \Omega_0)} \right)}{e^{j2\Omega} - 2\cos(\Omega_0) e^{j\Omega} + 1}$$

$$+ \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)$$

$$= \frac{e^{j2\Omega} - e^{j\Omega} \cos(\Omega_0)}{e^{j2\Omega} - 2\cos(\Omega_0) e^{j\Omega} + 1} + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k).$$

(f) To derive pair 16, we first note that

$$\sin(\Omega_0 n)u[n] = \frac{e^{j\Omega_0 n}}{2j}u[n] - \frac{e^{-j\Omega_0 n}}{2j}u[n].$$

We next apply the frequency-shifting property of Eq. (9.32) to pair 10 to obtain

$$\begin{aligned}
\text{DTFT}\{\sin(\Omega_0 n)u[n]\} &= \frac{1}{2j}\left(\frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)\right)\\
&\quad -\frac{1}{2j}\left(\frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega+\Omega_0-2\pi k)\right)\\
&= \frac{\frac{1}{2j}\left(e^{j2\Omega} - e^{j(\Omega-\Omega_0)} - e^{j2\Omega} + e^{j(\Omega+\Omega_0)}\right)}{e^{j2\Omega} - 2\cos(\Omega_0)e^{j\Omega} + 1}\\
&\quad + \frac{\pi}{2j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k) - \delta(\Omega+\Omega_0-2\pi k)\\
&= \frac{e^{j\Omega}\sin(\Omega_0)}{e^{j2\Omega} - 2\cos(\Omega_0)e^{j\Omega} + 1} + \frac{\pi}{2j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k) - \delta(\Omega+\Omega_0-2\pi k).
\end{aligned}$$

Solution 9.3-7

From the time-shifting property of Eq. (9.31), we know that

$$x[n+k] \Longleftrightarrow e^{j\Omega k}X(\Omega) \qquad \text{and} \qquad x[n-k] \Longleftrightarrow e^{-j\Omega k}X(\Omega).$$

Thus,

$$x[n+k] + x[n-k] \Longleftrightarrow e^{j\Omega k}X(\Omega) + e^{-j\Omega k}X(\Omega) = 2X(\Omega)\left(\frac{e^{j\Omega k} + e^{-j\Omega k}}{2}\right).$$

Simplifying using Euler's formula, we obtain the desired result of

$$x[n+k] + x[n-k] \Longleftrightarrow 2X(\Omega)\cos(k\Omega).$$

(a) To determine the DTFT of signal $y_a[n]$, let us start by defining a signal $x[n] = u[n-2]-u[n-3]$. From pair 7 of Table 9.1 and the time-shifting property, we know that

$$X(\Omega) = \left(\frac{\sin(5\Omega/2)}{\sin(\Omega/2)}e^{-j2\Omega}\right)e^{j2\Omega} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)}.$$

Because $y_a[n] = x[n+4] + x[n-4]$, we know from our previous derivation that

$$Y_a(\Omega) = 2X(\Omega)\cos(4\Omega) = 2\frac{\sin(5\Omega/2)}{\sin(\Omega/2)}\cos(4\Omega).$$

(b) To determine the DTFT of signal $y_b[n]$, let us start by defining a signal $z[n] = u[n] - u[n-4]$. From pair 7 of Table 9.1, we know that

$$Z(\Omega) = \frac{\sin(2\Omega)}{\sin(\Omega/2)}e^{-j3\Omega/2}.$$

Next, we define $x[n] = z[n] * z[-n]$, which is a triangle function described vector-style as

$$x[n] = z[n] * z[-n] = [1,\, 2,\, 3,\, \overset{\underset{\downarrow}{n=0}}{4},\, 3,\, 2,\, 1].$$

Using the convolution property of Eq. (9.34) and the reversal property of Eq. (9.29), we see that

$$X(\Omega) = Z(\Omega)Z(-\Omega) = \frac{\sin^2(2\Omega)}{\sin^2(\Omega/2)}.$$

Because $y_{\mathrm{b}}[n] = x[n+8] + x[n-8]$, we know from our previous derivation that

$$Y_{\mathrm{b}}(\Omega) = 2X(\Omega)\cos(8\Omega) = 2\frac{\sin^2(2\Omega)}{\sin^2(\Omega/2)}\cos(8\Omega).$$

Solution 9.3-8

From the time-shifting property of Eq. (9.31), we know that

$$x[n+k] \Longleftrightarrow e^{j\Omega k}X(\Omega) \qquad \text{and} \qquad x[n-k] \Longleftrightarrow e^{-j\Omega k}X(\Omega).$$

Thus,

$$x[n+k] - x[n-k] \Longleftrightarrow e^{j\Omega k}X(\Omega) - e^{-j\Omega k}X(\Omega) = 2jX(\Omega)\left(\frac{e^{j\Omega k} - e^{-j\Omega k}}{2j}\right).$$

Simplifying using Euler's formula, we obtain the desired result of

$$x[n+k] - x[n-k] \Longleftrightarrow 2jX(\Omega)\sin(k\Omega).$$

To determine the DTFT of signal $y[n]$, let us start by defining a signal $x[n] = u[n-2] - u[n-3]$. From pair 7 of Table **??** and the time-shifting property, we know that

$$X(\Omega) = \left(\frac{\sin(5\Omega/2)}{\sin(\Omega/2)}e^{-j2\Omega}\right)e^{j2\Omega} = \frac{\sin(5\Omega/2)}{\sin(\Omega/2)}.$$

Because $y[n] = x[n+6] - x[n-6]$, we know from our previous derivation that

$$Y(\Omega) = 2jX(\Omega)\sin(6\Omega) = 2j\frac{\sin(5\Omega/2)}{\sin(\Omega/2)}\sin(6\Omega).$$

Solution 9.3-9

To begin, notice that we can express $y[n]$ in the more convenient form of

$$y[n] = \left(\frac{1}{2} + \frac{1}{2}(-1)^n\right)x[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{j\pi n}x[n].$$

Using the frequency-shifting property of Eq. (9.32), the spectrum of $y[n]$ is easily computed as

$$Y(\Omega) = \frac{1}{2}X(\Omega) + \frac{1}{2}X(\Omega - \pi).$$

Since the problem restricts $X(\Omega)$ to a bandwidth of $\pi/2$ rad/sample, the spectrum $Y(\Omega)$ will contain two undistorted (nonoverlapping) copies of $X(\Omega)$.

To illustrate, consider the fundamental band spectrum

$$X(\Omega) = \begin{cases} |2\Omega/\pi| & -\pi/2 \le \Omega \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

Figure S9.3-9 shows $X(\Omega)$ as well as $Y(\Omega)$. Notice that $Y(\Omega)$ contains two copies of $X(\Omega)$, each scaled by $\frac{1}{2}$. The first copy is centered at $\Omega = 0$, and the second copy is centered at $\Omega = \pi$.
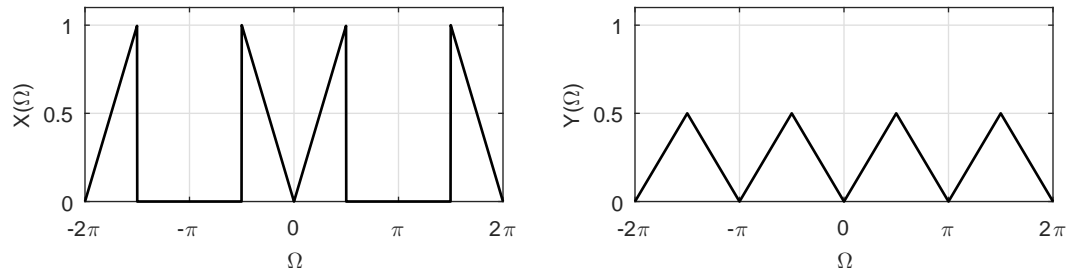
Figure S9.3-9

Solution 9.3-10

To begin, notice that we can express $y[n]$ in the more convenient form of

$$y[n] = \left(\frac{1}{2} - \frac{1}{2}(-1)^n\right)x[n] = \frac{1}{2}x[n] - \frac{1}{2}e^{j\pi n}x[n].$$

Using the frequency-shifting property of Eq. (9.32), the spectrum of $y[n]$ is easily computed as

$$Y(\Omega) = \frac{1}{2}X(\Omega) - \frac{1}{2}X(\Omega - \pi).$$

Since the problem restricts $X(\Omega)$ to a bandwidth of $\pi/2$ rad/sample, the spectrum $Y(\Omega)$ will contain two undistorted (nonoverlapping) copies of $X(\Omega)$.

To illustrate, consider the fundamental band spectrum

$$X(\Omega) = \begin{cases} |2\Omega/\pi| & -\pi/2 \le \Omega \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

Figure S9.3-10 shows $X(\Omega)$ as well as $Y(\Omega)$. Notice that $Y(\Omega)$ contains two copies of $X(\Omega)$, each scaled by $\frac{1}{2}$. The first copy is centered at $\Omega = 0$, and the second (negated) copy is centered at $\Omega = \pi$.
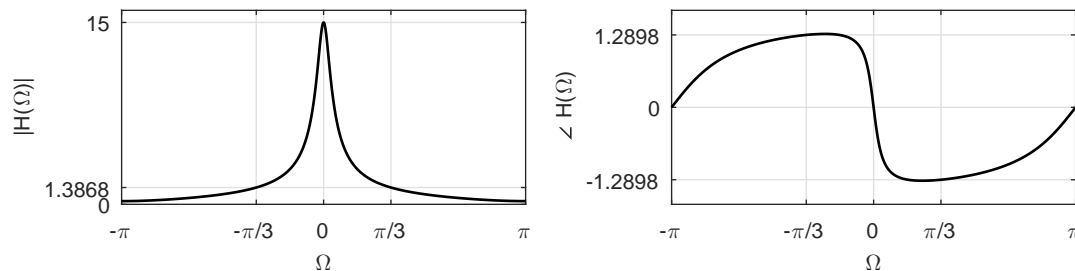
Figure S9.3-10

Solution 9.3-11

To begin, let us define

$$W(\Omega) = e^{j\Omega}/(e^{j\Omega} - \gamma).$$

Thus,

$$X(\Omega) = e^{2j\Omega}/(e^{j\Omega} - \gamma)^2 = W^2(\Omega).$$

Using pair 2 in Table 9.1 and the time convolution property of the DTFT, the inverse DTFT of

$X(\Omega)$ is

$$x[n] = \gamma^n u[n] * \gamma^n u[n]$$

$$= \sum_{m=0}^{n} \gamma^m \gamma^{n-m}$$

$$= \gamma^n \sum_{m=0}^{n} 1$$

$$= (n+1)\gamma^n u[n].$$

Solution 9.3-12

In this problem, we derive pairs 2, 3, 4, 5, 6, and 7 in Table 9.1 using only pair 1 and properties of the DTFT.

(a) To derive pair 2, we notice that

$$\gamma^n u[n] = \delta[n] + \gamma\delta[n-1] + \gamma^2\delta[n-2] + \cdots .$$

Therefore,

$$\mathrm{DTFT}\{\gamma^n u[n]\} = 1 + \gamma e^{-j\Omega} + \gamma^2 e^{-j2\Omega} + \gamma^3 e^{-j3\Omega} + \cdots$$

$$= \sum_{k=0}^{\infty} \left(\gamma e^{-j\Omega}\right)^k = \frac{1}{1 - \gamma e^{-j\Omega}}$$

$$= \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}, \qquad |\gamma| < 1.$$

We require that $|\gamma| < 1$ to ensure convergence of the sum as $k \to \infty$.

(b) To derive pair 3, we apply the time-reversal property to the result of part (a) to obtain

$$\lambda^{-n} u[-n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega} - \lambda}, \qquad |\lambda| < 1.$$

Using this result and pair 1, we see that

$$\lambda^{-n} u[-(n+1)] = \lambda^{-n} u[-n] - \delta[n] \Longleftrightarrow \frac{-e^{-j\Omega}}{e^{-j\Omega} - \lambda} + 1 = \frac{-\lambda}{e^{-j\Omega} - \lambda}, \qquad |\lambda| < 1.$$

Letting $\lambda = 1/\gamma$, we obtain

$$-\gamma^n u[-(n+1)] \Longleftrightarrow \frac{-1/\gamma}{e^{-j\Omega} - 1/\gamma} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}, \qquad |\gamma| > 1.$$

(c) To derive pair 4, we first note that

$$\gamma^{|n|} = \gamma^n u[n] + \left(\frac{1}{\gamma}\right)^n u[-(n+1)].$$

Using the results from parts (a) and (b), we obtain

$$\mathrm{DTFT}\{\gamma^{|n|}\} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} - \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{\gamma}}$$

$$= \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} + \frac{\gamma}{e^{-j\Omega} - \gamma}$$

$$= \frac{1 - \gamma^2}{1 - 2\gamma\cos(\Omega) + \gamma^2}, \qquad |\gamma| < 1.$$

(d) To derive pair 5, we apply the differentiation property of Eq. (**??**) to the result of part (a) to obtain

$$\text{DTFT}\{n\gamma^n u[n]\} = j\frac{d}{d\Omega}\left(\frac{e^{j\Omega}}{e^{j\Omega} - \gamma}\right)$$

$$= j\left(\frac{je^{j\Omega}}{e^{j\Omega} - \gamma} - \frac{je^{j2\Omega}}{(e^{j\Omega} - \gamma)^2}\right)$$

$$= \frac{e^{j2\Omega} - e^{j\Omega}(e^{j\Omega} - \gamma)}{(e^{j\Omega} - \gamma)^2}$$

$$= \frac{\gamma e^{j\Omega}}{(e^{j\Omega} - \gamma)^2}, \qquad |\gamma| < 1.$$

(e) To derive pair 6, we first note that

$$\gamma^n \cos(\Omega_0 n + \theta)u[n] = \frac{e^{j\theta}}{2}\left(\gamma e^{j\Omega_0}\right)^n u[n] + \frac{e^{-j\theta}}{2}\left(\gamma e^{-j\Omega_0}\right)^n u[n].$$

Using the result of part (a), we therefore obtain

$$\text{DTFT}\{\gamma^n \cos(\Omega_0 n + \theta)u[n]\} = \frac{e^{j\theta}}{2}\left(\frac{e^{j\Omega}}{e^{j\Omega} - \gamma e^{j\Omega_0}}\right) + \frac{e^{-j\theta}}{2}\left(\frac{e^{j\Omega}}{e^{j\Omega} - \gamma e^{-j\Omega_0}}\right)$$

$$= \frac{\frac{e^{j\theta}}{2}e^{j\Omega}\left(e^{j\Omega} - \gamma e^{-j\Omega_0}\right) + \frac{e^{-j\theta}}{2}e^{j\Omega}\left(e^{j\Omega} - \gamma e^{j\Omega_0}\right)}{e^{j2\Omega} - 2\gamma\cos(\Omega_0)e^{j\Omega} + \gamma^2}$$

$$= \frac{e^{j\Omega}\left[e^{j\Omega}\cos(\theta) - \gamma\cos(\Omega_0 - \theta)\right]}{e^{j2\Omega} - 2\gamma\cos(\Omega_0)e^{j\Omega} + \gamma^2}, \qquad |\gamma| < 1.$$

(f) To derive pair 7, we first note that

$$u[n] - u[n - M] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots + \delta[n - M + 1] = \sum_{k=0}^{M-1}\delta[n-k].$$

Using pair 1, we obtain

$$\text{DTFT}\{u[n] - u[n-M]\} = \sum_{k=0}^{M-1}e^{-jk\Omega} = \sum_{k=0}^{M-1}\left(e^{-j\Omega}\right)^k = \frac{1 - e^{-jM\Omega}}{1 - e^{-j\Omega}}$$

$$= \frac{e^{-jM\Omega/2}\left(e^{jM\Omega/2} - e^{-jM\Omega/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)}$$

$$= \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}e^{-j\Omega(M-1)/2}.$$

Solution 9.3-13

Throughout this solution, we assume that $|\Omega_0| < \pi$. We solve this problem in the fundamental band and then periodically extend the result. Using information from the problem statement, let us first define the (fundamental band) transform pair

$$x[n] = e^{j(\Omega_0/2)n} \iff 2\pi\delta(\Omega - \frac{\Omega_0}{2}) = X(\Omega).$$

Thus,

$$x^2[n] = \left(e^{j\frac{\Omega_0 n}{2}}\right)\left(e^{j\frac{\Omega_0 n}{2}}\right) = e^{j\Omega_0 n}.$$

To find the DTFT of $x^2[n]$, we use the frequency convolution property whereby we convolve $2\pi\delta(\Omega - \frac{\Omega_0}{2})$ with itself, multiply by $1/2\pi$, and then extend the result periodically. Over the fundamental band, we therefore have

$$
\begin{aligned}
\text{DTFT}\{x^2[n]\} &= \frac{1}{2\pi}\left[2\pi\delta\left(\Omega - \frac{\Omega_0}{2}\right) * 2\pi\delta\left(\Omega - \frac{\Omega_0}{2}\right)\right] \\
&= 2\pi\int_{-\infty}^{\infty}\delta\left(\lambda - \frac{\Omega_0}{2}\right)\delta\left(\Omega - \lambda + \frac{\Omega_0}{2}\right)d\lambda \\
&= 2\pi\delta\left(\Omega - \frac{\Omega_0}{2} - \frac{\Omega_0}{2}\right) \\
&= 2\pi\delta\left(\Omega - \Omega_0\right).
\end{aligned}
$$

Periodically extending the result, we obtain

$$
X(\Omega) = 2\pi\sum_{k-\infty}^{\infty}\delta\left(\Omega - \Omega_0 - 2\pi k\right).
$$

This result matches pair 12 of Table 9.1, as expected.

Solution 9.3-14

(a) This case constrains $\Omega_c < \pi$. Let

$$
x[n] = \text{sinc}(\Omega_c n).
$$

From pair 8 of Table 9.1, we have

$$
X(\Omega) = \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega}{2\Omega_c}\right), \qquad |\Omega| \leq \pi.
$$

From the definition of the DTFT, we know that

$$
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}.
$$

Hence,

$$
X(0) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} \text{sinc}(\Omega_c n) = \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega}{2\Omega_c}\right)\bigg|_{\Omega=0} = \frac{\pi}{\Omega_c}.
$$

This proves the desired result of

$$
\sum_{n=-\infty}^{\infty} \text{sinc}(\Omega_c n) = \frac{\pi}{\Omega_c}.
$$

(b) This case constrains $\Omega_c < \pi$. From part (a), we know that

$$
x[n] = \text{sinc}(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega}{2\Omega_c}\right) = X(\Omega), \qquad |\Omega| \leq \pi.
$$

Next, define

$$
y[n] = (-1)^n x[n] = (-1)^n \text{sinc}(\Omega_c n).
$$

Now, if $x[n] \Longleftrightarrow X(\Omega)$, then $(-1)^n x[n] \Longleftrightarrow X(\Omega - \pi)$. To prove this fact, we use the definition of the DTFT to write

$$\text{DTFT}\{(-1)^n x[n]\} = \sum_{n=-\infty}^{\infty} (-1)^n x[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{j\pi n} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega - \pi)n} = X(\Omega - \pi).$$

Thus,

$$y[n] = (-1)^n \text{sinc}(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c} \text{rect}\left(\frac{\Omega - \pi}{2\Omega_c}\right) = Y(\Omega), \qquad |\Omega| \le \pi.$$

From the definition of the DTFT, we know that

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\Omega n}.$$

Hence,

$$Y(0) = \sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} (-1)^n \text{sinc}(\Omega_c n).$$

Since $\text{rect}\left(\frac{\Omega - \pi}{2\Omega_c}\right)\Big|_{\Omega=0} = 0$ for $0 < \Omega_c < \pi$, we see that $Y(0) = 0$. This proves the desired result of

$$\sum_{n=-\infty}^{\infty} (-1)^n \text{sinc}(\Omega_c n) = 0.$$

(c) This case constrains $\Omega_c < \pi/2$. Let

$$x[n] = \text{sinc}^2(\Omega_c n).$$

From pair 9 of Table 9.1, we have

$$X(\Omega) = \frac{\pi}{\Omega_c} \Delta\left(\frac{\Omega}{4\Omega_c}\right), \qquad |\Omega| \le \pi.$$

From the definition of the DTFT, we know that

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}.$$

Hence,

$$X(0) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} \text{sinc}^2(\Omega_c n) = \frac{\pi}{\Omega_c} \Delta\left(\frac{\Omega}{4\Omega_c}\right)\Big|_{\Omega=0} = \frac{\pi}{\Omega_c}.$$

This proves the desired result of

$$\sum_{n=-\infty}^{\infty} \text{sinc}^2(\Omega_c n) = \frac{\pi}{\Omega_c}.$$

(d) This case constrains $\Omega_c < \pi/2$. From part (c), we know that

$$x[n] = \text{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c} \Delta\left(\frac{\Omega}{4\Omega_c}\right) = X(\Omega), \qquad |\Omega| \le \pi.$$

Next, define

$$y[n] = (-1)^n x[n] = (-1)^n \mathrm{sinc}^2(\Omega_c n).$$

From part (b), we know that if $x[n] \Longleftrightarrow X(\Omega)$, then $(-1)^n x[n] \Longleftrightarrow X(\Omega - \pi)$. Thus,

$$y[n] = (-1)^n \mathrm{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c} \Delta\left(\frac{\Omega - \pi}{4\Omega_c}\right) = Y(\Omega), \qquad |\Omega| \le \pi.$$

From the definition of the DTFT, we know that

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\Omega n}.$$

Hence,

$$Y(0) = \sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} (-1)^n \mathrm{sinc}^2(\Omega_c n).$$

Since $\Delta\left(\frac{\Omega - \pi}{4\Omega_c}\right)\Big|_{\Omega=0} = 0$ for $0 < \Omega_c < \pi/2$, we see that $Y(0) = 0$. This proves the desired result of

$$\sum_{n=-\infty}^{\infty} (-1)^n \mathrm{sinc}^2(\Omega_c n) = 0.$$

(e) By the definition of the inverse DTFT, we know that

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega.$$

Hence

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) d\Omega$$

or

$$\int_{-\pi}^{\pi} X(\Omega) d\Omega = 2\pi x[0].$$

Using pair 7 of Table 9.1, if we left-shift $x[n]$ by $\frac{M-1}{2}$ units (where M is odd), then we obtain

$$x[n] = u\left[n + \frac{M-1}{2}\right] - u\left[n - \frac{M+1}{2}\right] \Longleftrightarrow \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}.$$

Since $x[0] = 1$, applying this pair to $\int_{-\pi}^{\pi} X(\Omega) d\Omega = 2\pi x[0]$ produces the desired result of

$$\int_{-\pi}^{\pi} \frac{\sin(M\Omega/2)}{\sin(\Omega/2)} d\Omega = 2\pi.$$

(f) This case constrains $\Omega_c < \pi/2$. From part (c), we know that

$$x[n] = \mathrm{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c} \Delta\left(\frac{\Omega}{4\Omega_c}\right) = X(\Omega), \qquad |\Omega| \le \pi.$$

Application of Parseval's theorem [Eq. (9.36)] yields

$$
\sum_{n=-\infty}^{\infty} \left| \mathrm{sinc}^2(\Omega_c n) \right|^2 = \frac{2}{2\pi} \int_0^{2\Omega_c} \frac{\pi^2}{\Omega_c^2} \left| 1 - \frac{\Omega}{2\Omega_c} \right|^2 d\Omega
$$

$$
= \frac{\pi}{\Omega_c^2} \int_0^{2\Omega_c} \left(1 - \frac{\Omega}{\Omega_c} + \frac{\Omega^2}{4\Omega_c^2} \right) d\Omega
$$

$$
= \frac{\pi}{\Omega_c^2} \left(\Omega - \frac{\Omega^2}{2\Omega_c} + \frac{\Omega^3}{12\Omega_c^2} \Big|_{\Omega=0}^{2\Omega_c} \right)
$$

$$
= \frac{\pi}{\Omega_c^2} \left(2\Omega_c - 2\Omega_c + \frac{2}{3}\Omega_c \right)
$$

$$
= \frac{2\pi}{3\Omega_c}
$$

This proves the desired result of

$$
\sum_{n=-\infty}^{\infty} |\mathrm{sinc}(\Omega_c n)|^4 = \frac{2\pi}{3\Omega_c}.
$$

Solution 9.3-15

We know from Eq. (8.6) that the bandlimited interpolation of $x[n]$ is

$$
x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \, \mathrm{sinc}\left(\pi \frac{t - nT}{T} \right).
$$

The energy of $x_c(t)$ is computed as

$$
E_{x_c} = \int_{-\infty}^{\infty} |x_c(t)|^2 \, dt = \int_{-\infty}^{\infty} x_c(t) x_c^*(t) \, dt
$$

$$
= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x[n] \mathrm{sinc}\left(\pi \frac{t - nT}{T} \right) \right] \left[\sum_{m=-\infty}^{\infty} x[n]^*(mT) \mathrm{sinc}\left(\pi \frac{t - mT}{T} \right) \right] dt
$$

Applying a change of variable from t to t/T to the problem statement hint, we see that

$$
\int_{-\infty}^{\infty} \mathrm{sinc}(\pi \frac{t}{T} - \pi m) \mathrm{sinc}(\pi \frac{t}{T} - \pi n) \, dt = \begin{cases} 0 & m \neq n \\ T & m = n \end{cases}.
$$

Because of this property, all the $m \neq n$ cross-product terms vanish. Moreover when $m = n$, the integral is T. Hence, we obtain the desired result that

$$
E_{x_c} = T \sum_{n=-\infty}^{\infty} |x[n]|^2 = T E_x.
$$

Solution 9.4-1

Using pair 2 of Table 9.1

$$
X(\Omega) = \frac{1}{1 + 0.5e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} + 0.5}.
$$

Thus,

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{e^{j\Omega}\left(e^{j\Omega} + 0.32\right)}{\left(e^{j\Omega} + 0.5\right)\left(e^{j\Omega} + 0.8\right)\left(e^{j\Omega} + 0.2\right)}$$

$$= e^{j\Omega}\left(\frac{e^{j\Omega} + 0.32}{\left(e^{j\Omega} + 0.5\right)\left(e^{j\Omega} + 0.8\right)\left(e^{j\Omega} + 0.2\right)}\right)$$

$$= e^{j\Omega}\left(\frac{2}{e^{j\Omega} + 0.5} - \frac{8/3}{e^{j\Omega} + 0.8} + \frac{2/3}{e^{j\Omega} + 0.2}\right)$$

$$= 2\frac{e^{j\Omega}}{e^{j\Omega} + 0.5} - \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega} + 0.8} + \frac{2}{3}\frac{e^{j\Omega}}{e^{j\Omega} + 0.2}.$$

Inverting, we obtain

$$y[n] = \left[2\left(-0.5\right)^n - \frac{8}{3}\left(-0.8\right)^n + \frac{2}{3}\left(-0.2\right)^n\right]u[n].$$

Solution 9.4-2

Using pair 10 of Table 9.1, the DTFT of $x[n] = u[n]$ is

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega), \qquad |\Omega| \leq \pi.$$

From Eq. (9.37), the DTFT of the output $y[n]$ is

$$Y(\Omega) = X(\Omega)H(\Omega) = \left(\frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega)\right)\left(\frac{e^{j\Omega} + 0.32}{e^{j2\Omega} + e^{j\Omega} + 0.16}\right), \qquad |\Omega| \leq \pi.$$

Using the fact that $f(x)\delta(x) = f(0)\delta(x)$, we obtain

$$Y(\Omega) = \frac{1.32\pi}{2.16}\delta(\Omega) + e^{j\Omega}\left(\frac{e^{j\Omega} + 0.32}{(e^{j\Omega} - 1)(e^{j\Omega} + 0.2)(e^{j\Omega} + 0.8)}\right), \qquad |\Omega| \leq \pi.$$

To invert $Y(\Omega)$, we first use MATLAB to perform the needed partial fraction expansion.

```
>>  [r,p,k] = residue(poly(-0.32),poly([1,-0.2,-0.8]))
    r =  0.6111  -0.4444  -0.1667
    p =  1.0000  -0.8000  -0.2000
    k = []
```

Thus,

$$Y(\Omega) = \frac{11\pi}{18}\delta(\Omega) + \frac{11}{18}\left(\frac{e^{j\Omega}}{e^{j\Omega} - 1}\right) - \frac{4}{9}\left(\frac{e^{j\Omega}}{e^{j\Omega} + \frac{4}{5}}\right) - \frac{1}{6}\left(\frac{e^{j\Omega}}{e^{j\Omega} + \frac{1}{5}}\right), \qquad |\Omega| \leq \pi.$$

Using Table 9.1 to compute the IDTFT, the system output is given as

$$y[n] = \left(\frac{11}{18} - \frac{4}{9}\left(-\frac{4}{5}\right)^n - \frac{1}{6}\left(-\frac{1}{5}\right)^n\right)u[n].$$

Solution 9.4-3

Using pairs 2 and 3 of Table 9.1, the DTFT of $x[n] = (0.8)^n u[n] + 2(2)^n u[-n-1]$ is

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - \frac{4}{5}} - 2\frac{e^{j\Omega}}{e^{j\Omega} - 2}.$$

From Eq. (9.37), the DTFT of the output $y[n]$ is

$$Y(\Omega) = X(\Omega)H(\Omega) = \left(\frac{e^{j\Omega}}{e^{j\Omega} - \frac{4}{5}} - 2\frac{e^{j\Omega}}{e^{j\Omega} - 2}\right)\left(\frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}\right).$$

Thus,

$$\frac{Y(\Omega)}{e^{j\Omega}} = \frac{e^{j\Omega}}{(e^{j\Omega} - \frac{4}{5})(e^{j\Omega} - \frac{1}{2})} - 2\frac{e^{j\Omega}}{(e^{j\Omega} - 2)(e^{j\Omega} - \frac{1}{2})}.$$

To invert $Y(\Omega)$, we first use MATLAB to perform the needed partial fraction expansions.

```
>> [r,p,k] = residue(poly(0),poly([4/5,1/2]))
   r =  2.6667  -1.6667
   p =  0.8000   0.5000
   k = []
>> [r,p,k] = residue(-2*poly(0),poly([2,1/2]))
   r = -2.6667   0.6667
   p =  2.0000   0.5000
   k = []
```

Thus,

$$Y(\Omega) = \frac{\frac{8}{3}e^{j\Omega}}{e^{j\Omega} - \frac{4}{5}} + \frac{(-\frac{5}{3} + \frac{2}{3})e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}} + \frac{-\frac{8}{3}e^{j\Omega}}{e^{j\Omega} - 2}.$$

Using Table 9.1 to compute the IDTFT, the system output is given as

$$y[n] = \left(\frac{8}{3}\left(\frac{4}{5}\right)^n - \left(\frac{1}{2}\right)^n\right)u[n] + \frac{8}{3}\left(2\right)^n u[-n-1].$$

Solution 9.4-4

Taking the DTFT of the system's difference equation yields

$$Y(\Omega)\left(1 + \frac{1}{2}e^{-j\Omega}\right) = X(\Omega)\left(1 - \frac{9}{10}e^{-j\Omega}\right).$$

The system frequency response is therefore

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{e^{j\Omega} + \frac{1}{2}}{e^{j\Omega} - \frac{9}{10}}.$$

The corresponding magnitude and phase responses are shown in Fig. S9.4-4.
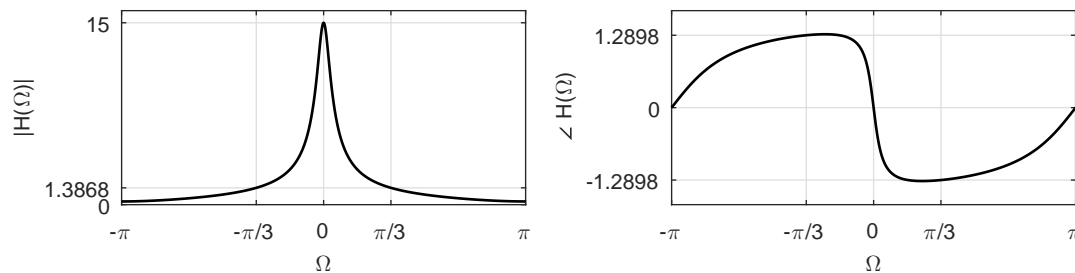
Figure S9.4-4

The sinusoidal input $x[n] = \cos(\frac{\pi n}{3} + 0.5)$ has frequency $\Omega_0 = \frac{\pi}{3}$. Since $H(\pi/3) = 1.3868e^{-j1.2898}$, the system output is

$$y[n] = 1.3868\cos(\frac{\pi n}{3} + 0.5 - 1.2898) = 1.3868\cos(\frac{\pi n}{3} - 0.7898).$$

Solution 9.4-5

Taking the DTFT of the system's difference equation yields

$$Y(\Omega)\left(1 - \frac{1}{2}e^{-j\Omega}\right) = X(\Omega)\left(1 + \frac{9}{10}e^{-j\Omega}\right).$$

The system frequency response is therefore

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{e^{j\Omega} - \frac{1}{2}}{e^{j\Omega} + \frac{9}{10}}.$$

The corresponding magnitude and phase responses are shown in Fig. S9.4-5.
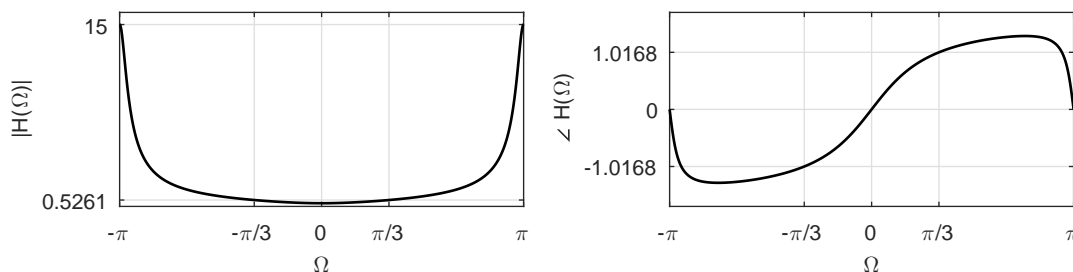
Figure S9.4-5

The sinusoidal input $x[n] = \cos(\frac{\pi n}{3} + 0.5)$ has frequency $\Omega_0 = \frac{\pi}{3}$. Since $H(\pi/3) = 0.5261e^{j1.0168}$, the system output is

$$y[n] = 0.5261 \cos(\frac{\pi n}{3} + 0.5 + 1.0168) = 0.5261 \cos(\frac{\pi n}{3} + 1.5168).$$

Solution 9.4-6

(a) The accumulator system of this problem has an input-output relationship given by

$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

Now, when $x[n] = \delta[n]$, the output is $h[n]$. Making these substitutions into the input-output equation yields the impulse response as

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n].$$

Using pair 10 of Table 9.1, the system's (fundamental-band) frequency response is

$$H(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega).$$

(b) Since the impulse response of the accumulator system is $h[n] = u[n]$, we can use the system frequency response found in part (a) to determine the DTFT of $u[n]$ as

$$\text{DTFT}(u[n]) = \text{DTFT}(h[n]) = H(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega).$$

Solution 9.4-7

(a) The (noncausal) 7-point moving average difference equation is given as

$$y[n] = \frac{1}{7} \sum_{k=-3}^{3} x[n-k].$$

Taking the DTFT of this equation yields

$$Y(\Omega) = \frac{1}{7} \sum_{k=-3}^{3} X(\Omega)e^{-j\Omega k}.$$

The frequency response is therefore

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{7} \sum_{k=-3}^{3} e^{-j\Omega k} = \frac{1}{7} \left(\frac{e^{j\Omega 3} - e^{-j\Omega 4}}{1 - e^{-j\Omega}} \right) = \frac{1}{7} \left(\frac{\sin(7\Omega/2)}{\sin(\Omega/2)} \right).$$

The corresponding magnitude and phase responses, plotted using MATLAB, are shown in Fig. S9.4-7a.

```
>>  Om = linspace(-pi,pi,1000);   H = @(Om) 1/7*sin(7*Om/2)./sin(Om/2);
>>  subplot(121); plot(Om,abs(H(Om))); axis([-pi pi 0 1.1]);
>>  xlabel('\Omega'); ylabel('|H(\Omega)|'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'xticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'});
>>  subplot(122); plot(Om,angle(H(Om))); axis([-pi pi -1.1*pi 1.1*pi]);
>>  xlabel('\Omega'); ylabel('\angle H(\Omega)'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'xticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'});
>>  set(gca,'ytick',-pi:pi:pi,'yticklabel',{'-\pi','0','\pi'});
```

Despite appearances, the phase response of Fig. S9.4-7a does satisfies the requirement of odd symmetry since the angle π is equivalent to $-\pi$.
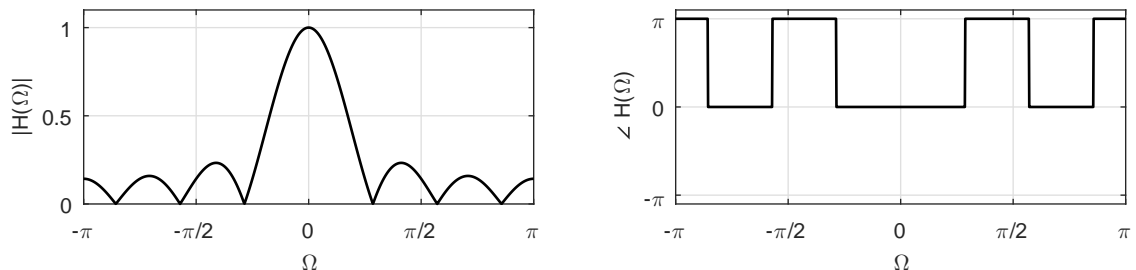
Figure S9.4-7a

(b) To make the system causal, we adjust the difference equations so that the output no longer depends on future inputs. This is accomplished by delaying the input terms by 3 to obtain

$$y[n] = \frac{1}{7} \sum_{k=0}^{6} x[n-k].$$

Taking the DTFT of this equation yields

$$Y(\Omega) = \frac{1}{7} \sum_{k=0}^{6} X(\Omega)e^{-j\Omega k}.$$

The frequency response of this causal realization is therefore

$$H_{\text{causal}}(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{7} \sum_{k=0}^{6} e^{-j\Omega k} = \frac{1}{7} \left(\frac{1 - e^{-j\Omega 7}}{1 - e^{-j\Omega}} \right) = \frac{1}{7} \left(\frac{\sin(7\Omega/2)}{\sin(\Omega/2)} \right) e^{-j\Omega 3} = H(\Omega)e^{-j\Omega 3}.$$

The corresponding magnitude and phase responses, plotted using MATLAB, are shown in Fig. S9.4-7b.

```
>>  Om = linspace(-pi,pi,1000);
>>  Hcausal = @(Om) 1/7*sin(7*Om/2)./sin(Om/2).*exp(-3j*Om);
>>  subplot(121); plot(Om,abs(Hcausal(Om))); axis([-pi pi 0 1.1]);
>>  xlabel('\Omega'); ylabel('|H_{causal}(\Omega)|'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'xticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'});
>>  subplot(122); plot(Om,angle(Hcausal(Om))); axis([-pi pi -1.1*pi 1.1*pi]);
>>  xlabel('\Omega'); ylabel('\angle H_{causal}(\Omega)'); grid on
>>  set(gca,'xtick',-pi:pi/2:pi,'xticklabel',{'-\pi','-\pi/2','0','\pi/2','\pi'});
>>  set(gca,'ytick',-pi:pi:pi,'yticklabel',{'-\pi','0','\pi'});
```

As can be seen by comparing the frequency responses of the two systems, the causal system differs from the noncausal system only in the addition of a linear phase term. The magnitude responses of the two systems are identical.
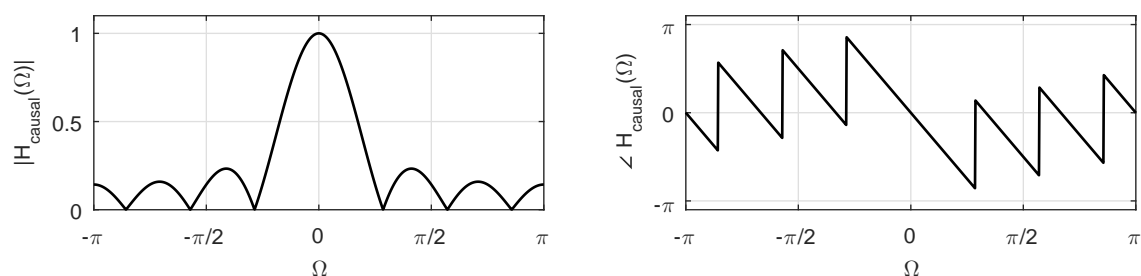
Figure S9.4-7b

Solution 9.4-8

Over the fundamental band, the system frequency response is $H(\Omega) = \mathrm{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega}$. This corresponds to a distortionless system over the passband $|\Omega| < \pi/2$.

(a) For $x_{\mathrm{a}}[n] = \mathrm{sinc}(\pi n/2)$, we know from pair 8 of Table 9.1 that

$$X_{\mathrm{a}}(\Omega) = 2\mathrm{rect}\left(\frac{\Omega}{\pi}\right).$$

Thus,

$$Y_{\mathrm{a}}(\Omega) = X_{\mathrm{a}}(\Omega)H(\Omega) = 2\mathrm{rect}\left(\frac{\Omega}{\pi}\right)\mathrm{rect}\left(\frac{\Omega}{\pi}\right)e^{-j2\Omega} = X_{\mathrm{a}}(\Omega)e^{-j2\Omega}.$$

Consequently, the system output is

$$y_{\mathrm{a}}[n] = x_{\mathrm{a}}[n-2] = \mathrm{sinc}\left(\pi\frac{n-2}{2}\right).$$

(b) For $x_{\mathrm{b}}[n] = \mathrm{sinc}(\pi n)$, we know from pair 8 of Table 9.1 that

$$X_{\mathrm{b}}(\Omega) = \mathrm{rect}\left(\frac{\Omega}{2\pi}\right).$$

Thus,

$$Y_{\mathrm{b}}(\Omega) = X_{\mathrm{b}}(\Omega)H(\Omega) = \mathrm{rect}\left(\frac{\Omega}{2\pi}\right)\mathrm{rect}\left(\frac{\Omega}{\pi}\right)e^{-j2\Omega} = \frac{1}{2}X_{\mathrm{a}}(\Omega)e^{-j2\Omega}.$$

Consequently, the system output is

$$y_{\mathrm{b}}[n] = \frac{1}{2}x_{\mathrm{a}}[n-2] = \frac{1}{2}\mathrm{sinc}\left(\pi\frac{n-2}{2}\right).$$

(c) For $x_c[n] = \text{sinc}^2(\pi n/4)$, we know from pair 9 of Table 9.1 that

$$X_c(\Omega) = 4\Delta\left(\frac{\Omega}{\pi}\right).$$

Thus,

$$Y_c(\Omega) = X_c(\Omega)H(\Omega) = 4\Delta\left(\frac{\Omega}{\pi}\right)\text{rect}\left(\frac{\Omega}{\pi}\right)e^{-j2\Omega} = X_c(\Omega)e^{-j2\Omega}.$$

Consequently, the system output is

$$y_c[n] = x_c[n-2] = \text{sinc}^2\left(\pi\frac{n-2}{4}\right).$$

Solution 9.4-9

(a) To begin, we notice that
$$(-1)^n x[n] = e^{j\pi n}x[n].$$

Using the frequency-shifting property of Eq. (9.32), we immediately obtain the desired result of

$$\text{DTFT}\{(-1)^n x[n]\} = X(\Omega - \pi).$$

Thus, the spectrum $X(\Omega - \pi)$ is just a π-shifted version of the spectrum $X(\Omega)$. Thus, the dc content of $X(\Omega)$ becomes the high-frequency content of $X(\Omega - \pi)$, and the high-frequency content of $X(\Omega)$ becomes the low-frequency content of $X(\Omega - \pi)$. With the roles of high and low frequencies reversed, it is reasonable to describe this as a spectral inversion system.

(b) The signals $x_1[n] = (0.8)^n u[n]$ and $x_2[n] = (-0.8)^n u[n]$ are shown in Figs. S9.4-9b. Since $x_2[n] = (-1)^n x_1[n]$, we can obtain $X_2(\Omega) = X_1(\Omega - \pi)$. Applying a π-shift to Figs. 9.5b and 9.5c, Fig. S9.4-9b shows the magnitude and phase spectra of $x_2[n]$.
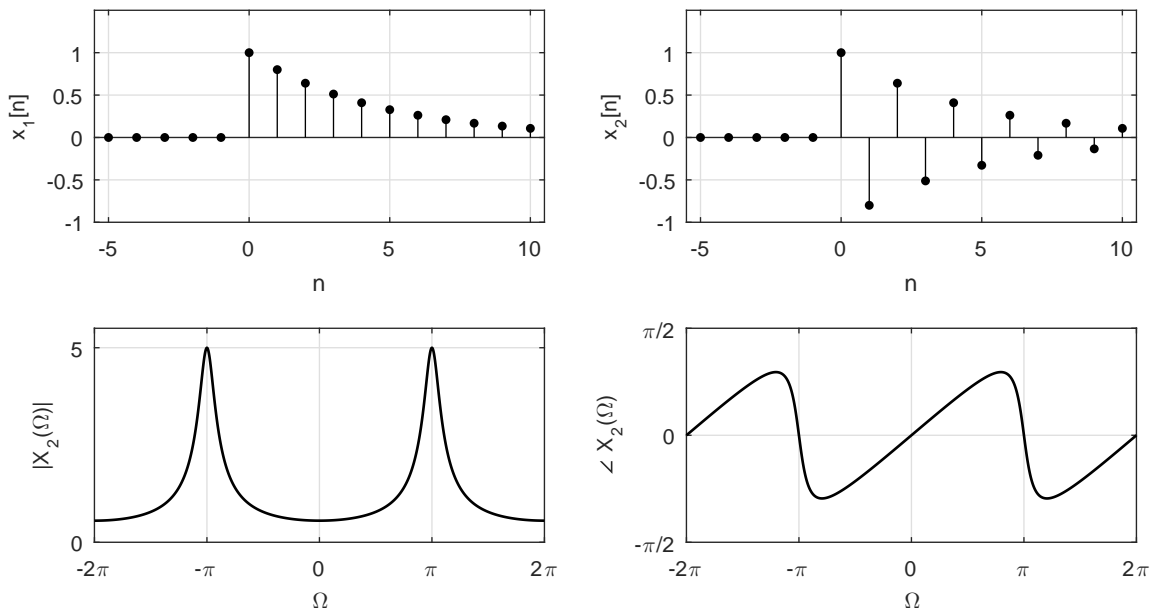
Figure S9.4-9b

(c) From pair 8 of Table 9.1, the impulse response $h[n]$ that corresponds to frequency response $H(\Omega) = \text{rect}\,(\Omega/2\Omega_c)$ is

$$h[n] = \frac{\Omega_c}{\pi}\text{sinc}\,(\Omega_c n)\,.$$

Since $h'[n] = (-1)^n h[n]$, we know that $H'(\Omega) = H(\Omega - \pi)$. Figure S9.4-9c shows $H'(\Omega)$ for $\Omega_c = \pi/3$. Clearly, the filter with impulse response $h'[n] = (-1)^n h[n]$ is a highpass filter.
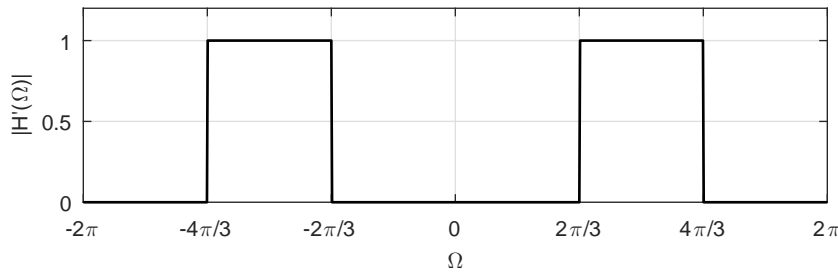
Figure S9.4-9c

Solution 9.4-10

Taking the DTFT of the backward difference system's difference equation yields

$$Y(\Omega) = \frac{1}{T}X(\Omega)\left(1 - e^{-j\Omega}\right).$$

The system frequency response is therefore

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{T}\left(1 - e^{-j\Omega}\right).$$

The corresponding magnitude and phase responses are shown in Fig. S9.4-10. The (magnitude) response of an ideal differentiator is shown dotted in Fig. S9.4-10. Clearly, this digital approximation to a differentiator works best for low frequencies (near $\Omega = 0$) and works least well at high frequencies (near $\Omega = \pm\pi$).
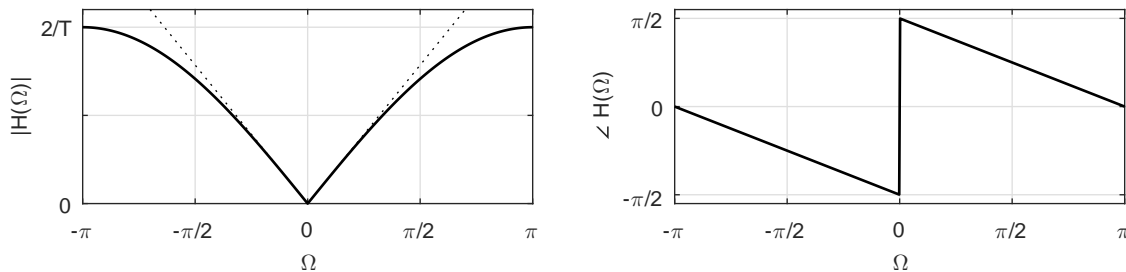
Figure S9.4-10

Solution 9.4-11

Figure S9.4-11 shows the system with additional signal path labels of $x_0[n]$ and $y_0[n]$.

To begin, we show that multiplying a signal by $(-1)^n$ simply shifts its spectrum by π. Since $(-1)^n x[n] = e^{j\pi n}x[n]$, the frequency-shifting property of Eq. (9.32) tells us that $\text{DTFT}\,\{(-1)^n x[n]\} = X(\Omega - \pi)$. Using this fact, we see that

$$x_0[n] = (-1)^n x[n] \Longleftrightarrow X_0(\Omega) = X(\Omega - \pi) \quad \text{and} \quad y[n] = (-1)^n y_0[n] \Longleftrightarrow Y(\Omega) = Y_0(\Omega - \pi).$$
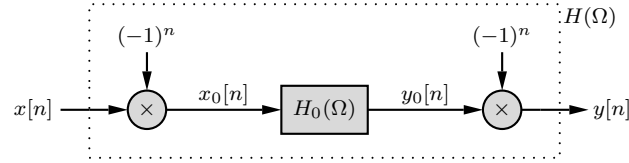
Figure S9.4-11

From Eq. (9.37), we know that the spectrum of $y_0[n]$ is

$$Y_0(\Omega) = X_0(\Omega)H_0(\Omega) = X(\Omega - \pi)H_0(\Omega).$$

Thus,

$$Y(\Omega) = Y_0(\Omega - \pi) = X(\Omega - 2\pi)H_0(\Omega - \pi) = X(\Omega)H_0(\Omega - \pi).$$

Since $Y(\Omega) = X(\Omega)H(\Omega)$, it follows that

$$H(\Omega) = H_0(\Omega - \pi) \qquad \text{and} \qquad h[n] = (-1)^n h_0[n].$$

Solution 9.4-12

(a) Setting $a_0 = 1$, system $\mathcal{S}_1$ can be specified by Eq. (3.16) as

$$y_1[n] + \sum_{i=1}^{N} a_i y_1[n - i] = \sum_{i=0}^{N} b_i x[n - i].$$

Taking the DTFT of this equation yields

$$Y_1(\Omega)\left[1 + \sum_{i=1}^{N} a_i e^{-ji\Omega}\right] = X(\Omega)\sum_{i=0}^{N} b_i e^{-ji\Omega}.$$

The frequency response of this system is therefore given by

$$H_1(\Omega) = \frac{Y_1(\Omega)}{X(\Omega)} = \frac{\sum_{i=0}^{N} b_i e^{-ji\Omega}}{1 + \sum_{i=1}^{N} a_i e^{-ji\Omega}}.$$

Next, coefficients a_i ($i = 1, 2, \ldots, N$) are replaced by coefficients $(-1)^i a_i$ and all coefficients b_i ($i = 0, 1, 2, \ldots, N$) are replaced by coefficients $(-1)^i b_i$ to produce a new system $\mathcal{S}_2$ with frequency response

$$\begin{aligned}
H_2(\Omega) &= \frac{\sum_{i=0}^{N}(-1)^i b_i e^{-ji\Omega}}{1 + \sum_{i=1}^{N}(-1)^i a_i e^{-ji\Omega}} \\
&= \frac{\sum_{i=0}^{N} e^{j\pi i} b_i e^{-ji\Omega}}{1 + \sum_{i=1}^{N} e^{j\pi i} a_i e^{-ji\Omega}} \\
&= \frac{\sum_{i=0}^{N} b_i e^{-ji(\Omega - \pi)}}{1 + \sum_{i=1}^{N} a_i e^{-ji(\Omega - \pi)}} \\
&= H_1(\Omega - \pi).
\end{aligned}$$

(b) As shown in part (a), the frequency response $H_2(\Omega)$ is just a π-shifted version of the frequency response $H_1(\Omega)$. Thus, the low frequency content of $H_1(\Omega)$ becomes the high-frequency content of $H_2(\Omega)$, and vice-versa. If $H_1(\Omega)$ is a lowpass filter, then $H_2(\Omega) = H_1(\Omega - \pi)$ is a highpass filter.

(c) The DTFT of the first system equation $y[n] - 0.8y[n-1] = x[n]$ is

$$Y(\Omega)\left[1 - 0.8e^{-j\Omega}\right] = X(\Omega).$$

The corresponding frequency response is

$$H_1(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - 0.8e^{-j\Omega}}.$$

The magnitude spectra for this system, shown in Fig. 9.5b, shows that this is a lowpass system.

The system equation $y[n] + 0.8y[n-1] = x[n]$ is the system $y[n] - 0.8y[n-1] = x[n]$ where coefficients a_i ($i = 1, 2, \ldots, N$) are replaced by coefficients $(-1)^i a_i$ and all coefficients b_i ($i = 0, 1, 2, \ldots, N$) are replaced by coefficients $(-1)^i b_i$. Using the results of part (a), the frequency response of this second system is

$$H_2(\Omega) = H_1(\Omega - \pi) = \frac{1}{1 - 0.8e^{-j(\Omega - \pi)}} = \frac{1}{1 + 0.8e^{-j\Omega}}.$$

Since $H_1(\Omega)$ is a lowpass system, $H_2(\Omega) = H(\Omega - \pi)$ will be a highpass system.

Solution 9.4-13

To facilitate our derivation, Fig. S9.4-13 shows the system with additional signal path labels of p_1, p_2, p_3, p_4, p_5, and p_6.
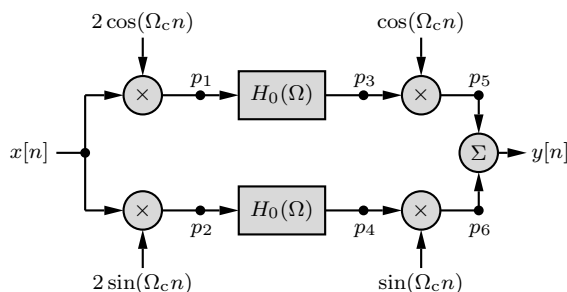
Figure S9.4-13

(a) Let us compute the response $h[n]$ to the unit impulse input $\delta[n]$. Because the system contains time-varying multipliers, however, we must test whether it is a time-variant or a time-invariant system. It is therefore appropriate to consider the system response to an input $\delta[n-k]$. This is an impulse at $n = k$. Using the fact that $x[n]\,\delta[n-k] = x[k]\delta[n-k]$, we can express the signals at the labeled points as follows:

$$\begin{aligned}
\text{point } p_1: &\quad 2\cos(\Omega_c k)\delta[n-k] \\
\text{point } p_2: &\quad 2\sin(\Omega_c k)\delta[n-k] \\
\text{point } p_3: &\quad 2\cos(\Omega_c k)h_0[n-k] \\
\text{point } p_4: &\quad 2\sin(\Omega_c k)h_0[n-k] \\
\text{point } p_5: &\quad 2\cos(\Omega_c k)\cos(\Omega_c n)h_0[n-k] \\
\text{point } p_6: &\quad 2\sin(\Omega_c k)\sin(\Omega_c n)h_0[n-k]
\end{aligned}$$

Thus, the system output is

$$\begin{aligned}
y[n] &= 2h_0[n-k]\left[\cos(\Omega_c k)\cos(\Omega_c n) + \sin(\Omega_c k)\sin(\Omega_c n)\right] \\
&= 2h_0[n-k]\cos(\Omega_c[n-k]).
\end{aligned}$$

Thus, the system response to the input $\delta[n-k]$ is $2h_0[n-k]\cos(\Omega_c[n-k])$. Clearly, the system is time-invariant with impulse response

$$h[n] = 2h_0[n]\cos(\Omega_c n).$$

The system is also linear, a fact that easily follows since the system components of H_0, multiplication, and additional are all linear operators.

(b) From the modulation property of Eq. (9.33), it follows that

$$H(\Omega) = H_0(\Omega - \Omega_c) + H_0(\Omega + \Omega_c).$$

If $H_0(\Omega) = \text{rect}(\Omega/2W)$, then

$$H(\omega) = \text{rect}\left(\frac{\Omega - \Omega_c}{2W}\right) + \text{rect}\left(\frac{\Omega + \Omega_c}{2W}\right).$$

As long as $\Omega_c + W \leq \pi$, $H(\omega)$ is an ideal bandpass response (see Fig. 9.15b) with passband bandwidth of $2W$ centered at Ω_c.

Solution 9.5-1

Referencing pair 10 of Table 7.1, the CTFT of $x_c(t) = \sin(\omega_0 t)$ is given as

$$X_c(\omega) = j\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right].$$

We are interested in finding the DTFT of

$$x[n] = x_c(nT) = \sin(\omega_0 nT) = \sin(\Omega_0 n).$$

Now, the DTFT is determined from the CTFT as [see Eq. (8.2) or (9.42)]

$$\begin{aligned}
X(\Omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\Omega - 2\pi k}{T}\right) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} j\pi\left[\delta\left(\frac{\Omega - 2\pi k}{T} + \omega_0\right) - \delta\left(\frac{\Omega - 2\pi k}{T} - \omega_0\right)\right] \\
&= j\pi\sum_{k=-\infty}^{\infty} \frac{1}{T}\left[\delta\left(\frac{\Omega + \Omega_0 - 2\pi k}{T}\right) - \delta\left(\frac{\Omega - \Omega_0 - 2\pi k}{T}\right)\right].
\end{aligned}$$

Noting that $\frac{1}{|a|}\delta\left(\frac{x}{a}\right) = \delta(x)$, this expression simplifies to

$$X(\Omega) = j\pi\sum_{k=-\infty}^{\infty} \delta\left(\Omega + \Omega_0 - 2\pi k\right) - \delta\left(\Omega - \Omega_0 - 2\pi k\right).$$

This result matches pair 14 of Table 9.1, as expected.

Solution 9.5-2

From the problem statement, we know that a CT signal $x_c(t)$, bandlimited to 25 kHz, is sampled at 50 kHz to produce

$$x[n] = \delta[n+4] - 2\delta[n+2] + \delta[n+1] - 3\delta[n] - \delta[n-1] - 2\delta[n-2] - \delta[n-4].$$

Taking the DTFT, we obtain

$$\begin{aligned}
X(\Omega) &= e^{j4\Omega} - 2e^{j2\Omega} + e^{j\Omega} - 3 - e^{-j\Omega} - 2e^{-j2\Omega} - e^{-j4\Omega} \\
&= -3 + 2j\sin(\Omega) - 4\cos(2\Omega) + 2j\sin(4\Omega).
\end{aligned}$$

Since $x(t)$ is sampled at the Nyquist rate ($f_s = 50$ kHz), no aliasing occurs and $X_c(\omega)$ can be recovered from $X(\Omega)$. Using Eq. (9.42) and the fact that $X_c(\omega) = T\overline{X}_c(\omega)$ and $\Omega = \omega T$, we obtain

$$X_c(\omega) = \begin{cases} TX(\omega T) & |\omega| \le \pi/T \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{50000}\left[-3 + 2j\sin\left(\frac{\omega}{50000}\right) - 4\cos\left(\frac{2\omega}{50000}\right) + 2j\sin\left(\frac{4\omega}{50000}\right)\right]\text{rect}\left(\frac{\omega}{2\pi 50000}\right).$$

As Fig. S9.5-2 shows, the resulting magnitude spectrum is bandlimited to 25 kHz, as required.
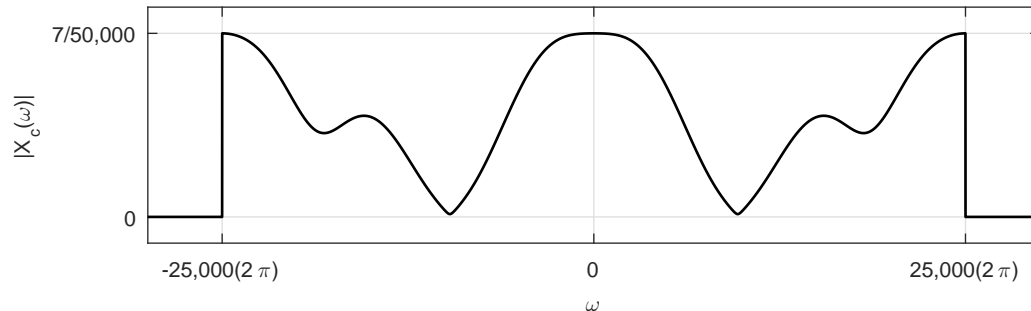
Figure S9.5-2

Solution 9.7-1

(a) The inverse DTFS is given by $x[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n}$. Like the DTFS, the IDTFS can be computed using a matrix based approach. First, define $W_{N_0} = e^{j\Omega_0}$, which is a constant for a given N_0. Substituting W_{N_0} into the IDTFS equation yields $x[n] = \sum_{n=0}^{N_0-1} \mathcal{D}_r W_{N_0}^{nr}$. An inner product of two vectors computes $x[n]$.

$$x[n] = \begin{bmatrix} 1, W_{N_0}^n, W_{N_0}^{2n}, \cdots, W_{N_0}^{(N_0-1)r} \end{bmatrix} \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{N_0-1} \end{bmatrix}.$$

Stacking the results for all n yields:

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N_0-1] \end{bmatrix} = \begin{bmatrix} 1, & 1, & 1, & \cdots, & 1 \\ 1, & W_{N_0}^1, & W_{N_0}^2, & \cdots, & W_{N_0}^{(N_0-1)} \\ 1, & W_{N_0}^2, & W_{N_0}^4, & \cdots, & W_{N_0}^{2(N_0-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1, & W_{N_0}^{(N_0-1)}, & W_{N_0}^{2(N_0-1)}, & \cdots, & W_{N_0}^{(N_0-1)^2} \end{bmatrix} \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{N_0-1} \end{bmatrix}.$$

In matrix notation, the IDTFS is compactly written as $\mathbf{x} = \mathbf{W}_{N_0}^* \mathcal{D}$. Notice, $\mathbf{W}_{N_0}^*$ is just the conjugate of the DFT matrix $\mathbf{W}_{N_0}$.

In MATLAB, the N_0-by-N_0 IDTFS matrix is easily computed according to

```
>> Wconj= (exp(j*2*pi/N_0)).^((0:N_0-1)'*(0:N_0-1));
```

(b) MATLAB code, similar to that presented in Sec. 9.7, is used to test the execution speed of the matrix IDTFS approach to the inverse FFT approach. First, test vectors and IDTFS matrices are created.

```
>>  X10 = fft(randn(10,1)); X100 = fft(randn(100,1)); X1000 = fft(randn(1000,1));
>>  W10 = (exp(j*2*pi/10)).^((0:10-1)'*(0:10-1));
>>  W100 = (exp(j*2*pi/100)).^((0:100-1)'*(0:100-1));
>>  W1000 = (exp(j*2*pi/1000)).^((0:1000-1)'*(0:1000-1));
```

Next, execution speeds are measured by repeating calculations within a loop. Notice, output from MATLAB's `ifft` command must be scaled by $1/N_0$ to compute the IDTFS.

```
>>  tic; for t=1:50000, ifft(X10)/10; end; T10ifft =toc;
>>  tic; for t=1:50000, W10*X10; end; T10mat = toc;
>>  tic; for t=1:5000, ifft(X100)/100; end; T100ifft =toc;
>>  tic; for t=1:5000, W100*X100; end; T100mat = toc;
>>  tic; for t=1:500, ifft(X1000)/1000; end; T1000ifft =toc;
>>  tic; for t=1:500, W1000*X1000; end; T1000mat = toc;
>>  [T10mat/T10ifft, T100mat/T100ifft, T1000mat/T1000ifft]
    ans = 1.0323    3.5000   101.4754
```

For these trials, these results indicate that the IFFT approach is about as fast as the matrix approach for $N_0 = 10$, about an order of magnitude faster than the matrix approach for $N_0 = 100$, and about two orders of magnitude faster than the matrix approach for $N_0 = 1000$. While actual times will vary considerably from computer to computer and from trial to trial, the general trend is clear: the matrix based approach is less efficient than the inverse FFT approach, and this difference grows rapidly as N_0 increases.

(c) Substituting $\mathcal{D} = \frac{1}{N_0}\mathbf{W}_{N_0}\mathbf{x}$ into $\mathbf{x} = \mathbf{W}_{N_0}^*\mathcal{D}$ yields $\mathbf{x} = \mathbf{W}_{N_0}^*\frac{1}{N_0}\mathbf{W}_{N_0}\mathbf{x} = \frac{1}{N_0}\mathbf{W}_{N_0}^*\mathbf{W}_{N_0}\mathbf{x}$. For equality, $\frac{1}{N_0}\mathbf{W}_{N_0}^*\mathbf{W}_{N_0} = I_{N_0}$, where I_{N_0} is the N_0-by-N_0 identity matrix. Thus, $\mathbf{W}_{N_0}^*\mathbf{W}_{N_0} = N_0 I_{N_0} = \mathbf{W}_{N_0}\mathbf{W}_{N_0}^*$.

Thus, multiplying the DFT matrix $\mathbf{W}_{N_0}$ by the inverse DTFS matrix $\mathbf{W}_{N_0}^*$, or vice versa, yields the scaled identity matrix $N_0 I_{N_0}$:

$$\mathbf{W}_{N_0}^*\mathbf{W}_{N_0} = \mathbf{W}_{N_0}\mathbf{W}_{N_0}^* = N_0 I_{N_0}.$$

This result is consistent with the fact that the DTFS represents a signal using an orthogonal set of basis functions. Since the columns (or rows) of $\mathbf{W}_{N_0}$ are orthogonal, $\mathbf{W}_{N_0}^*\mathbf{W}_{N_0}$ must be a diagonal matrix. The scale factor of N_0 results from mixing matrices from the DFT and DTFS (recall, the DFT is N_0 times the DFTS).

Solution 9.7-2

(a) We know that

$$\begin{aligned}
|H(e^{j\Omega_c})|^2 = \frac{1}{2} &= \frac{(1+\alpha)^2}{4}\frac{|1-e^{-j\Omega}|^2}{|1-\alpha e^{-j\Omega}|^2} \\
&= \frac{1+2\alpha+\alpha^2}{4}\frac{(1-\cos(-\Omega_c))^2 + (-\sin(-\Omega_c))^2}{(1-\alpha\cos(-\Omega_c))^2 + (\alpha\sin(-\Omega_c))^2} \\
&= \frac{1+2\alpha+\alpha^2}{4}\frac{1-2\cos(\Omega_c)+\cos^2(\Omega_c)+\sin^2(\Omega_c)}{1-2\alpha\cos(\Omega_c)+\alpha^2\cos^2(\Omega_c)+\alpha^2\sin^2(\Omega_c)} \\
&= \frac{1+2\alpha+\alpha^2}{4}\frac{2-2\cos(\Omega_c)}{1+\alpha^2-2\alpha\cos(\Omega_c)} \\
&= \frac{1+2\alpha+\alpha^2}{4}\frac{2-2\cos(\Omega_c)}{1+\alpha^2-2\alpha\cos(\Omega_c)}.
\end{aligned}$$

Thus,

$$2(1 + \alpha^2 - 2\alpha\cos(\Omega_c)) = (1 + 2\alpha + \alpha^2)(2 - 2\cos(\Omega_c))$$

or

$$2 + 2\alpha^2 - 4\alpha\cos(\Omega_c) = 2 + 4\alpha + 2\alpha^2 - 2\cos(\Omega_c) - 4\alpha\cos(\Omega_c) - 2\alpha^2\cos(\Omega_c).$$

This simplifies to $0 = -2\alpha^2\cos(\Omega_c) + 4\alpha - 2\cos(\Omega_c)$ or $\cos(\Omega_c)\alpha^2 - 2\alpha + \cos(\Omega_c)$. Solving with the quadratic formula yields $\alpha = \frac{2 \pm \sqrt{4 - 4\cos^2(\Omega_c)}}{2\cos(\Omega_c)} = \frac{1 \pm \sin(\Omega_c)}{\cos(\Omega_c)}$. For $0 \leq \Omega_c \leq \pi$, $\left|\frac{1 + \sin(\Omega_c)}{\cos(\Omega_c)}\right| \geq 1$ and $\left|\frac{1 - \sin(\Omega_c)}{\cos(\Omega_c)}\right| \leq 1$. Since a stable system is desired,

$$\alpha = \frac{1 - \sin(\Omega_c)}{\cos(\Omega_c)}.$$

(b) For this part, $\Omega_c = 2\pi\frac{f_c}{f_s} = \frac{2\pi}{5}$. Using this value and the results of part (a), we use MATLAB to compute α.

```
>>  Omega_c = 2*pi/5; alpha = (1-sin(Omega_c))/cos(Omega_c)
    alpha = 0.1584
```

Using MATLAB, the corresponding difference equation is determined from $H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \left(\frac{1+\alpha}{2}\right)\left(\frac{1 - z^{-1}}{1 - \alpha z^{-1}}\right)$.

```
>>  B = (1+alpha)/2*[1,-1], A = [1,-alpha]
    B = 0.5792   -0.5792
    A = 1.0000   -0.1584
```

Thus, the difference equation is

$$y[n] - 0.1584y[n-1] = 0.5792x[n] - 0.5792x[n-1].$$

This first order system has one pole at $z = \alpha = 0.1584$. Since this pole is inside the unit circle, the system is stable. The system's frequency response is computed using MATLAB.

```
>>  Omega = -pi:2*pi/2000:pi; H = freqz(B,A,Omega);
>>  H3dB = freqz(B,A,[-Omega_c,Omega_c]);
>>  plot(Omega,(abs(H)),'k'); grid on;
>>  axis([-pi,pi,0,1.1]); xlabel('\Omega'); ylabel('|H(\Omega)|');
>>  set(gca,'ytick',[0,1/sqrt(2),1],'xtick',[-pi,-2*pi/5,0,2*pi/5,pi],...
>>      'xticklabel',{'-\pi','-2\pi/5','0','2\pi/5','\pi'});
```
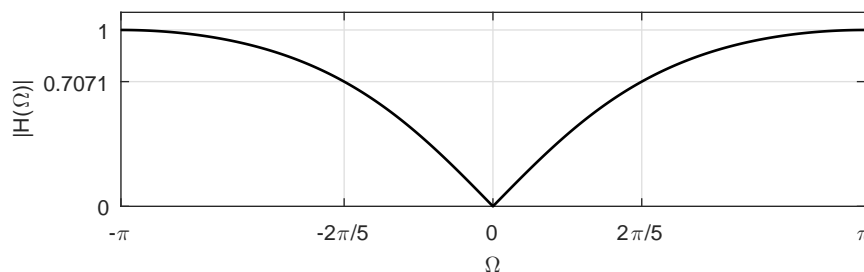

Figure S9.7-2

As seen if Fig. S9.7-2, the filter is highpass with a cutoff frequency $\Omega_c = 2\pi/5$, as desired.

(c) Since α, and therefore $H(z)$, is held constant, the cutoff frequency Ω_c remains constant as well. That is, changing the sampling frequency does not affect the digital cutoff frequency Ω_c of the filter. However, the cut-off frequency expressed in hertz scales directly with the sampling frequency. That is, as $f_s = 5$ kHz is increased to $f_s = 50$ kHz, $f_c = 1$ kHz is increased to $f_c = 10$ kHz.

(d) The inverse to $H(z) = \left(\frac{1+\alpha}{2}\right)\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$ is

$$H^{-1}(z) = \left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha z^{-1}}{1-z^{-1}}\right).$$

Since the inverse has a root on the unit circle, it is not BIBO stable and therefore not well behaved.

(e) For $\Omega_c = \pi/2$, $\alpha = \frac{1-\sin(\Omega_c)}{\cos(\Omega_c)} = \frac{1-1}{0}$, which is indeterminant. Using L'Hospital's rule, $\lim_{\Omega_c \to \pi/2} \frac{1-\sin(\Omega_c)}{\cos(\Omega_c)} = \lim_{\Omega_c \to \pi/2} \frac{d}{dt}\left(\frac{1-\sin(\Omega_c)}{\cos(\Omega_c)}\right) = \lim_{\Omega_c \to \pi/2} \frac{-\cos(\Omega_c)}{-\sin(\Omega_c)} = \frac{0}{1} = 0$. Using $\alpha = 0$, the system function is $H(z) = 0.5(1 - z^{-1})$. What is particularly interesting is that when $\Omega_c = \pi/2$ this normally IIR filter $H(z)$ becomes an FIR filter. Notice that the impulse response is $h[n] = 0.5\delta[n] - 0.5\delta[n - 1]$, which is a finite duration signal.

Solution 9.7-3

We follow the frequency sampling method presented in Sec. 9.7 to design a length-35 linear phase FIR highstop (lowpass) filter that has cutoff frequency $\Omega_c = 2\pi/3$. To ensure proper filter behavior, we specify the desired response to have linear phase characteristics. Plots of the designed filter's impulse response $h[n]$ and magnitude response $|H(\Omega)|$ are shown in Fig. S9.7-3

Figure S9.7-3

```
>>  Hd = @(Omega) (mod(Omega,2*pi)<=2*pi/3)+(mod(Omega,2*pi)>=2*pi-2*pi/3);
>>  N = 35; Omega = linspace(0,2*pi*(1-1/N),N)'; H = Hd(Omega).*exp(-j*Omega*((N-1)/2));
>>  H(fix(N/2)+2:N,1) = H(fix(N/2)+2:N,1)*((-1)^(N-1)); h = real(ifft(H));
>>  subplot(211); stem(0:N-1,h,'k.'); xlabel('n'); ylabel('h[n]');
>>  axis([-.5 N-.5 -0.3 0.8]); grid on; samples = linspace(0,2*pi*(1-1/N),N)';
>>  Omega = 0:2*pi/2001:2*pi; H = freqz(h,1,Omega);
>>  subplot(212); plot(samples,Hd(samples),'k.',Omega,abs(H),'k');
>>  ylabel('|H(\Omega)|'); xlabel('\Omega'); grid on; axis([0 2*pi 0 1.2]);
>>  set(gca,'xtick',[0 2*pi/3 4*pi/3 2*pi],'xticklabel',{'0','2\pi/3','4\pi/3','2\pi'});
```

Solution 9.7-4

We follow the frequency sampling method presented in Sec. 9.7 to design a length-71 linear phase FIR bandstop filter that has stopband ($\pi/3 < |\Omega| < \pi/2$). To ensure proper filter behavior, we specify the desired response to have linear phase characteristics. Plots of the designed filter's impulse response $h[n]$ and magnitude response $|H(\Omega)|$ are shown in Fig. S9.7-4

```
>>  Hd = @(Omega) (mod(Omega,2*pi)<=pi/3)+(mod(Omega,2*pi)>=2*pi-pi/3)+...
>>      ((mod(Omega,2*pi)>pi/2)&(mod(Omega,2*pi)<3*pi/2));
>>  N = 71; Omega = linspace(0,2*pi*(1-1/N),N)';
>>  H = Hd(Omega).*exp(-j*Omega*((N-1)/2));
>>  H(fix(N/2)+2:N,1) = H(fix(N/2)+2:N,1)*((-1)^(N-1));
>>  h = real(ifft(H));
>>  subplot(211); stem(0:N-1,h,'k.'); xlabel('n'); ylabel('h[n]');
>>  axis([-.5 N-.5 -0.3 1]); grid on;
>>  samples = linspace(0,2*pi*(1-1/N),N)';
>>  Omega = 0:2*pi/2001:2*pi; H = freqz(h,1,Omega);
>>  subplot(212); plot(samples,Hd(samples),'k.',Omega,abs(H),'k');
>>  ylabel('|H(\Omega)|'); xlabel('\Omega'); grid on; axis([0 2*pi 0 1.2]);
>>  set(gca,'xtick',[0 pi/3 pi/2 3*pi/2 5*pi/3 2*pi],...
>>      'xticklabel',{'0','\pi/3','\pi/2','3\pi/2','5\pi/3','2\pi'});
```
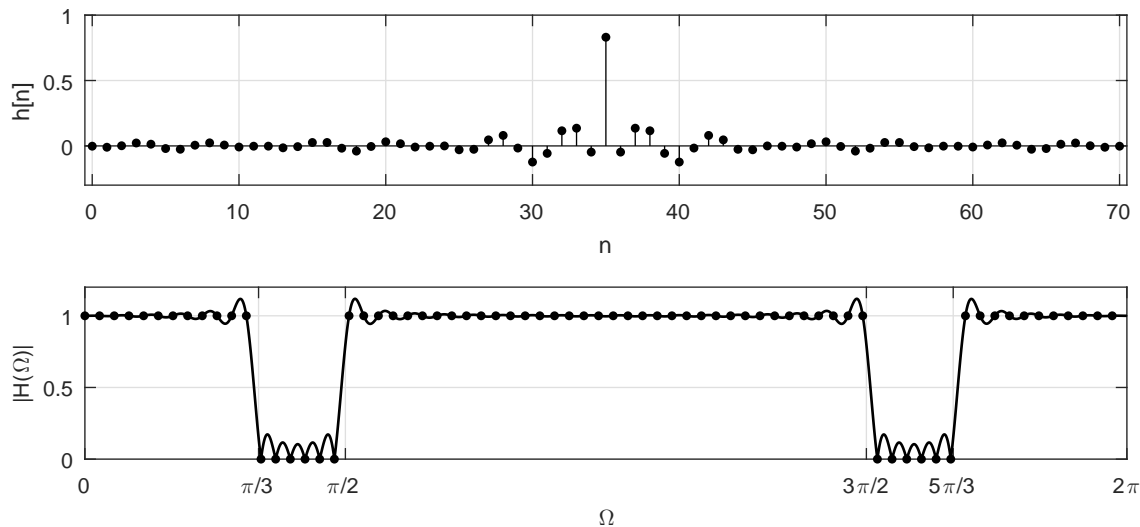

Figure S9.7-4

Solution 9.7-5

(a) It would be unlikely if not impossible to achieve this exact magnitude response with a practical FIR filter. Since the magnitude response has points of derivative discontinuities, an infinite length filter would be required, which is not practical.

(b) We follow the frequency sampling method presented in Sec. 9.7 to design a length-71 linear phase FIR filter that reasonably approximates the desired response. Different approximations are easily accomplished by changing N. Plots of the designed filter's impulse response $h[n]$ and magnitude response $|H(\Omega)|$ are shown in Fig. S9.7-5

```
>>  Hd = @(Omega) ((mod(Omega,2*pi)>=0)&(mod(Omega,2*pi)<pi/4)).*(4*mod(Omega,2*pi)/pi)+...
>>      ((mod(Omega,2*pi)>=pi/4)&(mod(Omega,2*pi)<pi/2)).*(2-4*mod(Omega,2*pi)/pi)+...
>>      ((mod(Omega,2*pi)>7*pi/4)&(mod(Omega,2*pi)<=2*pi)).*(-4*(mod(Omega,2*pi)-2*pi)/pi)+...
```
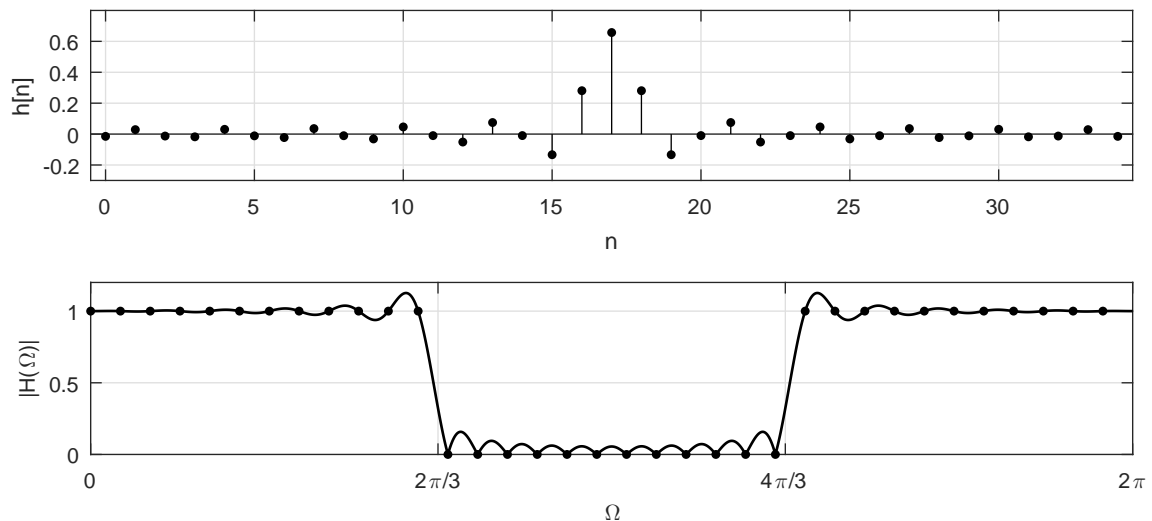
```
>>        ((mod(Omega,2*pi)>3*pi/2)&(mod(Omega,2*pi)<=7*pi/4)).*(2+4*(mod(Omega,2*pi)-2*pi)/pi);
>>    N = 71; Omega = linspace(0,2*pi*(1-1/N),N)';
>>    H = Hd(Omega).*exp(-j*Omega*((N-1)/2));
>>    H(fix(N/2)+2:N,1) = H(fix(N/2)+2:N,1)*((-1)^(N-1));
>>    h = real(ifft(H));
>>    subplot(211); stem(0:N-1,h,'k.'); xlabel('n'); ylabel('h[n]');
>>    axis([-.5 N-.5 -0.15 .35]); grid on;
>>    samples = linspace(0,2*pi*(1-1/N),N)';
>>    Omega = 0:2*pi/2001:2*pi; H = freqz(h,1,Omega);
>>    subplot(212); plot(samples,Hd(samples),'k.',Omega,abs(H),'k');
>>    ylabel('|H(\Omega)|'); xlabel('\Omega'); grid on; axis([0 2*pi 0 1.1]);
>>    set(gca,'xtick',[0 pi/2 pi 3*pi/2 2*pi],...
>>        'xticklabel',{'0','\pi/2','\pi','3\pi/2','2\pi'});
```
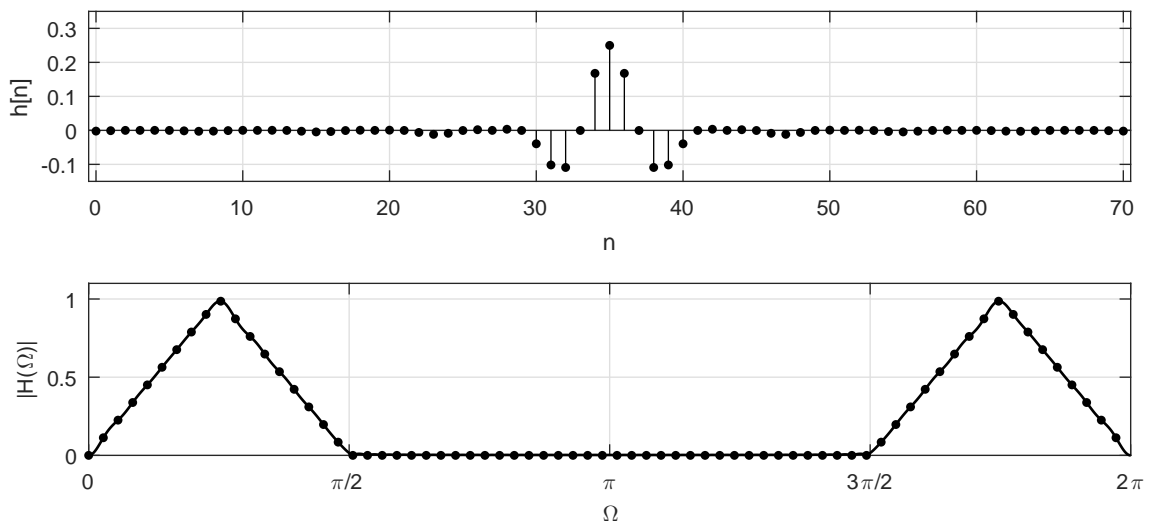

Figure S9.7-5

Solution 9.7-6

A simple first-order highpass filter is given by $H_{\text{hp}}(z) = k(1 - z^{-1})$. To achieve a gain of 3, solve $3 = k|1 - e^{-j\pi}| = 2k$. Thus, $k = 3/2$. To realize the desired comb filter, the HPF response needs to be compressed by a factor of 4, which effectively replicates the original response four times over $[0, 2\pi)$. Compression is achieved by letting $z = z^4$. Thus,

$$H_{\text{comb}}(z) = 1.5(1 - z^{-4}).$$

The corresponding impulse response is

$$h_{\text{comb}} = 1.5\delta[n] - 1.5\delta[n - 4].$$

MATLAB is used to verify operation:

```
>>   Omega =0:2*pi/2000:2*pi;
>>   B = [1.5 0 0 0 -1.5]; A = 1; H = freqz(B,A,Omega);
>>   plot(Omega,(abs(H)),'k'); grid on;
>>   axis([0,2*pi,0,3.2]); xlabel('\Omega'); ylabel('|H(\Omega)|');
>>   set(gca,'ytick',0:3,'xtick',0:pi/2:2*pi,...
>>       'xticklabel',{'0','\pi/2','\pi','3\pi/2','2\pi'});
```
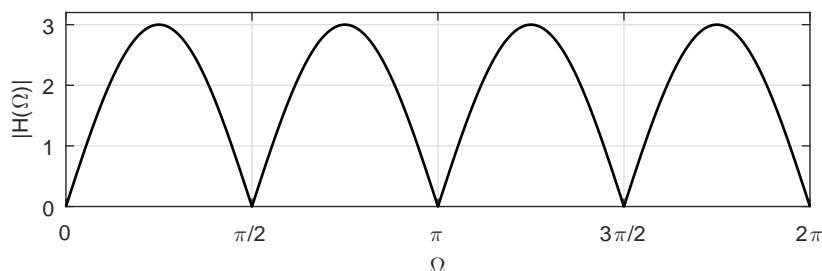
Figure S9.7-6

Solution 9.7-7

(a) Reversing the order of the elements of column vector $\mathbf{x}$ can be accomplished using a N_0-by-N_0 permutation matrix R_{N_0} that is simply a 90-degree rotated N_0-by-N_0 identity matrix. For example, R_5 is

$$R_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Let integer $i \in \{0, 1, \ldots, N_0 - 1\}$ be used to designate row or column of $\mathbf{W}_{N_0}$. Row i of $\mathbf{W}_{N_0}$ is represented as $\mathbf{r}_i = e^{j2\pi[0,1,\ldots,N_0-1]i/N_0}$. Column i of $\mathbf{W}_{N_0}$ is represented as $\mathbf{c}_i = e^{j2\pi[0,1,\ldots,N_0-1]^T i/N_0}$. For $i \geq 1$, notice that column $N_0 - i$ is $\mathbf{c}_{N_0-i} = e^{j2\pi[0,1,\ldots,N_0-1]^T(N_0-i)/N_0} = e^{j2\pi[0,1,\ldots,N_0-1]^T}e^{-j2\pi[0,1,\ldots,N_0-1]^T i/N_0} = e^{-j2\pi[0,1,\ldots,N_0-1]^T i/N_0} = \mathbf{r}_i^H$. That is, for $i \geq 1$, column $(N_0 - i)$ is the complex-conjugate transpose of row i. Also notice that $\mathbf{W}_{N_0}$ is composed of orthogonal rows,

$$\mathbf{r}_i \mathbf{r}_k^H = \begin{cases} 0 & i \neq k \\ N_0 & i = k \end{cases}.$$

Combining these facts yields

$$\mathbf{W}_{N_0}^2 = N_0 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R_{N_0-1} & \\ 0 & & & \end{bmatrix}.$$

For example, if $N_0 = 5$ then

$$\mathbf{W}_5^2 = 5 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection, it is clear that $\mathbf{W}_{N_0}^2$ is a scaled permutation matrix. The operation $\mathbf{W}_{N_0}^2 \mathbf{x}$ scales and reorders the vector $\mathbf{x}$: the first element of $\mathbf{x}$ is not moved, but the order of the remaining $N_0 - 1$ elements are reversed.

MATLAB is used to confirm these conclusions.

```
>>  x = [1 2 3 4 5]';
>>  W_5 = dftmtx(5);
>>  real(W_5*W_5*x)'
    ans = 5.0000    25.0000    20.0000    15.0000    10.0000
```

The last line includes the `real` command to remove minute imaginary components that result due to computer round-off. As expected, vector **x** is scaled by $N_0 = 5$ and the order of the last four elements is reversed.

(c) Using the previous result, $\mathbf{W}_{N_0}^4 \mathbf{x} = (\mathbf{W}_{N_0}^2)(\mathbf{W}_{N_0}^2)\mathbf{x} = N_0^2 \mathbf{x}$. The first multiplication by $(\mathbf{W}_{N_0}^2)$ scales **x** by N_0 and reverses the order of the last $N_0 - 1$ elements. The second multiplication again scales **x** by N_0 for a total of N_0^2 and reverses the previously reversed last $N_0 - 1$ elements, effectively leaving the order of **x** unchanged. MATLAB is used to confirm these conclusions.

```
>>  x = [1 2 3 4 5]';
>>  W_5 = dftmtx(5);
>>  real(W_5*W_5*W_5*W_5*x)'
    ans = 25.0000    50.0000    75.0000    100.0000    125.0000
```

The result is just **x** scaled by $N_0^2 = 25$.

Chapter 10 Solutions

(a) Here,
$$\ddot{y} + 10\dot{y} + 2y = x.$$

Choosing $q_1 = y$ and $q_2 = \dot{y} = \dot{q}_1 \implies \dot{q}_2 = \ddot{y}$, we obtain

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -2q_1 - 10q_2 + x$$

In matrix form we get

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -10 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x.$$

(b) In this case,
$$\ddot{y} + 2e^y \dot{y} + \log y = x.$$

Choosing $q_1 = y$ and $\quad q_2 = \dot{y} = \dot{q}_1$, we obtain

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -2e^{q_1} q_2 - \log q_1 + x$$

It is easy to see that this set is nonlinear.

(c) Here,
$$\ddot{y} + \phi_1(y)\dot{y} + \phi_2(y)y = x.$$

Choosing $q_1 = y$ and $q_2 = \dot{y}$, we obtain

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -\phi_1(q_1)q_2 - \phi_2(q_1)q_1 + x$$

This case is also a nonlinear set, since $\phi_2(q_1)$ and $\phi_1(q_1)$ are not constants.

Refer to Fig. S10.3-1. Writing the loop equations we get

$$x = q_1 + 2i + 3i_2,$$

where
$$i_2 = \frac{x - q_1 - \dot{q}_2}{2} - q_2 \qquad \text{and} \qquad i = \frac{x - q_1 - \dot{q}_2}{2}.$$

Also we have
$$\frac{1}{2}\dot{q}_1 = \frac{x - q_1 - \dot{q}_2}{2} - q_1.$$

Therefore,

$$\dot{q}_1 = x - q_1 - \dot{q}_2 - 2q_1 = -3q_1 - \dot{q}_2 + x \qquad (10.3\text{-}1a)$$

We can also write

$$\dot{q}_2 = 3i_2 = 3\left[\frac{x - q_1 - \dot{q}_2}{2} - q_2\right] = \frac{3}{2}x - \frac{3}{2}q_1 - \frac{3}{2}\dot{q}_2 - 3q_2.$$

Hence, $\frac{5}{2}\dot{q}_2 = -\frac{3}{2}q_1 - 3q_2 + \frac{3}{2}x$, or

$$\dot{q}_2 = -\frac{3}{5}q_1 - \frac{6}{5}q_2 + \frac{3}{5}x \qquad (10.3\text{-}1b)$$

Substituting Eq. (10.3-1b) in Eq. (10.3-1a) we obtain

$$\dot{q}_1 = -3q_1 + x - \left[-\frac{3}{5}q_1 - \frac{6}{5}q_2 + \frac{3}{5}x\right] = -\frac{12}{5}q_1 + \frac{6}{5}q_2 + \frac{2}{5}x.$$

Hence the state equations are:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} & \frac{6}{5} \\ -\frac{3}{5} & -\frac{6}{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} x(t).$$
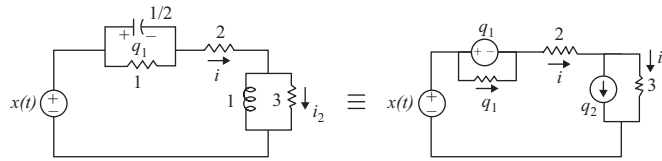
Figure S10.3-1

Solution 10.3-2

Refer to Fig. S10.3-2. In the first loop, the current i_1 can be computed as

$$x = \frac{1}{3}i_1 + q_1 \implies i_1 = 3(x - q_1).$$

Using node equation, we also have

$$\frac{1}{2}\dot{q}_1 = -2q_1 - q_2 - 3q_1 + 3x = -5q_1 - q_2 + 3x.$$

Hence,

$$\dot{q}_1 = -10q_1 - 2q_2 + 6x. \qquad (10.3\text{-}2a)$$

Writing the equations in the rightmost loop we get

$$q_1 = q_2 + \dot{q}_2 \quad \text{and} \quad \dot{q}_2 = q_1 - q_2. \qquad (10.3\text{-}2b)$$

Hence from Eq. (10.3-2a) and Eq. (10.3-2b) the state equations are found as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -10 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} x.$$

The output equation is $y = \dot{q}_2 = q_1 - q_2$ or

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$
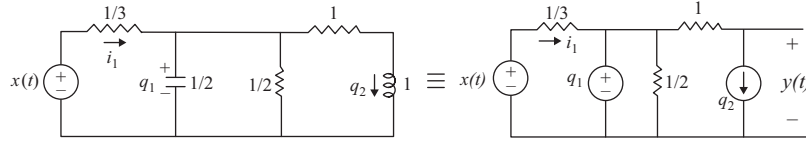
Figure S10.3-2

Solution 10.3-3

Refer to Fig. S10.3-3. Let's choose the voltage across the capacitor and the current through the inductor as state variables q_1 and q_2, respectively. Writing the loop equations we get

$$x_1 = q_1 + \frac{1}{5}[\dot{q}_1 - q_2].$$

Here we use the fact that $\dot{q}_1 = i_1$ and $q_2 = i_2$. Further,

$$x_2 = -\frac{1}{2}\dot{q}_2 - q_2 + \frac{1}{5}[\dot{q}_1 - q_2].$$

Thus,

$$\dot{q}_1 = -5q_1 + q_2 + 5x_1$$
$$\dot{q}_2 = -2q_1 - 2q_2 + 2x_1 - 2x_2 \quad .$$

Hence the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The output equation is

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + 0x(t).$$
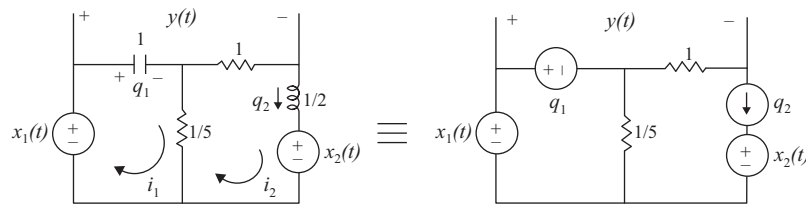
Figure S10.3-3

Solution 10.3-4

Refer to Fig. S10.3-4. The loop equations yield, with $i_2 = \dot{q}_2$ and $i_1 = q_1 + i_2 = q_1 + \dot{q}_2$,

$$x = 2i_1 + q_1 + \dot{q}_1 = 2q_1 + 2\dot{q}_2 + q_1 + \dot{q}_1 = 3q_1 + \dot{q}_1 + 2\dot{q}_2 \tag{10.3-4a}$$

and

$$x = 2i_1 + \dot{q}_2 + q_2 = 2q_1 + 2\dot{q}_2 + \dot{q}_2 + q_2 = 2q_1 + q_2 + 3\dot{q}_2. \tag{10.3-4b}$$

The last equation gives

$$\dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x. \tag{10.3-4c}$$

Substituting $\dot{q}_2$ in Eq. (10.3-4a) we get

$$\dot{q}_1 = -\frac{5}{3}q_1 + \frac{2}{3}q_2 + \frac{1}{3}x. \tag{10.3-4d}$$

From Eq. (10.3-4c) and Eq. (10.3-4d) the state equations are obtained as

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} x(t).
$$

Further, the output equations are $y_1 = q_1$ and $y_2 = i_2 = \dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x$ so that

$$
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} x(t).
$$
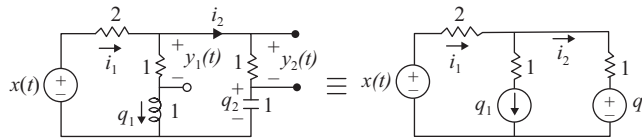
Figure S10.3-4

Solution 10.3-5

Refer to Fig. S10.3-5. We have

$$
i = q_1 + \dot{q}_1 = \frac{x - q_1}{2} + \frac{\dot{x} - \dot{q}_1}{2}.
$$

Multiplying both sides of this equations by 2, we get

$$
2q_1 + 2\dot{q}_1 = x - q_1 + \dot{x} - \dot{q}_1 \qquad \text{or} \qquad 3\dot{q}_1 = -3q_1 + x + \dot{x}.
$$

Thus the only state equation is

$$
\dot{q}_1 = -q_1 + \frac{x}{3} + \frac{\dot{x}}{3}.
$$

The output equation is $y = -q_1 + x$.

Note that although there are two capacitors, there is only one independent capacitor voltage. This is because the two capacitors form a loop with the voltage source. In such a case the state equation contains the terms x as well as $\dot{x}$. A similar situation exists with inductors with a current source.
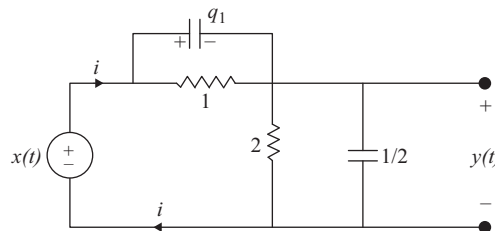
Figure S10.3-5

Solution 10.3-6

Let us choose q_1, q_2 and q_3 as the outputs of the subsystem shown in Fig. S10.3-6.

From the block diagram we obtain:

$$
5q_2 = \dot{q}_1 + 10q_1 \Longrightarrow \dot{q}_1 = -10q_1 + 5q_2
$$
$$
q_1 = \dot{q}_3 + q_3 \Longrightarrow \dot{q}_3 = q_1 - q_3
$$
$$
w = \dot{q}_2 + 2q_2 \Longrightarrow \dot{q}_2 = w - 2q_2
$$
$$
\dot{q}_2 = -2q_2 - q_3 + x
$$
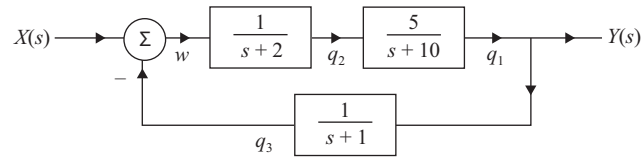
Figure S10.3-6

From these equations, the state equations can be written as

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -10 & 5 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x,
$$

and the output equation is

$$
y = q_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.
$$

Solution 10.3-7

From Fig. P10.3-7, it is easy to write the state equations as

$$
\begin{aligned}
\dot{q}_1 &= \lambda_1 q_1 \\
\dot{q}_2 &= \lambda_2 q_2 + x_1 \\
\dot{q}_3 &= \lambda_3 q_3 + x_2 \\
\dot{q}_4 &= \lambda_4 q_4 + x_2
\end{aligned}
$$

or

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$

The output equation is

$$
\begin{aligned} y_1 &= q_1 + q_2 \\ y_2 &= q_2 + q_3 \end{aligned} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}.
$$

Solution 10.3-8

Here,

$$
H(s) = \frac{3s + 10}{s^2 + 7s + 12}.
$$

Direct Form II:

We can write the state and output equations straightforward from the transfer function $H(s)$. Thus we get

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x
$$

and

$$
y = \begin{bmatrix} 10 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.
$$

Transposed Direct Form II:

In this case the block diagram can be drawn as shown in Fig. S10.3-8a. Hence,

$$\dot{q}_1 = -7q_1 + q_2 + 3x$$
$$\dot{q}_2 = -12q_1 + 10x$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 10 \end{bmatrix} x.$$

The output equation is

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$
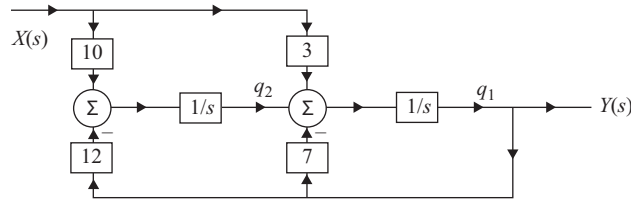
Figure S10.3-8a

Cascade Form:

Writing the transfer function in factored form yields

$$H(s) = \frac{3s + 10}{s^2 + 7s + 12} = \left(\frac{3s + 10}{s + 4} \right) \left(\frac{1}{s + 3} \right).$$

Referring to the left side of Fig. S10.3-8b, we can write

$$\left. \begin{array}{c} \dot{q}_1 + 4q_1 = 3\dot{q}_2 + 10q_2 \\ \dot{q}_2 = -3q_2 + x \end{array} \right\} \implies \begin{array}{c} \dot{q}_1 = -4q_1 - 9q_2 + 10q_2 + 3x \\ \dot{q}_2 = -3q_2 + x \end{array}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} x$$

and

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Figure S10.3-8b

Parallel Form:

Performing a partial fraction expansion on $H(s)$ yields

$$H(s) = \frac{2}{s + 4} + \frac{1}{s + 3}$$

Referring to the right side of Fig. S10.3-8b, the state equations are

$$\begin{array}{l} \dot{q}_1 = -4q_1 + x \\ \dot{q}_2 = -3q_1 + x \end{array} \implies \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x$$

and the output equation is

$$y = 2q_1 + q_2 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Solution 10.3-9

(a) Here,

$$H(s) = \frac{4s}{(s+1)(s+2)^2} = \frac{4s}{s^3 + 5s^2 + 8s + 4}.$$

Direct Form II:
We can write the state and output equations straightforward from the transfer function $H(s)$.
Thus

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

and

$$y = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

Transposed Direct Form II:
In this case the block diagram can be drawn as shown in Fig. S10.3-9a-1. Hence, the state equations are

$$\begin{array}{l} \dot{q}_1 = -5q_1 + q_2 \\ \dot{q}_2 = -8q_1 + q_3 + 4x \\ \dot{q}_3 = -q_1 \end{array}$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 0 \\ -8 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} x.$$

The output equation is

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$
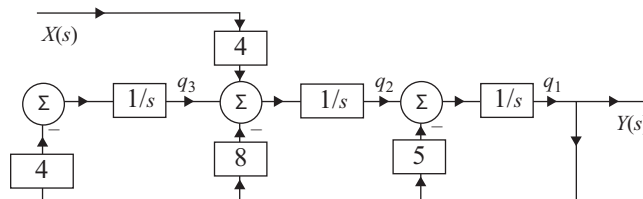
Figure S10.3-9a-1

Cascade Form:
Writing the transfer function in factored form yields

$$H(s) = \left(\frac{1}{s+1} \right) \left(\frac{4s}{s+2} \right) \left(\frac{1}{s+2} \right).$$

Referring to Fig. S10.3-9a-2, we can write the state equations as

$$\begin{aligned}
\dot{q}_1 &= -2q_1 + q_2 \\
\dot{q}_2 + 2q_2 &= 4\dot{q}_3 \\
\dot{q}_3 &= -q_3 + x
\end{aligned} \implies \left\{ \begin{aligned}
\dot{q}_1 &= -2q_1 + q_2 \\
\dot{q}_2 &= -4q_3 - 2q_2 + 4x \\
\dot{q}_3 &= -q_3 + x
\end{aligned} \right.$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} x.$$

The output equation is

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$
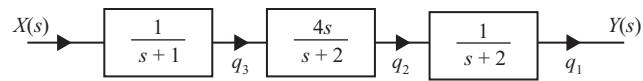
Figure S10.3-9a-2

Parallel Form:

Performing a partial fraction expansion on $H(s)$ yields

$$H(s) = \frac{-4}{s+1} + \frac{4}{s+2} + \frac{8}{(s+2)^2}.$$

Referring to Fig. S10.3-9a-3, the state equations are

$$\begin{aligned}
\dot{q}_1 &= -q_1 + x \\
\dot{q}_2 &= -2q_2 + q_3 \\
\dot{q}_3 &= -2q_3 + x
\end{aligned}$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x.$$

The output equation is

$$y = -4q_1 + 8q_2 + 4q_3 = \begin{bmatrix} -4 & 8 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$
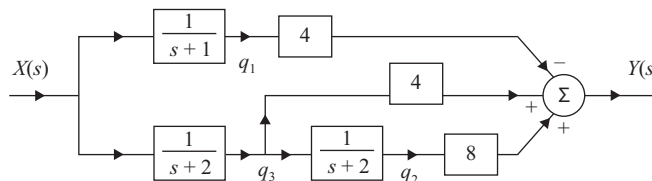
Figure S10.3-9a-3

(b) In this case, the transfer function is

$$H(s) = \frac{s^3 + 7s^2 + 12s}{(s+1)^3(s+2)} = \frac{s^3 + 7s^2 + 12s}{s^4 + 5s^3 + 9s^2 + 7s + 2}.$$

Direct Form II:

Straightforward from $H(s)$, the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -7 & -9 & -5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x$$

and the output equation is

$$y = \begin{bmatrix} 0 & 12 & 7 & \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}.$$

Transposed Direct Form II:

Similar to the DFII case, we can write the state equation directly from $H(s)$ as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 0 & 0 \\ -9 & 0 & 1 & 0 \\ -7 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 7 \\ 12 \\ 0 \end{bmatrix} x.$$

The output equation is

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}.$$

Cascade Form:

Writing the transfer function in factored form yields

$$H(s) = \frac{s(s+3)(s+4)}{(s+2)(s+1)^3} = \left(\frac{1}{s+2} \right) \left(\frac{s}{s+1} \right) \left(\frac{s+3}{s+1} \right) \left(\frac{s+4}{s+1} \right)$$

Referring to Fig. S10.3-9b-1, we can write

$$\begin{aligned} \dot{q}_1 + q_1 &= \dot{q}_2 + 4q_2 \\ \dot{q}_2 + q_2 &= \dot{q}_3 + 3q_3 \\ \dot{q}_3 &= -q_3 + \dot{q}_4 \\ \dot{q}_4 &= -2q_4 + x \end{aligned} \implies \begin{cases} \dot{q}_1 = -q_1 + 4q_2 - q_2 + 2q_3 - 2q_4 + x \\ \dot{q}_2 = -q_2 + 3q_3 - q_3 - 2q_4 + x \\ \dot{q}_3 = -q_3 - 2q_4 + x \\ \dot{q}_4 = -2q_4 + x \end{cases}.$$

Hence

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x$$

and

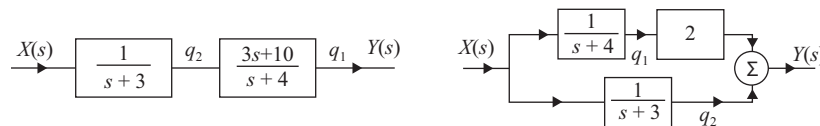$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}.$$
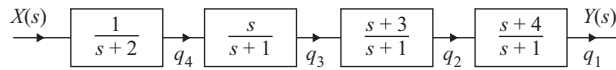
Figure S10.3-9b-1

Parallel Form:

Performing a partial fraction expansion on $H(s)$ yields

$$H(s) = \frac{6}{s+2} + \frac{11}{s+1} + \frac{7}{(s+1)^2} - \frac{6}{(s+1)^3}.$$

Referring to Fig. S10.3-9b-2, the state equations are

$$\dot{q}_1 = -2q_1 + x$$
$$\dot{q}_2 = -q_2 + q_3$$
$$\dot{q}_3 = -q_3 + q_4$$
$$\dot{q}_4 = -q_4 + x$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x.$$

The output equation can be written as $y = 6q_1 - 6q_2 + 7q_3 + 11q_4$ or

$$y = \begin{bmatrix} 6 & -6 & 7 & 11 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}.$$
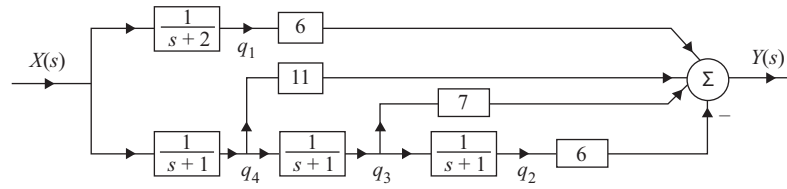
Figure S10.3-9b-2

Solution 10.4-1

The problem provides

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}x$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{q}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad x(t) = 0.$$

The solution of the state equation in the frequency domain is given by

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)\mathbf{q}(0) + \mathbf{\Phi}(s)\mathbf{B}\mathbf{X}(s).$$

In the current case $x(t) = 0 \implies X(s) = 0$, so $\mathbf{Q}(s) = \mathbf{\Phi}(s)\mathbf{q}(0)$ where $\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$.
Computing:

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \qquad\qquad (s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \implies \mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \frac{1}{s^2 + 3s + 2}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{2}{s^2+3s+2} \\ \frac{-1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Hence

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)\mathbf{q}(0) = \begin{bmatrix} \frac{2(s+3)+2}{(s+1)(s+2)} \\ \frac{-2+s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2s+8}{(s+1)(s+2)} \\ \frac{s-2}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{6}{s+1} - \frac{4}{s+2} \\ \frac{-3}{s+1} + \frac{4}{s+2} \end{bmatrix}.$$

Finally,

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \mathcal{L}^{-1} \begin{bmatrix} \mathbf{Q}(s) \end{bmatrix} = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}.$$

Solution 10.4-2

The problem provides

$$\mathbf{A} = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{q}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \qquad x(t) = \sin(100t)u(t).$$

The solution of the state equation in the frequency domain is given by

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)\mathbf{q}(0) + \mathbf{\Phi}(s)\mathbf{B}X(s) = \mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{B}X(s)].$$

Computing:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+5 & 6 \\ -1 & s \end{bmatrix} \quad \text{and} \quad \mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s & -6 \\ 1 & s+5 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s}{(s+3)(s+2)} & \frac{-6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{bmatrix}$$

Hence

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)\left[\mathbf{q}(0) + \mathbf{B}X(s)\right] = \begin{bmatrix} \frac{s}{(s+3)(s+2)} & \frac{-6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{bmatrix} \begin{bmatrix} 5 + \frac{100}{s^2+10^4} \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-34.02}{s+2} + \frac{39.03}{s+3} - \frac{10^{-2}s}{s^2+10^4} \\ \frac{17.01}{s+2} - \frac{13.01}{s+3} - \frac{0}{s^2+10^4} \end{bmatrix}.$$

Inverting, we obatin

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} \left(-34.02e^{-2t} + 39.03e^{-3t} - 0.01\cos 100t\right)u(t) \\ \left(17.01e^{-2t} - 13.01e^{-3t}\right)u(t) \end{bmatrix}.$$

Solution 10.4-3

The problem provides

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{q}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad x(t) = u(t).$$

The solution of the state equation in the frequency domain is given by

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)].$$

Computing:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix}$$

and

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix}$$

Since $x(t) = u(t) \implies X(s) = \frac{1}{s}$, we see that

$$\mathbf{B}\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{q}(0) + \mathbf{B}\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ -1 \end{bmatrix}.$$

Thus

$$\mathbf{Q}(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s+2)} \\ \frac{1}{s(s+1)(s+2)} - \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2s} - \frac{1}{2(s+2)} \\ \frac{1}{2s} - \frac{2}{s+1} - \frac{1}{2(s+2)} \end{bmatrix}.$$

Inverting, we obatin

$$\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} - \frac{1}{2}e^{-2t})u(t) \\ (\frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t})u(t) \end{bmatrix}.$$

Solution 10.4-4

The problem provides

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{q}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix}.$$

The solution of the state equation in the frequency domain is given by

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)].$$

Computing:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix} \implies \mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$\mathbf{BX}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s} \\ 1 \end{bmatrix}$$

$$\mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} \frac{s+1}{s} + 1 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{s} \\ 3 \end{bmatrix}$$

Thus,

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} \frac{2s+1}{s} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(2s+1)(s+2)+3s}{s(s+1)(s+2)} \\ \frac{3}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} + \frac{4}{s+1} - \frac{3}{s+2} \\ \frac{3}{s+2} \end{bmatrix}.$$

Inverting, we obtain

$$\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} (1 + 4e^{-t} - 3e^{-2t})u(t) \\ 3e^{-2t}u(t) \end{bmatrix}.$$

Solution 10.4-5

The problem defines

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{Bx}(t)$$
$$y = \mathbf{Cq} + \mathbf{Dx}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \mathbf{D} = 0$$

and

$$x(t) = u(t) \qquad \mathbf{q}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The Laplace transform method solution states

$$\mathbf{Y}(s) = \mathbf{CQ}(s) + \mathbf{DX}(s) = \mathbf{C\Phi}(s)\mathbf{q}(0) + [\mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}]\mathbf{X}(s).$$

Computing:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix} \qquad \mathbf{BX}(s) = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

Since $\mathbf{D} = 0 \implies Y(s) = \mathbf{C\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)]$, we see that

$$\mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} 2 + \frac{1}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{s} \\ 0 \end{bmatrix}$$

and

$$\mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}\begin{bmatrix} \frac{2s+1}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{(s+1)(s+2)} \\ \frac{-2(2s+1)}{s(s+1)(s+2)} \end{bmatrix}.$$

Combining everything together, we see that

$$\mathbf{Y}(s) = \mathbf{C\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s+1}{(s+1)(s+2)} \\ \frac{-2(2s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

$$= \frac{-4s - 2}{s(s+1)(s+2)} = \frac{-1}{s} - 2 \cdot \frac{1}{s+1} + \frac{3}{s+2}.$$

Inverting, we obtain

$$y(t) = \mathcal{L}^{-1}[y(s)] = (-1 - 2e^{-t} + 3e^{-2t})u(t).$$

Solution 10.4-6

The problem provides

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \mathbf{D} = 1$$

$$x(t) = u(t) \quad \mathbf{q}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The Laplace transform method solution is

$$\mathbf{Y}(s) = \mathbf{CQ}(s) + \mathbf{DX}(s) = \mathbf{C\Phi}(s)\mathbf{q}(0) + [\mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}]\mathbf{X(s)}$$
$$= \mathbf{C}\{\mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)]\} + \mathbf{DX}(s).$$

Computing:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 2s + 2} \begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s^2+2s+2} & \frac{1}{s^2+2s+2} \\ \frac{-1}{s^2+2s+2} & \frac{s+1}{s^2+2s+2} \end{bmatrix}$$

$$\mathbf{BX}(s) = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} \quad \text{and} \quad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} 2 \\ \frac{s+1}{s} \end{bmatrix}$$

Hence,

$$\mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix} \begin{bmatrix} 2 \\ \frac{s+1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(s+1)}{(s+1)^2+1} + \frac{s+1}{s[(s+1)^2+1]} \\ \frac{-2}{(s+1)^2+1} + \frac{(s+1)^2}{s[(s+1)^2+1]} \end{bmatrix} = \begin{bmatrix} \frac{2s^2+3s+1}{s[(s+1)^2+1]} \\ \frac{s^2+1}{s[(s+1)^2+1]} \end{bmatrix}.$$

Now,

$$\mathbf{C\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{2s^2 + 3s + 1 + s^2 + 1}{s\{(s+1)^2 + 1\}} \end{bmatrix}.$$

Combined with $\mathbf{DX}(s) = \frac{1}{s}$, we see that

$$\mathbf{Y}(s) = \mathbf{C\Phi}(s)[\mathbf{q}(0) + \mathbf{BX}(s)] + \mathbf{DX}(s) = \frac{3s^2 + 3s + 2}{s\{(s+1)^2 + 1\}} + \frac{1}{s} = \frac{4s^2 + 5s + 4}{s\{(s+1)^2 + 1\}}.$$

Using partial fractions and clearing fractions we get

$$Y(s) = \frac{2}{s} + \frac{2s+1}{(s+1)^2 + 1^2} = \frac{2}{s} + 2\frac{(s+1)}{(s+1)^2 + 1^2} - \frac{1}{(s+1)^2 + 1^2}.$$

Inverting yields

$$y(t) = \mathcal{L}^{-1}[Y(s)] = (2 + 2e^{-t}\cos t - e^{-t}\sin t)u(t).$$

Solution 10.4-7

Expanding the cascade, we see that

$$H(s) = \left(\frac{1}{s+3}\right)\left(\frac{3s+10}{s+4}\right) = \frac{3s+10}{s^2+7s+12}.$$

This is the same transfer function as in Prob. 10.3-8, where the cascade form state equations were found to be

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}x$$

and

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

In this case:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+4 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+3)(s+4)}\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$$

Further,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{D} = 0.$$

Hence

$$\mathbf{\Phi}(s)\mathbf{B} = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+3} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3(s+3)+1}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix} = \begin{bmatrix} \frac{3s+10}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix}$$

and

$$\mathbf{C}\phi(s)\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{3s+10}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix} = \frac{3s+10}{(s+3)(s+4)}.$$

Since $\mathbf{H}(s) = \mathbf{C}\phi(s)\mathbf{B} + \mathbf{D}$, we see that

$$H(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} = \frac{3s+10}{s^2+7s+12}.$$

This confirms our earlier expansion of $H(s)$.

Solution 10.4-8

We know that

$$H(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D}.$$

The solutions to Prob. 10.4-5 shows that

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}.$$

Thus,

$$\mathbf{\Phi}(s)\mathbf{B} = \left[\begin{array}{cc} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{array}\right]\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} \frac{s}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} \end{array}\right]$$

and

$$\mathbf{C\Phi}(s)\mathbf{B} = \left[\begin{array}{cc} 0 & 1 \end{array}\right]\mathbf{\Phi}(s)\mathbf{B} = \frac{-2}{(s+1)(s+2)}.$$

Since $\mathbf{D} = 0$, we see that

$$H(s) = \mathbf{C\Phi}(s)\mathbf{B} = \frac{-2}{s^2 + 3s + 2}.$$

Solution 10.4-9

From Prob. 10.4-6,

$$\mathbf{\Phi}(s)\mathbf{B} = \left[\begin{array}{cc} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{array}\right]\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} \frac{1}{(s+1)^2+1} \\ \frac{s+1}{(s+1)^2+1} \end{array}\right]$$

and

$$\mathbf{C\Phi}(s)\mathbf{B} = \left[\begin{array}{cc} 1 & 1 \end{array}\right]\left[\begin{array}{c} \frac{1}{(s+1)^2+1} \\ \frac{s+1}{(s+1)^2+1} \end{array}\right] = \frac{s+1+1}{(s+1)^2+1} = \frac{s+2}{(s+1)^2+1}.$$

Thus,

$$H(s) = \mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D} = \frac{s+2}{(s+1)^2+1} + 1 = \frac{s^2+3s+4}{s^2+2s+2}.$$

Solution 10.4-10

The problem provides

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{Bx}$$
$$y = \mathbf{Cq} + \mathbf{Dx}$$

where

$$\mathbf{A} = \left[\begin{array}{cc} 0 & 1 \\ -1 & -2 \end{array}\right] \quad \mathbf{B} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \quad \mathbf{x} = \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right]$$

$$\mathbf{C} = \left[\begin{array}{cc} 1 & 2 \\ 4 & 1 \\ 1 & 1 \end{array}\right] \quad \mathbf{D} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{array}\right].$$

Computing:

$$s\mathbf{I} - \mathbf{A} = \left[\begin{array}{cc} s & -1 \\ 1 & s+2 \end{array}\right]$$

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)^2}\left[\begin{array}{cc} s+2 & 1 \\ -1 & s \end{array}\right] = \left[\begin{array}{cc} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{array}\right]$$

Furthermore,

$$\mathbf{\Phi}(s)\mathbf{B} = \left[\begin{array}{cc} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{cc} \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \\ \frac{s}{(s+1)^2} & \frac{-1}{(s+1)^2} \end{array}\right]$$

and

$$\mathbf{C\Phi}(s)\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \\ \frac{s}{(s+1)^2} & \frac{-1}{(s+1)^2} \end{bmatrix}.$$

The transfer function is thus

$$H(s) = \mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D} = \begin{bmatrix} \frac{2s+1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{4+s}{(s+1)^2} & \frac{4s+7}{(s+1)^2} \\ \frac{s+2}{s+1} & \frac{1}{s+1} \end{bmatrix}.$$

Solution 10.4-11

In the time domain, the solution $\mathbf{q}(t)$ is given by

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bx}(\tau)\,d\tau$$
$$= e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{Bx}(t),$$

where

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathcal{L}^{-1}(\mathbf{\Phi}(s)).$$

Using $\mathbf{\Phi}(s)$ from the solution to Prob. 10.4-1, we see that

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ \frac{-1}{s+1} + \frac{1}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}.$$

Thus,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} (2e^{-t} - e^{-2t})u(t) & (2e^{-t} - 2e^{-2t})u(t) \\ (-e^{-t} + e^{-2t})u(t) & (-e^{-t} + 2e^{-2t})u(t) \end{bmatrix}$$

and

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} (4e^{-t} - 2e^{-2t} + 2e^{-t} - 2e^{-2t})u(t) \\ (-2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t})u(t) \end{bmatrix} = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}.$$

Since $\mathbf{Bx}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times 0 = 0$, we therefore see that

$$\mathbf{q}(t) = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}.$$

This matches the result of Prob. 10.4-1.

Solution 10.4-12

From Prob. 10.4-2,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \\ \frac{1}{(s+2)(s+3)} & \frac{s+5}{(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{-2}{s+2} + \frac{3}{s+3} & \frac{-6}{s+2} + \frac{6}{s+3} \\ \frac{1}{s+2} - \frac{1}{s+3} & \frac{3}{s+2} - \frac{2}{s+3} \end{bmatrix}.$$

Hence,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} (-2e^{-2t} + 3e^{-3t})u(t) & (-6e^{-2t} + 6e^{-3t})u(t) \\ (e^{-2t} - e^{-3t})u(t) & (3e^{-2t} - 2e^{-3t})u(t) \end{bmatrix}$$

and

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} (-10e^{-2t} + 15e^{-3t} - 24e^{-2t} + 24e^{-3t})u(t) \\ (5e^{-2t} - 5e^{-3t} + 12e^{-2t} - 8e^{-3t})u(t) \end{bmatrix} = \begin{bmatrix} (-34e^{-2t} + 39e^{-3t})u(t) \\ (17e^{-2t} - 13e^{-3t})u(t) \end{bmatrix}.$$

Also,

$$\mathbf{Bx}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(100t)u(t) = \begin{bmatrix} \sin(100t)u(t) \\ 0 \end{bmatrix}$$

and

$$e^{\mathbf{A}t} * \mathbf{Bx}(t) = \begin{bmatrix} -2e^{-2t}u(t) * \sin(100t)u(t) + 3e^{-3t}u(t) * \sin(100t)u(t) \\ e^{-2t}u(t) * \sin(100t)u(t) - e^{-3t}u(t) * \sin(100t)u(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2e^{-2t}u(t)}{100} + \frac{2\cos(100t)u(t)}{100} + \frac{3e^{-3t}u(t)}{100} - \frac{3\cos(100t)u(t)}{100} \\ +\frac{e^{-2t}u(t)}{100} - \frac{\cos(100t)u(t)}{100} - \frac{e^{-3t}u(t)}{100} + \frac{\cos(100t)u(t)}{100} \end{bmatrix}$$

$$= \begin{bmatrix} -0.02e^{-2t}u(t) + 0.03e^{-3t}u(t) - 0.01\cos(100t)u(t) \\ 0.01e^{-2t}u(t) - 0.01e^{-3t}u(t) \end{bmatrix}.$$

Hence

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{Bx}(t) = \begin{bmatrix} \left(-34.02e^{-2t} + 39.03e^{-3t} - 0.01\cos 100t\right)u(t) \\ \left(17.01e^{-2t} - 13.01e^{-3t}\right)u(t) \end{bmatrix}.$$

This matches the result of Prob. 10.4-2.

Solution 10.4-13

From Prob. 10.4-3,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} \end{bmatrix}.$$

Hence,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} e^{-2t}u(t) & 0 \\ (e^{-t} - e^{-2t})u(t) & e^{-t}u(t) \end{bmatrix}$$

and

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} e^{-2t}u(t) & 0 \\ (e^{-t} - e^{-2t})u(t) & e^{-t}u(t) \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{-t}u(t) \end{bmatrix}.$$

Also,

$$\mathbf{Bx}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} u(t) \\ 0 \end{bmatrix}$$

and

$$e^{\mathbf{A}t} * \mathbf{Bx}(t) = \begin{bmatrix} e^{-2t}u(t) * u(t) \\ e^{-t}u(t) * u(t) - e^{-2t}u(t) * u(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 - e^{-2t})u(t) \\ (1 - e^{-t})u(t) - \frac{1}{2}(1 - e^{-2t})u(t) \end{bmatrix}.$$

Since $\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{Bx}(t)$, we see that

$$\mathbf{q}(t) = \begin{bmatrix} 0 \\ -e^{-t}u(t) \end{bmatrix} + \begin{bmatrix} (\frac{1}{2} - \frac{1}{2}e^{-2t})u(t) \\ (\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t})u(t) \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} - \frac{1}{2}e^{-2t})u(t) \\ (\frac{1}{2} + \frac{1}{2}e^{-2t} - 2e^{-t})u(t) \end{bmatrix}.$$

This matches the result of Prob. 10.4-3.

Solution 10.4-14

From Prob. 10.4-4,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}.$$

Hence,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} e^{-t}u(t) & (e^{-t} - e^{-2t})u(t) \\ 0 & e^{-2t}u(t) \end{bmatrix},$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} e^{-t}u(t) & (e^{-t} - e^{-2t})u(t) \\ 0 & e^{-2t}u(t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} (3e^{-t} - 2e^{-2t})u(t) \\ 2e^{-2t}u(t) \end{bmatrix},$$

$$\mathbf{Bx}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} u(t) + \delta(t) \\ \delta(t) \end{bmatrix},$$

and

$$e^{\mathbf{A}t} * \mathbf{Bx}(t) = \begin{bmatrix} e^{-t}u(t) * u(t) + e^{-t}u(t) * \delta(t) + e^{-t}u(t) * \delta(t) - e^{-2t}u(t) * \delta(t) \\ e^{-2t}u(t) * \delta(t) \end{bmatrix}$$

$$= \begin{bmatrix} (1 - e^{-t})u(t) + e^{-t}u(t) + e^{-t}u(t) - e^{-2t}u(t) \\ e^{-2t}u(t) \end{bmatrix} = \begin{bmatrix} (1 + e^{-t} - e^{-2t})u(t) \\ e^{-2t}u(t) \end{bmatrix}.$$

Thus,

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{Bx}(t) = \begin{bmatrix} (3e^{-t} - 2e^{-2t} + 1 + e^{-t} - e^{-2t})u(t) \\ (2e^{-2t} + e^{-2t})u(t) \end{bmatrix}$$

$$= \begin{bmatrix} (1 + 4e^{-t} - 3e^{-2t})u(t) \\ 3e^{-2t}u(t) \end{bmatrix}.$$

This matches the result of Prob. 10.4-4.

Solution 10.4-15

From Prob. 10.4-5,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}.$$

Now, output $y(t)$ is given by

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t}\mathbf{B} * x(t)] + \mathbf{Dx}(t),$$

where

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} (-e^{-t} + 2e^{-2t})u(t) & (e^{-t} - e^{-2t})u(t) \\ (-2e^{-t} + 2e^{-2t})u(t) & (2e^{-t} - e^{-2t})u(t) \end{bmatrix},$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} (-2e^{-t} + 4e^{-2t})u(t) \\ (-2e^{-t} + 4e^{-2t})u(t) \end{bmatrix},$$

and

$$e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-e^{-t} + 2e^{-2t})u(t) \\ (-2e^{-t} + 2e^{-2t})u(t) \end{bmatrix}.$$

Further,

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} (-e^{-t} + 2e^{-2t})u(t) \\ (-2e^{-t} + 2e^{-2t})u(t) \end{bmatrix} * u(t) = \begin{bmatrix} -e^{-t}u(t) * u(t) + e^{-2t}u(t) * u(t) \\ -2e^{-t}u(t) * u(t) + 2e^{-2t}u(t) * u(t) \end{bmatrix}$$

$$= \begin{bmatrix} (e^{-t} - e^{-2t})u(t) \\ (-1 + 2e^{-t} - e^{-2t})u(t) \end{bmatrix}.$$

Since $\mathbf{D} = 0 \implies y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)]$ and

$$e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} (-2e^{-t} + 4e^{-2t})u(t) \\ (-4e^{-t} + 4e^{-2t})u(t) \end{bmatrix} + \begin{bmatrix} (e^{-t} - e^{-2t})u(t) \\ (-1 + 2e^{-t} - e^{-2t})u(t) \end{bmatrix}$$

$$= \begin{bmatrix} (-e^{-t} + 3e^{-2t})u(t) \\ (-1 - 2e^{-t} + 3e^{-2t})u(t) \end{bmatrix},$$

we see that

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} (-e^{-t} + 3e^{-2t})u(t) \\ (-1 - 2e^{-t} + 3e^{-2t})u(t) \end{bmatrix} = (-1 - 2e^{-t} + 3e^{-2t})u(t).$$

This matches the result of Prob. 10.4-5.

Solution 10.4-16

To begin, recall that

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] + \mathbf{D}\mathbf{x}(t).$$

From Prob. 10.4-6,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}.$$

Hence

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}u(t),$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t}\cos t + e^{-t}\sin t \\ -2e^{-t}\sin t + e^{-t}\cos t \end{bmatrix}u(t),$$

and

$$e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}u(t).$$

Further,

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-t}\sin t\, u(t) * u(t) \\ e^{-t}\cos t\, u(t) * u(t) \end{bmatrix} = \begin{bmatrix} \frac{\cos(\frac{\pi}{2} - \phi)}{\sqrt{2}} - \frac{e^{-t}}{\sqrt{2}}\cos(t - \frac{\pi}{2} - \phi) \\ \frac{\cos(-\phi)}{\sqrt{2}} - \frac{e^{-t}}{\sqrt{2}}\cos(t - \phi) \end{bmatrix}u(t),$$

where $\phi = \tan^{-1}\frac{-1}{1} = -\frac{\pi}{4}$. Thus,

$$e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2} + \frac{3}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \\ \frac{1}{2} + \frac{1}{2}e^{-t}\cos t - \frac{3}{2}e^{-t}\sin t \end{bmatrix}u(t)$$

and

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] + \mathbf{D}\mathbf{x}(t)$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] + u(t)$$
$$= [1 + 2e^{-t}\cos t - e^{-t}\sin t + 1]u(t) = [2 + 2e^{-t}\cos t - e^{-t}\sin t]u(t).$$

This matches the result of Prob. 10.4-6.

Solution 10.4-17

Here,

$$H(s) = \frac{3s + 10}{s^2 + 7s + 12}.$$

From Eq. (10.45) we have

$$\mathbf{h}(t) = \mathbf{C}\phi(t)\mathbf{B} + \mathbf{D}\delta(t), \qquad \text{where } \phi(t) = e^{\mathbf{A}t}.$$

From Prob. 10.4-7,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+1} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{D} = 0.$$

Hence

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} e^{-4t} & e^{-3t} - e^{-4t} \\ 0 & e^{-3t} \end{bmatrix}u(t)$$

and

$$\phi(t)\mathbf{B} = \begin{bmatrix} 3e^{-4t} + e^{-3t} - e^{-4t} \\ e^{-3t} \end{bmatrix}u(t) = \begin{bmatrix} e^{-3t} + 2e^{-4t} \\ e^{-3t} \end{bmatrix}u(t).$$

Since $\mathbf{D} = 0$, we see that

$$h(t) = \mathbf{C}\phi(t)\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix}\phi(t)\mathbf{B} = (e^{-3t} + 2e^{-4t})u(t).$$

Solution 10.4-18

From Prob. 10.4-6,

$$\mathbf{\Phi}(s) = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}.$$

Hence

$$\phi(t) = \mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}u(t),$$

$$\phi(t)\mathbf{B} = \phi(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}u(t),$$

and

$$\mathbf{C}\phi(t)\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}\phi(t)\mathbf{B} = (e^{-t}\sin t + e^{-t}\cos t)u(t).$$

Finally,

$$h(t) = \mathbf{C}\phi(t)\mathbf{B} + \delta(t) = (e^{-t}\sin t + e^{-t}\cos t)u(t) + \delta(t).$$

Solution 10.4-19

From Prob. 10.4-10,

$$\phi(s) = \begin{bmatrix} \frac{2s+1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{4+s}{(s+1)^2} & \frac{4s+7}{(s+1)^2} \\ \frac{s+2}{s+1} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{(s+1)^2} & \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ \frac{1}{s+1} + \frac{3}{(s+1)^2} & \frac{4}{s+1} + \frac{3}{(s+1)^2} \\ 1 + \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}.$$

Inverting, the unit impulse response matrix $\mathbf{h}(t)$ is

$$\mathbf{h}(t) = \mathcal{L}^{-1}\{\mathbf{H}(s)\} = \begin{bmatrix} (2e^{-t} - te^{-t})u(t) & (e^{-t} - te^{-t})u(t) \\ (e^{-t} + 3te^{-t})u(t) & (4e^{-t} + 3te^{-t})u(t) \\ \delta(t) + e^{-t}u(t) & e^{-t}u(t) \end{bmatrix}.$$

Solution 10.5-1

Written in matrix form, the system state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x.$$

The new state vector $\mathbf{w}$ is

$$\mathbf{w} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{Pq}.$$

The new state equations of the system are given by

$$\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{P\dot{B}} = \hat{\mathbf{A}}\mathbf{w} + \hat{\mathbf{B}}x,$$

where

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \mathbf{PA} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix},$$

$$\mathbf{PAP}^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix},$$

and

$$\mathbf{PB} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x.$$

Hence the desired state equations are

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x.$$

Now, the eigenvalues are the roots of the characteristic equation. In the original system, we have

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 1 & s+1 \end{vmatrix} = (s+1)s + 1 = s^2 + s + 1 = 0.$$

The roots, and thus the eigenvalues, are thus

$$s_{1,2} = \frac{-1 \pm j\sqrt{3}}{2}.$$

In the transformed system, the characteristic equation is given by

$$|s\mathbf{I} - \hat{\mathbf{A}}| = \left| \begin{bmatrix} s+2 & -1 \\ 3 & s-1 \end{bmatrix} \right| = (s+2)(s-1) + 3 = s^2 - s + 2s - 2 + 3 = s^2 + s + 1$$

and the eigenvalues are given by

$$s_{1,\,2} = \frac{-1 \pm j\sqrt{3}}{2}.$$

Notice that the eigenvalues are the same for the original and transformed systems.

Solution 10.5-2

Written in matrix form, the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} x(t).$$

(a) The characteristic equation is given by

$$|s\mathbf{I} - \mathbf{A}| = 0 = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) + 2 = s^2 + 3s + 2 = (s+1)(s+2) = 0.$$

Clearly, $\lambda_1 = -1$ and $\lambda_2 = -2$ are the eigenvalues, and

$$\mathbf{\Lambda} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

The transformed state vector $\mathbf{w}$ has $\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x = \mathbf{\Lambda w} + \hat{\mathbf{B}}x$. To produce the desired diagonalized form, we therefore have to find $\mathbf{P}$ such that $\mathbf{PAP}^{-1} = \mathbf{\Lambda}$ or $\mathbf{\Lambda P} = \mathbf{PA}$. That is, we need

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Solving:

$$\left. \begin{aligned} -p_{11} &= -2p_{12} \\ -p_{12} &= p_{11} - 3p_{12} \\ -2p_{21} &= -2p_{22} \\ -2p_{22} &= p_{21} - 3p_{22} \end{aligned} \right\} \implies \begin{aligned} & p_{11} = 2p_{12} \quad \text{and} \quad p_{21} = p_{22} \\ & \text{If we choose } p_{11} = 2 \text{ and } p_{21} = 1, \\ & \text{then } p_{12} = 1 \text{ and } p_{22} = 1 \, . \end{aligned}$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2q_1 + q_2 \\ q_1 + q_2 \end{bmatrix}.$$

(b) The system output is given by $\mathbf{y} = \mathbf{Cq} + \mathbf{D}x$. Since $\mathbf{D} = 0$ we see that $\mathbf{y} = \mathbf{Cq}$. Further, $\mathbf{w} = \mathbf{Pq} \implies \mathbf{P}^{-1}\mathbf{w} = \mathbf{q}$ so that $\mathbf{y} = \mathbf{CP}^{-1}\mathbf{w}$. Now,

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix},$$

so that

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ 5w_2 - 3w_1 \end{bmatrix}.$$

Solution 10.5-3

From the problem statement, we have

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x.$$

The characteristic equation is given by:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{vmatrix} = s\{(s)(s+3) + 2\} = s(s+1)(s+2) = 0.$$

Clearly the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = -2$, so that

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

In the transformed system we have $\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x$. To achieve diagonalization, we need to find $\mathbf{P}$ such that $\mathbf{PAP}^{-1} = \mathbf{\Lambda}$ or $\mathbf{\Lambda P} = \mathbf{PA}$. That is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ -p_{21} & -p_{22} & -p_{23} \\ -2p_{31} & -2p_{32} & -2p_{33} \end{bmatrix} = \begin{bmatrix} 0 & p_{11} - 2p_{13} & p_{12} - 3p_{13} \\ 0 & p_{21} - 2p_{23} & p_{22} - 3p_{23} \\ 0 & p_{31} - 2p_{33} & p_{32} - 3p_{33} \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & p_{11} = 2p_{13} & p_{12} = 3p_{13} \\ p_{21} = 0 & p_{21} + p_{22} = 2p_{23} & p_{22} + p_{23} = 3p_{23} \\ p_{31} = 0 & p_{31} + 2p_{32} = 2p_{33} & p_{32} + 2p_{33} = 3p_{33} \end{bmatrix}.$$

Solving:

$$\begin{aligned} p_{21} &= 0 \qquad p_{31} = 0 \\ p_{11} &= 2p_{13} &&\text{if } p_{11} = 2, \text{ then } p_{13} = 1 \\ p_{12} &= 3p_{13} &&\text{and } p_{12} = 3 \\ p_{22} &= 2p_{23} - p_{21} &&\text{if } p_{23} = 1, \text{ then } p_{22} = 2 \\ p_{23} &= 3p_{23} - p_{22} \\ 2p_{33} &= 3p_{33} - p_{32} \implies p_{32} = p_{33} \\ 2p_{32} &= 2p_{33} - p_{31} &&\text{if } p_{32} = 1, \text{ then } p_{33} = 1 \end{aligned}$$

Thus one (non-unique) solution is

$$\mathbf{w} = \mathbf{Pq} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Solution 10.5-4

From the time-domain method, we know that

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{Bx}(t)],$$

where

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\boldsymbol{\phi}(s)).$$

In this problem,

$$(\boldsymbol{\phi}(s))^{-1} = [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+3 & 0 \\ 0 & 0 & s+2 \end{bmatrix}$$

and

$$\phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}.$$

Thus,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t}u(t) & 0 & 0 \\ 0 & e^{-3t}u(t) & 0 \\ 0 & 0 & e^{-2t}u(t) \end{bmatrix},$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}u(t) \\ 2e^{-3t}u(t) \\ e^{-2t}u(t) \end{bmatrix}, \quad \text{and} \quad e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}u(t) \\ e^{-3t}u(t) \\ e^{-2t}u(t) \end{bmatrix}.$$

Further,

$$e^{\mathbf{A}t} * \mathbf{B}x(t) = e^{\mathbf{A}t} * \mathbf{B}u(t) = \begin{bmatrix} e^{-t}u(t) * u(t) \\ e^{-3t}u(t) * u(t) \\ e^{-2t}u(t) * u(t) \end{bmatrix} = \begin{bmatrix} (1 - e^{-t})u(t) \\ \frac{1}{3}(1 - e^{-3t})u(t) \\ \frac{1}{2}(1 - e^{-2t})u(t) \end{bmatrix}.$$

Hence,

$$e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}x(t) = \begin{bmatrix} e^{-t} + 1 - e^{-t} \\ 2e^{-3t} + \frac{1}{3} - \frac{1}{3}e^{-3t} \\ e^{-2t} + \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} + \frac{5}{3}e^{-3t} \\ \frac{1}{2} + \frac{1}{2}e^{-2t} \end{bmatrix}.$$

Using $\mathbf{C} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$ with $y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}x(t)]$, we obtain

$$y(t) = \left(1 + 1 + 5e^{-3t} + \frac{1}{2} + \frac{1}{2}e^{-2t}\right)u(t) = \left(\frac{5}{2} + \frac{1}{2}e^{-2t} + 5e^{-3t}\right)u(t).$$

Solution 10.6-1

(a) Refer to the block diagram on the left side of Fig. S10.6-1. Deriving the state equations, we see that

$$\dot{q}_2 + bq_2 = (a - b)x \implies \dot{q}_2 = -bq_2 + (a - b)x$$
$$\dot{q}_1 + aq_1 = q_2 + x \implies \dot{q}_1 = -aq_1 + q_2 + x$$

so that

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ (a - b) \end{bmatrix}x.$$

The output equation is

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Now, the characteristic equation is

$$|s\mathbf{I} - \mathbf{A}| = 0 = \begin{vmatrix} s + a & -1 \\ 0 & s + b \end{vmatrix} = (s + a)(s + b) = 0.$$

Clearly, $\lambda_1 = -a$ and $\lambda_2 = -b$ are the eigenvalues and

$$\mathbf{\Lambda} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}.$$

Under transformation, $\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x$. We are looking for $\mathbf{P}$ such that $\mathbf{PAP}^{-1} = \mathbf{\Lambda}$ or $\mathbf{\Lambda P} = \mathbf{PA}$. That is, we require that

$$\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix}.$$

Solving:

$$\left. \begin{array}{l} -ap_{11} = -ap_{11} \\ -bp_{21} = -ap_{21} \Longrightarrow p_{21} = 0 \\ -ap_{12} = p_{11} - bp_{12} = 0 \\ -bp_{22} = p_{21} - bp_{22} \Longrightarrow p_{21} = 0 \end{array} \right\} \Longrightarrow \begin{array}{l} \text{If } p_{11} = (b-a), \text{ then } p_{12} = 1, p_{21} = 0, \\ \text{and } p_{22} \text{ can be anything.} \\ \text{Let us take } p_{22} = 1. \end{array}$$

Thus,

$$\mathbf{w} = \mathbf{Pq} = \begin{bmatrix} b-a & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Now, the output in terms of $\mathbf{w}$ is $y = \mathbf{Cq} = \mathbf{CP}^{-1}\mathbf{w} = \hat{\mathbf{C}}\mathbf{w}$, where

$$\mathbf{P}^{-1} = \frac{1}{b-a} \begin{bmatrix} 1 & -1 \\ 0 & b-a \end{bmatrix} = \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \\ 0 & 1 \end{bmatrix}.$$

Observability: In the new (diagonalized form) system,

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \end{bmatrix}.$$

We notice that in $\hat{\mathbf{C}}$, there is no column with all zeros, hence we conclude that the system is observable.

Controllability: In the new (diagonalized form) system,

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} b-a & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a-b \end{bmatrix} = \begin{bmatrix} 0 \\ a-b \end{bmatrix}.$$

Since the first row in $\hat{\mathbf{B}}$ is zero, we see that this system is not controllable.

(b) Refer to the block diagram on the right side of Fig. S10.6-1. The state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x,$$

and the output equation is

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

Although matrix A is already in the diagonal form, we can use it transform the system to another diagonal form using

$$\mathbf{P} = \mathbf{A} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix} \Longrightarrow \mathbf{P}^{-1} = \frac{1}{ab} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} -\frac{1}{b} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix}.$$

In the transformed system $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x = \mathbf{Aw} + \hat{\mathbf{B}}x$.

Observability: In the new (diagonalized form) system,

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{b} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix} = \begin{bmatrix} -\frac{1}{b} & 0 \end{bmatrix}.$$

Since the second column in $\hat{\mathbf{C}}$ is zero, this system is not observable. The same conclusion can also be drawn using the original matrix $\mathbf{C}$.

Controllability: In the new (diagonalized form) system,

$$\hat{\mathbf{B}} = \mathbf{PB} = \left[\begin{array}{cc} -b & 0 \\ 0 & -a \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \end{array} \right] = \left[\begin{array}{c} -b \\ -a \end{array} \right].$$

Since there is no row of zeros in $\hat{\mathbf{B}}$, this system is controllable.

X(s) —▶ [$\frac{a-b}{s+b}$]$_{q_2}$ —▶ (Σ) —▶ [$\frac{1}{s+a}$]$_{q_1}$ $\overset{Y(s)}{\longrightarrow}$ X(s) —▶ [$\frac{1}{s+a}$]$_{q_2}$ —▶ [$\frac{s+a}{s+b}$]$_{q_1}$ $\overset{Y(s)}{\longrightarrow}$
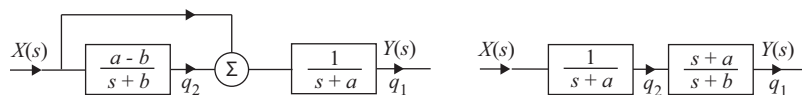
Figure S10.6-1

Solution 10.7-1

(a) Using the time-domain method, the output $y[n]$ is given by

$$y[n] = \mathbf{CA}^n \mathbf{q}[0] + \mathbf{CA}^{n-1} u[n-1] * \mathbf{B}x[n] + \mathbf{D}x[n].$$

The characteristic equation of $\mathbf{A}$ is

$$|\lambda \mathbf{I} - \mathbf{A}| = \left| \begin{array}{cc} \lambda - 2 & 0 \\ -1 & \lambda - 1 \end{array} \right| = (\lambda - 1)(\lambda - 2) = 0,$$

so $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of $\mathbf{A}$. Also,

$$\mathbf{A}^n = \beta_0 \mathbf{I} + \beta_1 \mathbf{A},$$

where

$$\left[\begin{array}{c} \beta_0 \\ \beta_1 \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right]^{-1} \left[\begin{array}{c} 1 \\ 2^n \end{array} \right] = \left[\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ 2^n \end{array} \right] = \left[\begin{array}{c} 2 - 2^n \\ -1 + 2^n \end{array} \right].$$

Hence

$$\mathbf{A}^n = \left[\begin{array}{cc} \beta_0 & 0 \\ 0 & \beta_0 \end{array} \right] + \left[\begin{array}{cc} 2\beta_1 & 0 \\ \beta_1 & \beta_1 \end{array} \right] = \left[\begin{array}{cc} 2^n & 0 \\ 2^n - 1 & 1 \end{array} \right]$$

and

$$\mathbf{CA}^n = \left[\begin{array}{cc} 0 & 1 \end{array} \right] \mathbf{A}^n = \left[\begin{array}{cc} 2^n - 1 & 1 \end{array} \right].$$

The zero-input response is

$$y_{\mathrm{zir}}[n] = \mathbf{CA}^n \mathbf{q}(0) = \mathbf{CA}^n \left[\begin{array}{c} 2 \\ 1 \end{array} \right] = (2^{n+1} - 1)u[n].$$

The zero-state component is given by

$$y_{\mathrm{zsr}}[n] = \mathbf{CA}^{n-1} u[n-1] * \mathbf{B}x[n] + \mathbf{D}x[n].$$

But

$$\mathbf{CA}^n u[n] * \mathbf{B}x[n] = \left[\begin{array}{cc} 2^n - 1 & 1 \end{array} \right] u[n] * \left[\begin{array}{c} 0 \\ u[n] \end{array} \right] = (n+1)u[n],$$

so that

$$y_{\mathrm{zsr}}[n] = nu[n-1] + \mathbf{D}x[n] = nu[n-1] + u[n] = (n+1)u[n].$$

Thus,

$$y[n] = y_{\mathrm{zir}}[n] + y_{\mathrm{zsr}}[n] = [2^{n+1} + n]u[n]$$

(b) Using the frequency-domain method, the output $Y[z]$ is given by

$$\mathbf{Y}[z] = \mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{q}[0] + [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]X[z].$$

Now,

$$(\mathbf{I} - z^{-1}\mathbf{A})^{-1} = \begin{bmatrix} 1 - 2z^{-1} & 0 \\ -z^{-1} & 1 - z^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} 1 - \frac{2}{z} & 0 \\ -\frac{1}{z} & 1 - \frac{1}{z} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{z-2}{z} & 0 \\ -\frac{1}{z} & \frac{z-1}{z} \end{bmatrix}^{-1}$$

$$= \frac{z^2}{(z-1)(z-2)} \begin{bmatrix} \frac{z-1}{z} & 0 \\ \frac{1}{z} & \frac{z-2}{z} \end{bmatrix} = \begin{bmatrix} \frac{z}{z-2} & 0 \\ \frac{z}{(z-1)(z-2)} & \frac{z}{z-1} \end{bmatrix}$$

and

$$(z\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} z-2 & 0 \\ -1 & z-1 \end{bmatrix}^{-1} = \frac{1}{(z-1)(z-2)} \begin{bmatrix} z-1 & 0 \\ 1 & z-2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{z-2} & 0 \\ \frac{1}{(z-1)(z-2)} & \frac{1}{z-1} \end{bmatrix}.$$

Thus,

$$\mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1} = \begin{bmatrix} \frac{z}{(z-1)(z-2)} & \frac{z}{z-1} \end{bmatrix}.$$

and

$$\mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{q}(0) = \begin{bmatrix} \frac{2z}{(z-1)(z-2)} + \frac{z}{z-1} \end{bmatrix} = \frac{z^2}{(z-1)(z-2)}.$$

Also,

$$\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{(z-1)(z-2)} & \frac{1}{z-1} \end{bmatrix} \quad \text{and} \quad \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{z-1}.$$

Hence

$$\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{1}{z-1} + \mathbf{D} = \frac{1}{z-1} + 1 = \frac{z}{z-1}.$$

Now, $x[n] = u[n]$ and $X[z] = \frac{z}{z-1}$ so that

$$(\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D})X[z] = \left[\frac{z}{z-1}\right]^2 = \frac{z^2}{(z-1)^2}.$$

Thus,

$$\mathbf{Y}[z] = \mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{q}(0) + [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]X[z] = \frac{z^2}{(z-1)(z-2)} + \frac{z^2}{(z-1)^2},$$

$$\frac{\mathbf{Y}[z]}{z} = \frac{1}{z-2} + \frac{z}{(z-1)^2} = \frac{2}{z-2} + \frac{1}{(z-1)^2},$$

and

$$\mathbf{Y}[z] = \frac{2z}{z-2} + \frac{z}{(z-1)^2}.$$

Inverting, we obtain

$$y[n] = z^{-1}[\mathbf{Y}[z]] = [2^n + 1]u[n] + (n+1)u[n] = [2^{n+1} + n]u[n].$$

As hoped, this result matches the result of part (a).

Solution 10.7-2

In advance operator form, the system is described as

$$(E^2 + E + 0.16)\{y[n]\} = (E + 0.32)\{x[n]\}.$$

(a) In this case,

$$H[z] = \frac{Y[z]}{X[z]} = \frac{z + 0.32}{z^2 + z + 0.16} = \frac{z + 0.32}{(z + 0.2)(z + 0.8)} = \frac{0.2}{z + 0.2} + \frac{0.8}{z + 0.8}.$$

The corresponding DFII, TDFII, cascade, and parallel system realizations are shown in Fig. S10.7-2.

Figure S10.7-2

(b) Direct Form II: Using the output of each delay as a state variable (see the controller canonical diagram in Fig. S10.7-2) we get

$$q_1[n + 1] = q_2[n]$$
$$q_2[n + 1] = -0.16q_1[n] - q_2[n] + x[n].$$

The DFII state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = 0.32q_1[n] + q_[n] = \begin{bmatrix} 0.32 & 1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}.$$

Transposed Direct Form II: Using the output of each delay as a state variable (see the observer canonical diagram in Fig. S10.7-2) we get

$$q_1[n+1] = -q_1[n] + q_2[n] + x[n]$$
$$q_2[n+1] = -0.16q_1[n] + 0.32x[n].$$

The TDFII state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -0.16 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0.32 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = q_1[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}.$$

Cascade Form: Using the output of each delay as a state variable (see the cascade diagram in Fig. S10.7-2) we get

$$q_1[n+1] = -0.8q_1[n] + q_2[n]$$
$$q_2[n+1] = -0.2q_2[n] + x[n].$$

The cascade-form state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.8 & 1 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = 0.32q_1[n] - 0.8q_1[n] + q_2[n] = \begin{bmatrix} -0.48 & 1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}.$$

Parallel Form: Using the output of each delay as a state variable (see the parallel diagram in Fig. S10.7-2) we get

$$q_1[n+1] = -0.2q_1[n] + x[n]$$
$$q_2[n+1] = -0.8q_2[n] + x[n].$$

The parallel-form state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = 0.2q_1[n] + 0.8q_2[n] = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}.$$

Solution 10.7-3

In advance operator form, the system is described as

$$E(2E+1)\{y[n]\} = (E^2 + E - 6)\{x[n]\}.$$

(a) In this case,

$$H[z] = \frac{Y[z]}{X[z]} = \frac{2z^2 + z}{z^2 + z - 6} = \left(\frac{z}{z-2}\right)\left(\frac{2z+1}{z+3}\right) = \frac{z}{z-2} + \frac{z}{z+3}.$$

The corresponding DFII, TDFII, cascade, and parallel system realizations are shown in Fig. S10.7-3.
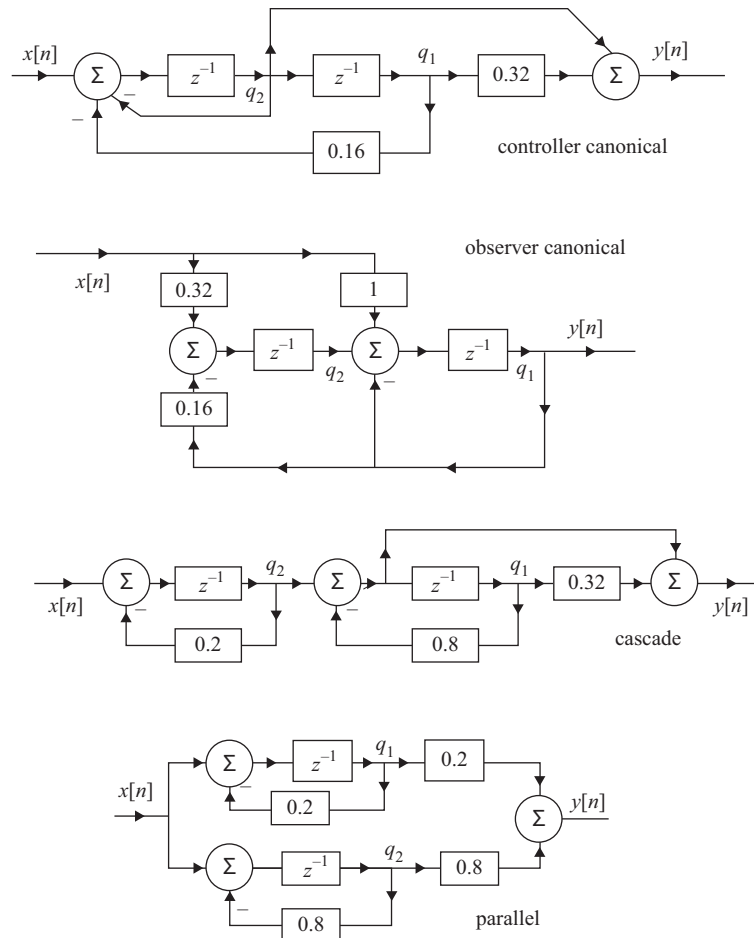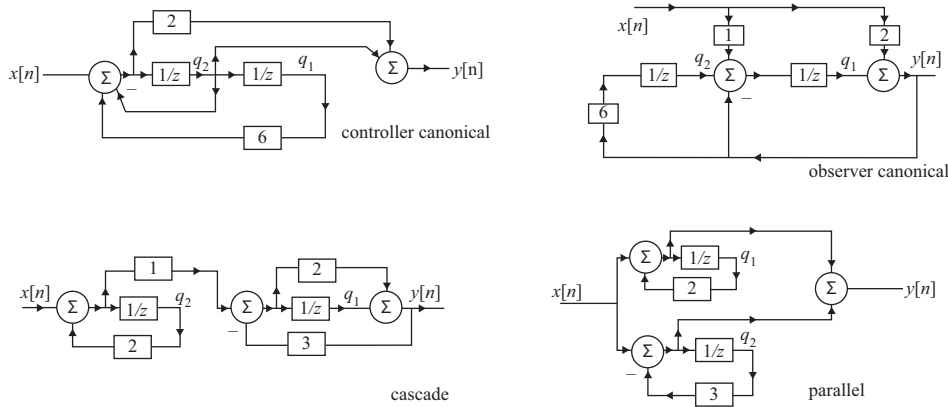

Figure S10.7-3

(b) **Direct Form II:** Using the output of each delay as a state variable (see the controller canonical diagram in Fig. S10.7-3) we get

$$q_1[n+1] = q_2[n]$$
$$q_2[n+1] = 6q_1[n] - q_2[n] + x[n].$$

The DFII state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = q_2[n] + 2[6q_1[n] - q_2[n] + x[n]] = 12q_1[n] - 2q_2[n] + 2x[n]$$

or

$$y[n] = \begin{bmatrix} 12 & -2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n].$$

Transposed Direct Form II: Using the output of each delay as a state variable (see the observer canonical diagram in Fig. S10.7-3) we get

$$q_1[n+1] = -q_1[n] + q_2[n] - x[n]$$
$$q_2[n+1] = 6q_1[n] + 12x[n].$$

The TDFII state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} -1 \\ 12 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = q_1[n] + 2x[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n].$$

Cascade Form: Using the output of each delay as a state variable (see the cascade diagram in Fig. S10.7-3) we get

$$q_1[n+1] = -\frac{3}{7}q_1[n] + \frac{2}{7}q_2[n] + \frac{1}{7}x[n]$$

$$q_2[n+1] = 2q_2[n] + x[n].$$

The cascade-form state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ 0 & 2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = \frac{1}{7}q_1[n] + \frac{4}{7}q_2[n] + \frac{2}{7}x[n] = \begin{bmatrix} \frac{1}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \frac{2}{7}x[n].$$

Parallel Form: Using the output of each delay as a state variable (see the parallel diagram in Fig. S10.7-3) we get

$$q_1[n+1] = 2q_1[n] + x[n]$$

$$q_2[n+1] = -3q_2[n] + x[n].$$

The parallel-form state equations are thus

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n],$$

and the output equation is

$$y[n] = 2q_1[n] + x[n] + x[n] - 3q_2[n] = \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n].$$

Solution 10.8-1

Figure S10.8-1 is used to help determine the state and output equations.
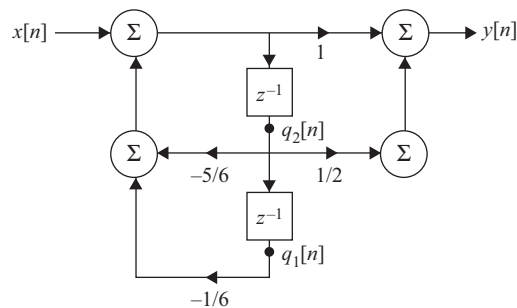
Figure S10.8-1

Directly from the diagram, note that $q_1[n+1] = q_2[n] + 0x[n]$ and $q_2[n+1] = -\frac{5}{6}q_2[n] - \frac{1}{6}q_1[n] + x[n]$. Taken together, the state equation is therefore

$$\mathbf{Q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n] = \mathbf{AQ}[n] + \mathbf{B}x[n].$$

The diagram is also used to write the output equation as $y[n] = \frac{1}{2}q_2[n] - \frac{5}{6}q_2[n] - \frac{1}{6}q_1[n] + x[n]$. Simplifying, the output equation is

$$y[n] = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 1x[n] = \mathbf{C}\mathbf{Q}[n] + \mathbf{D}x[n].$$

Solution 10.8-2

Figure S10.8-2 is used to help determine the state and output equations.
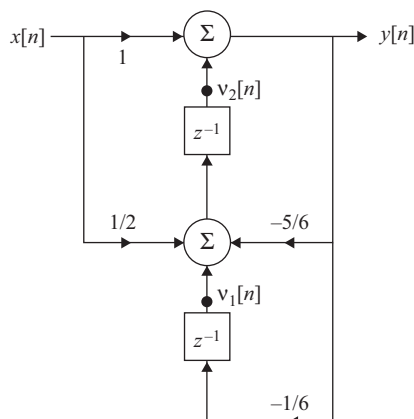
Figure S10.8-2

Directly from the diagram, note that $y[n] = v_2[n] + x[n]$. In standard form, the output equation is thus

$$y[n] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + 1x[n] = \mathbf{C}\mathbf{V}[n] + \mathbf{D}x[n].$$

Also using the diagram, note that $v_2[n+1] = v_1[n] + -\frac{5}{6}y[n] + \frac{1}{2}x[n]$ and $v_1[n+1] = -\frac{1}{6}y[n]$. Substituting $y[n] = v_2[n] + x[n]$ into each yields $v_2[n+1] = v_1[n] - \frac{5}{6}(v_2[n] + x[n]) + \frac{1}{2}x[n]$ and $v_1[n+1] = -\frac{1}{6}(v_2[n] + x[n])$. Simplifying to standard form, the state equations are represented in matrix form as

$$\mathbf{V}[n+1] = \begin{bmatrix} v_1[n+1] \\ v_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{6} \\ 1 & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} x[n] = \mathbf{A}\mathbf{V}[n] + \mathbf{B}x[n].$$

Chapter B Solutions

Solution B.1-1

Given $w = re^{j\theta} = r\left(\cos(\theta) + j\sin(\theta)\right) = x + jy$,

$$w^* = (x + jy)^* = x - jy = r\left(\cos(\theta) - j\sin(\theta)\right) = re^{-j\theta}.$$

Solution B.1-2

(a) For $w_a = 1 + j$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \arctan\left(\frac{1}{1}\right) = \pi/4 = 0.7854$. Thus,

$$w_a = 1 + j = \sqrt{2}e^{j\pi/4} = 1.414e^{j0.7854}.$$

(b) Here, $w_b = e^j + 1 = \cos(1) + j\sin(1) + 1$. Thus, $r = \sqrt{(\cos(1) + 1)^2 + (\sin(1))^2} = 1.7552$ and $\theta = \arctan\left(\frac{\sin(1)}{\cos(1) + 1}\right) = 0.500$, which yields

$$w_b = e^j + 1 = 1.7552e^{j/2}.$$

(c) For $w_c = -4 + j3$, $r = \sqrt{(-4)^2 + 3^2} = 5$ and $\theta = \arctan\left(\frac{3}{-4}\right) = -0.643 + \pi = 2.4981$. Thus,

$$w_c = -4 + j3 = 5e^{j2.4981}.$$

(d) Using the results from parts (a) and (c),

$$w_d = w_a w_c = (1 + j)(-4 + j3) = (\sqrt{2}e^{j\pi/4})(5e^{j2.4981}) = 7.0711e^{j3.2835} = 7.0711e^{-j2.9997}.$$

(e) Here, $w_e = e^{j\pi/4} + 2e^{-j\pi/4} = \frac{1+j}{\sqrt{2}} + \frac{2-j2}{\sqrt{2}} = \frac{3-j}{\sqrt{2}}$. Thus, $r = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{5} = 2.2361$ and $\theta = \arctan\left(\frac{-1}{3}\right) = -0.3218$, which yields

$$w_e = e^{j\pi/4} + 2e^{-j\pi/4} = 2.2361e^{-j0.3218}.$$

(f) For $w_f = \frac{1+j}{2j} = \frac{1}{2} - \frac{j}{2}$, $r = \sqrt{\frac{1}{4} + \frac{1}{4}} = 0.7071$ and $\theta = \arctan\left(\frac{-1/2}{1/2}\right) = -0.7854$. Thus,

$$w_f = \frac{1 + j}{2j} = 0.7071e^{-j0.7854}.$$

(g) Using the results from parts (a) and (c),

$$w_{\rm g} = \frac{w_{\rm a}}{w_{\rm c}} = \frac{1+j}{-4+j3} = \frac{\sqrt{2}e^{j\pi/4}}{5e^{j2.4981}} = 0.2828e^{-j1.7127}.$$

(h) Here, $w_{\rm h} = \frac{1-j}{\sin(j)}$. Following the procedure for part (a), we see that $1 - j = 1.414e^{-j0.7854}$. Using Euler's, we see that $\sin(j) = \frac{e^{j(j)}-e^{-j(j)}}{2j} = \frac{e^{-1}-e^{1}}{2j} = \frac{-2.3504}{2j} = 1.1752j = 1.1752e^{j\pi/2}$. Combining, we obtain

$$w_{\rm h} = \frac{1-j}{\sin(j)} = \frac{1.414e^{-j0.7854}}{1.1752e^{j\pi/2}} = 1.2034e^{-j2.3562}.$$

Solution B.1-3

(a) Using Euler's identity,

$$w_{\rm a} = j + e^j = j + \cos(1) + j\sin(1) = \cos(1) + j(1 + \sin(1)) = 0.5403 + j1.8415$$

(b) Using Euler's identity,

$$w_{\rm b} = 3e^{j\pi/4} = 3\cos(\pi/4) + j3\sin(\pi/4) = 2.1213 + j2.1213.$$

(c) Using Euler's identity,

$$w_{\rm c} = \frac{1}{e^j} = e^{-j} = \cos(-1) + j\sin(-1) = 0.5403 - j0.8415.$$

(d) Expanding,
$$w_{\rm d} = (1+j)(-4+j3) = (-4-3) + j(-4+3) = -7 - j.$$

(e) Using Euler's identity,

$$w_{\rm e} = e^{j\pi/4} + 2e^{-j\pi/4} = \frac{1+j}{\sqrt{2}} + \frac{2-j2}{\sqrt{2}} = \frac{3}{\sqrt{2}} + j\frac{-1}{\sqrt{2}} = 2.1213 - j0.7071.$$

(f) Using Euler's identity,

$$w_{\rm f} = e^j + 1 = \cos(1) + j\sin(1) + 1 = (\cos(1) + 1) + j\sin(1) = 1.5403 + j0.8415.$$

(g) Start by expressing the denominator in standard polar form, $\frac{1}{2^j} = \frac{1}{e^{j\ln(2)}} = e^{-j\ln(2)}$. Using Euler's identity,

$$w_{\rm g} = \frac{1}{2^j} = \cos(\ln(2)) - j\sin(\ln(2)) = 0.7692 - j0.6390.$$

(h) To begin, we notice that $j = e^{j\pi/2}$. Thus, $j^j = (e^{j\pi/2})^j = e^{-\pi/2}$. Continuing, we see that $j^{j^j} = (e^{j\pi/2})^{e^{-\pi/2}} = e^{j(\pi e^{-\pi/2}/2)}$, the last step of which is in standard polar form. Thus,

$$w_{\rm h} = j^{j^j} = \cos\left(\pi e^{-\pi/2}/2\right) + j\sin\left(\pi e^{-\pi/2}/2\right) = 0.9472 + j0.3208.$$

It is worthwhile noting that $a^{b^c} \neq (a^b)^c = a^{bc}$. Thus, $j^{j^j} \neq (j^j)^j = j^{-1} = -j$.

Solution B.1-4

(a) $\mathrm{Re}\,(w_a) = \mathrm{Re}\left(\frac{1}{j}(j - 5e^{2-3j})\right) = \mathrm{Re}\left(1 + 5je^2\left(\cos(-3) + j\sin(-3)\right)\right) = 1 + 5e^2\sin(3) = 6.2137.$

(b) $\mathrm{Re}\,(w_b) \quad = \quad \mathrm{Re}\,((1+j)\ln(1+j)) \quad = \quad \mathrm{Re}\left((1+j)\ln\left(\sqrt{2}e^{j(\frac{\pi}{4}+2\pi k)}\right)\right) \quad =$
$\mathrm{Re}\left((1+j)\ln\left(\sqrt{2}\right) + j\left(\frac{\pi}{4} + 2\pi k\right)\right) \quad = \quad \mathrm{Re}\left(\ln\left(\sqrt{2}\right) - \frac{\pi}{4} - 2\pi k + j\left(\ln\left(\sqrt{2}\right) + \frac{\pi}{4} + 2\pi k\right)\right) \quad =$
$\ln\left(\sqrt{2}\right) - \frac{\pi}{4} - 2\pi k = -0.4388 - 2\pi k.$

Solution B.1-5

(a) $\mathrm{Im}\,(w_a) = \mathrm{Im}\left(-je^{j\pi/4}\right) = \mathrm{Im}\left(-j\cos(\pi/4) + \sin(\pi/4)\right) = -\cos(\pi/4) = -0.7071.$

(b) $\mathrm{Im}\,(w_b) \quad = \quad \mathrm{Im}\left(1 - 2je^{2-4j}\right) \quad = \quad \mathrm{Im}\left(1 - 2je^2\left(\cos(4) - j\sin(4)\right)\right) \quad =$
$\mathrm{Im}\left(1 - 2e^2\sin(4) - j2e^2\cos(4)\right) = -2e^2\cos(4) = 9.6596.$

(c) $\mathrm{Im}\,(w_c) \quad = \quad \mathrm{Im}\,(\tan(j)) \quad = \quad \mathrm{Im}\left(\frac{\sin(j)}{\cos(j)}\right) \quad = \quad \mathrm{Im}\left(\frac{\frac{1}{2j}\left(e^{j(j)} - e^{-j(j)}\right)}{\frac{1}{2}\left(e^{j(j)} + e^{-j(j)}\right)}\right) \quad = \quad \mathrm{Im}\left(-j\frac{\left(e^{-1} - e^{1}\right)}{\left(e^{-1} + e^{1}\right)}\right) \quad =$
$\frac{\left(e^{1} - e^{-1}\right)}{\left(e^{1} + e^{-1}\right)} = 0.7616.$

Solution B.1-6

For each proof, substitute the Cartesian form $w = x + jy$.

(a)
$$\frac{w + w^*}{2} = \frac{x + jy + x - jy}{2} = x = \mathrm{Re}\,(x + jy) = \mathrm{Re}\,(w).$$

(b)
$$\frac{w - w^*}{2j} = \frac{x + jy - x + jy}{2j} = y = \mathrm{Im}\,(x + jy) = \mathrm{Im}\,(w).$$

Solution B.1-7

(a) Using the previous result that $\mathrm{Re}\,(w) = \frac{w+w^*}{2}$, $\mathrm{Re}\,(e^w) = \mathrm{Re}\left(e^{x-jy}\right) = \frac{e^x e^{-jy} + e^x e^{jy}}{2} = e^x \frac{e^{-jy} + e^{jy}}{2}$. Using Euler's identity yields

$$\mathrm{Re}\,(e^w) = e^x\cos(y).$$

(b) Using the previous result that $\mathrm{Im}\,(w) = \frac{w-w^*}{2j}$, $\mathrm{Im}\,(e^w) = \mathrm{Im}\left(e^{x-jy}\right) = \frac{e^x e^{-jy} - e^x e^{jy}}{2j} = e^x \frac{e^{-jy} - e^{jy}}{2j}$. Using Euler's identity yields

$$\mathrm{Re}\,(e^w) = -e^x\sin(y).$$

Solution B.1-8

For arbitrary complex constants w_1 and w_2,

(a) $\mathrm{Re}\,(jw_1) = \mathrm{Re}\,(j(x_1 + jy_1)) = \mathrm{Re}\,(-y_1 + jx_1) = -y_1$. Also, $-\mathrm{Im}\,(w_1) = -\mathrm{Im}\,(x_1 + jy_1) = -y_1$. Thus,

$$\text{True.} \qquad \mathrm{Re}\,(jw_1) = -\mathrm{Im}\,(w_1).$$

(b) $\mathrm{Im}\,(jw_1) = \mathrm{Im}\,(j(x_1 + jy_1)) = \mathrm{Im}\,(-y_1 + jx_1) = x_1$. Also, $\mathrm{Re}\,(w_1) = x_1$. Clearly,

$$\text{True.} \qquad \mathrm{Im}\,(jw_1) = \mathrm{Re}\,(w_1).$$

(c) $\operatorname{Re}(w_1) + \operatorname{Re}(w_2) = x_1 + x_2$. Also, $\operatorname{Re}(w_1 + w_2) = \operatorname{Re}(x_1 + jy_1 + x_2 + jy_2) = x_1 + x_2$. Thus,

$$\text{True.} \qquad \operatorname{Re}(w_1) + \operatorname{Re}(w_2) = \operatorname{Re}(w_1 + w_2).$$

(d) $\operatorname{Im}(w_1) + \operatorname{Im}(w_2) = y_1 + y_2$. Also, $\operatorname{Im}(w_1 + w_2) = \operatorname{Im}(x_1 + jy_1 + x_2 + jy_2) = y_1 + y_2$. Thus,

$$\text{True.} \qquad \operatorname{Im}(w_1) + \operatorname{Im}(w_2) = \operatorname{Im}(w_1 + w_2).$$

(e) $\operatorname{Re}(w_1)\operatorname{Re}(w_2) = x_1 x_2$. Also, $\operatorname{Re}(w_1 w_2) = \operatorname{Re}((x_1 + jy_1)(x_2 + jy_2)) = \operatorname{Re}(x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1)) = x_1 x_2 - y_1 y_2)$. In general $x_1 x_2 \neq x_1 x_2 - y_1 y_2$, so

$$\text{False.} \qquad \operatorname{Re}(w_1)\operatorname{Re}(w_2) \neq \operatorname{Re}(w_1 w_2).$$

(f) $\operatorname{Im}(w_1)/\operatorname{Im}(w_2) = y_1/y_2$. Also, $\operatorname{Im}(w_1/w_2) = \operatorname{Im}\left(\frac{x_1+jy_1}{x_2+jy_2}\right) = \operatorname{Im}\left(\frac{x_1+jy_1}{x_2+jy_2}\frac{x_2-jy_2}{x_2-jy_2}\right) = \operatorname{Im}\left(\frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}\right) = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$. In general $y_1/y_2 \neq \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$, so

$$\text{False.} \qquad \operatorname{Im}(w_1)/\operatorname{Im}(w_2) \neq \operatorname{Im}(w_1/w_2).$$

Solution B.1-9

First, express w_1 in both rectangular and polar coordinates. By inspection, $w_1 = x_1 + jy_1 = 3 + j4$. Next, $r_1 = \sqrt{3^2 + 4^2} = 5$ and $\theta_1 = \arctan\left(\frac{4}{3}\right) = 0.9273$ so $w_1 = r_1 e^{j\theta_1} = 5e^{j0.9273}$.

Second, express w_2 in both rectangular and polar coordinates. By inspection, $w_2 = r_2 e^{j\theta_2} = 2e^{j\pi/4} = 2e^{j0.7854}$. Next, $x_2 = r_2 \cos(\theta_2) = 2\cos(\pi/4) = \sqrt{2} = 1.4142$ and $y_2 = r_2 \sin(\theta_2) = 2\sin(\pi/4) = \sqrt{2} = 1.4142$. Thus, $w_2 = x_2 + jy_2 = 1.4142 + j1.4142$.

(a) From above,
$$w_1 = r_1 e^{j\theta_1} = 5e^{j0.9273}.$$

(b) From above,
$$w_2 = x_2 + jy_2 = 1.4142 + j1.4142.$$

(c)
$$|w_1|^2 = r_1^2 = 5^2 = 25.$$
Similarly,
$$|w_2|^2 = r_2^2 = 4.$$

(d)
$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (3 + 1.4142) + j(4 + 1.4142) = 4.4142 + j5.4142.$$

(e) $w_1 - w_2 = (x_1 + x_2) - j(y_1 + y_2) = (3 - 1.4142) + j(4 - 1.4142) = 1.5858 + j2.5858$. Converting to polar form, $r = \sqrt{(1.5858)^2 + (2.5858)^2} = 3.0333$ and $\theta = \arctan\left(\frac{2.5858}{1.5858}\right) = 1.0207$. Thus,
$$w_1 - w_2 = re^{j\theta} = 3.0333e^{j1.0207}.$$

(f) $w_1 w_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = 10e^{j1.7127}$. Converting to Cartesian form, $x = 10\cos(1.7127) = -1.4142$ and $y = 10\sin(1.7127) = 9.8995$. Thus,
$$w_1 w_2 = x + jy = -1.4142 + j9.8995.$$

(g)
$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1 - \theta_2)} = 2.5e^{j0.1419}.$$

Solution B.1-10

First, express w_1 in both rectangular and polar coordinates. For rectangular form, $w_1 = (3 + j4)^2 = 9 - 16 + j(12 + 12) = -7 + j24$. For polar form, $r_1 = \sqrt{(-7)^2 + 24^2} = 25$ and $\theta_1 = \arctan\left(\frac{24}{-7}\right) = -1.287 + \pi = 1.8546$. Thus, $w_1 = r_1 e^{j\theta_1} = 25e^{j1.8546}$.

Second, express w_2 in both rectangular and polar coordinates. Since $j = e^{j\pi/2}$ and $e^{-j40\pi} = 1$, rectangular form is $w_2 = x_2 + jy_2 = j2.5$. For polar form, $w_2 = r_2 e^{j\theta_2} = 2.5e^{j\pi/2} = 2.5e^{j1.5708}$.

(a) From above,
$$w_1 = r_1 e^{j\theta_1} = 25e^{j1.8546}.$$

(b) From above,
$$w_2 = x_2 + jy_2 = j2.5.$$

(c)
$$|w_1|^2 = r_1^2 = 25^2 = 625.$$

Similarly,
$$|w_2|^2 = r_2^2 = 2.5^2 = 6.25.$$

(d)
$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (-7 + 0) + j(24 + 2.5) = -7 + j26.5.$$

(e) $w_1 - w_2 = (x_1 + x_2) - j(y_1 + y_2) = (-7 - 0) + j(24 - 2.5) = -7 + j21.5$. Converting to polar form, $r = \sqrt{(-7)^2 + (21.5)^2} = 22.6108$ and $\theta = \arctan\left(\frac{21.5}{-7}\right) = -1.256 + \pi = 1.8856$. Thus,
$$w_1 - w_2 = re^{j\theta} = 22.6108e^{j1.8856}.$$

(f) $w_1 w_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = 62.5e^{j3.4254} = 62.5e^{-j2.8578}$. Converting to Cartesian form, $x = 62.5\cos(3.4254) = -60$ and $y = 62.5\sin(3.4254) = -17.5$. Thus,
$$w_1 w_2 = x + jy = -60 + j - 17.5.$$

(g)
$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = 10e^{j0.2838}.$$

Solution B.1-11

First, express w_1 in both rectangular and polar coordinates. By inspection, $w_1 = x_1 + jy_1 = e^{\pi/4} + j = 2.1933 + j$. Next, $r_1 = \sqrt{2.1933^2 + 1^2} = 2.4105$ and $\theta_1 = \arctan\left(\frac{1}{2.1933}\right) = 0.4278$ so $w_1 = r_1 e^{j\theta_1} = 2.4105e^{j0.4278}$.

Second, express w_2 in both rectangular and polar coordinates. Using Euler's identity, $w_2 = \cos(j) = \frac{e^{jj} + e^{-jj}}{2} = \frac{e^{-1} + e^1}{2} = \cosh(1) = 1.5431$. Thus, $w_2 = x_2 + jy_2 = 1.5431$. Polar form is $w_2 = r_2 e^{j\theta_2} = 1.5431e^{j0}$.

(a) From above,
$$w_1 = r_1 e^{j\theta_1} = 2.4105e^{j0.4278}.$$

(b) From above,
$$w_2 = x_2 + jy_2 = 1.5431.$$

(c)
$$|w_1|^2 = r_1^2 = 5.8105.$$

Similarly,
$$|w_2|^2 = r_2^2 = 2.3811.$$

(d)
$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (2.1933 + 1.5431) + j(1 + 0) = 3.7364 + j.$$

(e) $w_1 - w_2 = (x_1 + x_2) - j(y_1 + y_2) = (2.1933 - 1.5431) + j(1 - 0) = 0.6502 + j.$ Converting to polar form, $r = \sqrt{(0.6502)^2 + (1)^2} = 1.1928$ and $\theta = \arctan\left(\frac{1}{0.6502}\right) = 0.9943.$ Thus,

$$w_1 - w_2 = re^{j\theta} = 1.1928e^{j0.9943}.$$

(f) $w_1 w_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = 3.7196 e^{j0.4278}.$ Converting to Cartesian form, $x = 3.7196\cos(0.4278) = 3.3844$ and $y = 3.7196\sin(0.4278) = 1.5431.$ Thus,

$$w_1 w_2 = x + jy = 3.3844 + j1.5431.$$

(g)
$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = 1.5621 e^{j0.4278}.$$

Solution B.1-12

(a) $(w)^4 = -1 = e^{jj(\pi + 2\pi k)} \Rightarrow w = \left(e^{j(\pi + 2\pi k)}\right)^{1/4}.$ Thus,

$$w = e^{j\pi(1/4 + k/2)} \quad \text{for} \quad k = [0, 1, 2, 3].$$

```
>>  k = [0:3]; w = exp(j*pi*(1/4+k/2)); t = linspace(0,2*pi,200);
>>  plot(real(w),imag(w),'kx',cos(t),sin(t),'k:'); axis equal;
>>  xlabel('Real'); ylabel('Imag'); grid;
```

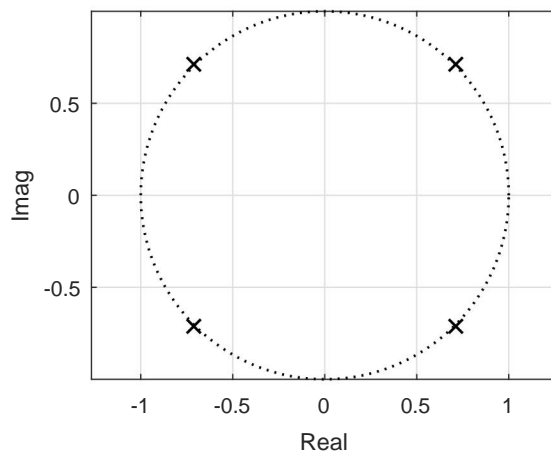The 4 unique solutions are shown in Fig. SB.1-12a.

Figure SB.1-12a

(b) Notice,

$$(w - (1 + j2))^5 = \frac{32}{\sqrt{2}}(1 + j) = 32e^{j(\pi/4 + 2\pi k)}.$$

This implies that

$$w - (1 + j2) = \left(32e^{j(\pi/4 + 2\pi k)}\right)^{1/5} = 2e^{j(\pi/20 + 2\pi k/5)}.$$

Thus,
$$w = (1 + j2) + 2e^{j(\pi/20 + 2\pi k/5)} \quad \text{for} \quad k = [0, 1, 2, 3, 4].$$

```
>>  k = [0:4]; w = (1+j*2)+2*exp(j*(pi/20+2*pi*k/5));
>>  t = linspace(0,2*pi,200);
>>  plot(real(w),imag(w),'kx',1+2*cos(t),2+2*sin(t),'k:');
>>  axis equal; xlabel('Real'); ylabel('Imag'); grid;
```

The 5 unique solutions are shown in Fig. SB.1-12b.
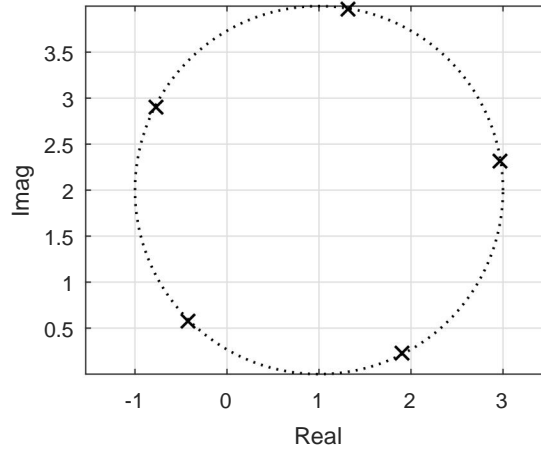
Figure SB.1-12b

(c) The solution set of $\mid w - 2j \mid = 3$ describes a circle. To see this, note that $\mid w - 2j \mid^2 = (w - 2j)(w - 2j)^* = (x + j(y - 2))(x + j(2 - y)) = x^2 + (y - 2)^2 = 3^2 = 9$. The circle has center $(0, 2)$ and radius $r = 3$.

```
>>  theta = linspace(0,2*pi,201); x = 3*cos(theta); y = 2+3*sin(theta);
>>  plot(x,y,'k-'); axis equal; grid; xlabel('Real'); ylabel('Imag');
```

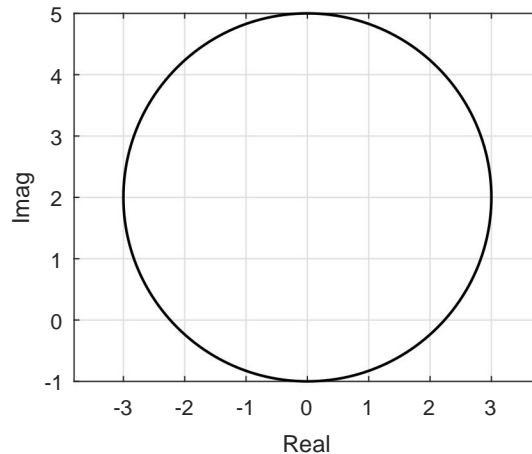The circle of solutions is shown in Fig. SB.1-12c.

Figure SB.1-12c

(d) Graph $w(t) = (1 + t)e^{jt}$ for $(-10 \leq t \leq 10)$.

```
>>  t = [-10:.01:10]; w = (1+t).*exp(j*t);
>>  plot(real(w(t==-10)),imag(w(t==-10)),'vk',...
```

```
>>        real(w(t<0)),imag(w(t<0)),'k-',...
>>        real(w(t==0)),imag(w(t==0)),'ok',...
>>        real(w(t>0)),imag(w(t>0)),'k:',...
>>        real(w(t==10)),imag(w(t==10)),'k^');
>> axis equal; xlabel('Real'); ylabel('Imag');
>> legend('t=-10','t<0','t=0','t>0','t=10','location','EastOutside')
```
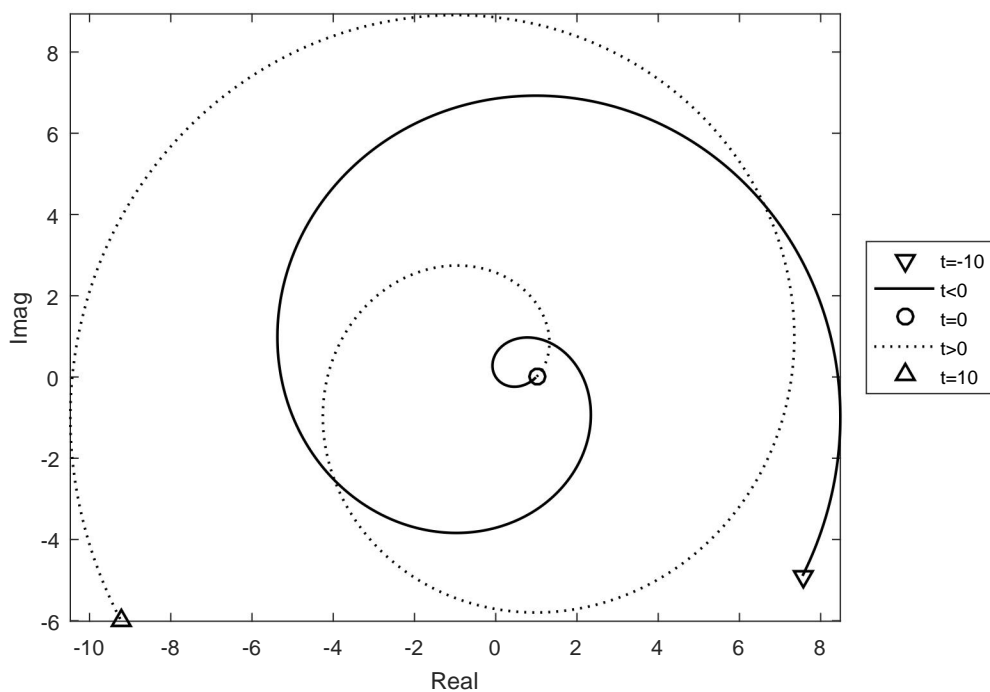

Figure SB.1-12d

Solution B.1-13

Since four distinct solutions are indicated, we know $n = 4$. The solutions to $w^n = w_2 = r_2 e^{j\theta_2}$ lie on a circle of radius $r_2^{1/n}$. The solutions to $(w - w_1)^n = w_2$ lie on the same circle shifted by w_1. To find w_1, drop perpendicular lines from the circle center to the real and imaginary axes, respectively. As shown, two similar triangles are formed. The circle center is $w_1 = A + jA$. Furthermore, we know that $A + B = \sqrt{3} + 1$ and $A - B = \sqrt{3} - 1$. Clearly, $A = \sqrt{3}$ and $B = 1$. Thus, $w_1 = \sqrt{3} + j\sqrt{3}$. The value of w_2 is now easily found by substitution: $w_2 = (\sqrt{3} + 1 - (\sqrt{3} + j\sqrt{3}))^4 = (1 + j\sqrt{3})^4 = 16e^{j2\pi/3}$. Thus,

$$n = 4, w_1 = \sqrt{3} + j\sqrt{3}, \text{ and } w_2 = 16e^{j2\pi/3}.$$
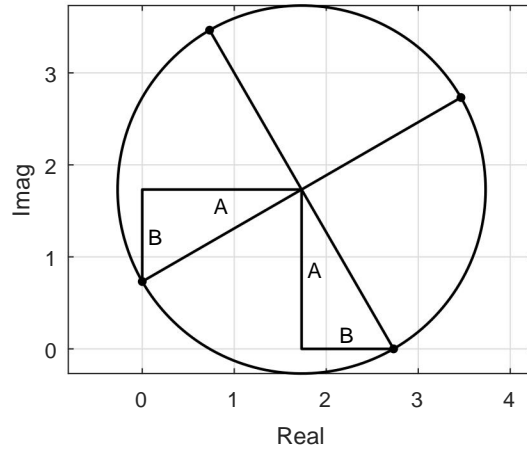
Figure SB.1-13

Solution B.1-14

(a) Expressing the righthand side in polar form, we see that $w^3 = -\frac{8}{27} = \frac{8}{27}e^{j\pi + j2\pi k}$. Taking the cube root yields three unique solutions,

$$w = \frac{2}{3}e^{j\pi(1+2k)/3} \quad \text{for} \quad k = [0, 1, 2].$$

Figure SB.1-14a graphs these solutions in the complex plane.

Figure SB.1-14a

(b) In this case, $(w + 1)^8 = 1 = e^{j2\pi k}$. Taking the eighth root and then subtracting one yields eight unique solutions,

$$w = e^{j\pi k/4} - 1 \quad \text{for} \quad k = [0, 1, \ldots, 7].$$

Figure SB.1-14b graphs these solutions in the complex plane.
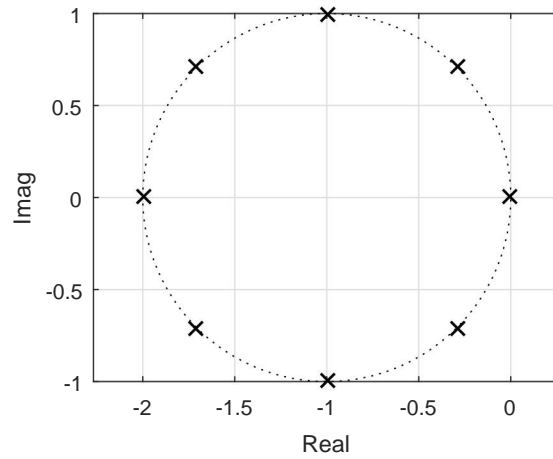
Figure SB.1-14b

(c) In this case, we rearrange $w^2 + j = 0$ as $w^2 = -j = e^{-j\pi/2 + j2\pi k}$. Taking the square root yields two unique solutions,

$$w = e^{-j\pi/4 + j\pi k} \quad \text{for} \quad k = [0, 1].$$

Figure SB.1-14c graphs these solutions in the complex plane.
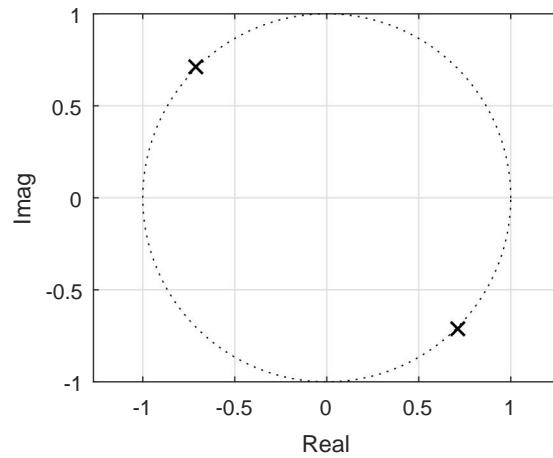
Figure SB.1-14c

(d) To begin, we rearrange $16(w-1)^4 + 81 = 0$ as $(w-1)^4 = -\frac{81}{16} = \frac{81}{16}e^{j\pi + j2\pi k}$. Taking the fourth root and then adding one yields four unique solutions,

$$w = \frac{3}{2}e^{j\pi/4 + j\pi k/2} + 1 \quad \text{for} \quad k = [0, 1, 2, 3].$$

Figure SB.1-14d graphs these solutions in the complex plane.

Figure SB.1-14d

(e) Here, $(w+2j)^3 = -8 = 8e^{j\pi+2\pi k}$. Taking the cube root and subtracting $2j$ yields three unique solutions,

$$w = 2e^{j\pi/3+j2\pi k/3} - 2j \quad \text{for} \quad k = [0, 1, 2].$$

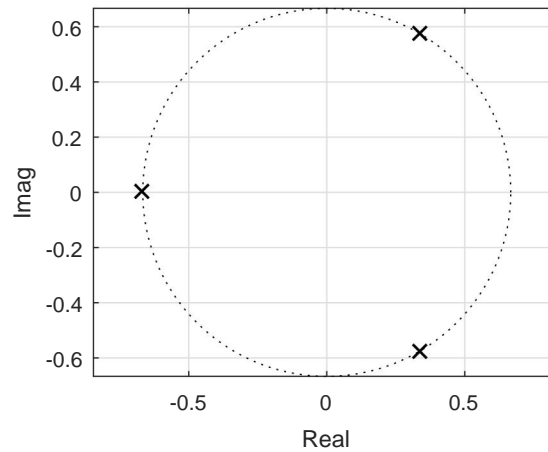Figure SB.1-14e graphs these solutions in the complex plane.
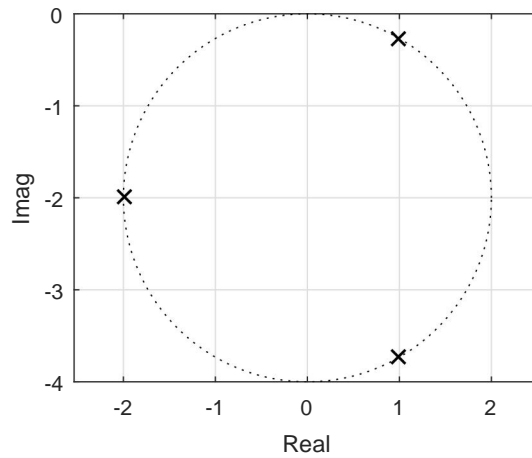
Figure SB.1-14e

(f) We can write $(j - w)^{1.5} = (j - w)^{3/2} = \sqrt{8}e^{j\pi/4}$. Squaring both sides yields $(j - w)^3 = 8e^{j(\pi/2+2\pi k)}$. Taking the third root of each side yields $(j - w) = 2e^{j(\pi/6+2\pi k/3)}$. Rearranging yields three distinct solutions

$$w = j - 2e^{j(\pi/6+2\pi k/3)} \quad \text{for} \quad k = [0, 1, 2].$$

Figure SB.1-14f graphs these solutions in the complex plane.
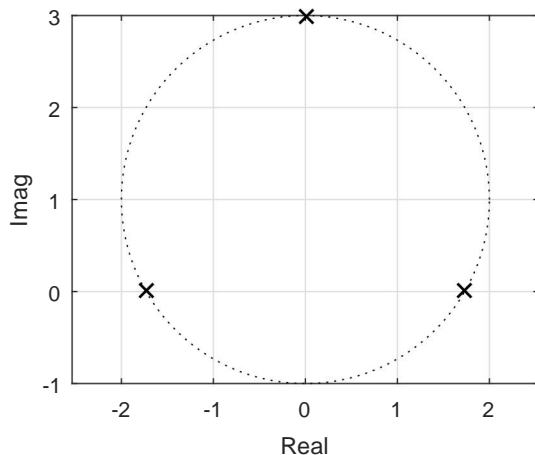
Figure SB.1-14f

(g) Here, we write $(w-1)^{2.5} = j4\sqrt{2}$ as $(w-1)^{5/2} = \sqrt{32}e^{j\pi/2}$. Squaring both sides yields $(w-1)^5 = 32e^{j\pi+j2\pi k}$. Taking the fifth root of each side yields $(w-1) = 2e^{j\pi/5+j2\pi k/5}$. Solving for w, the five unique solutions are

$$w = 2e^{j\pi/5+j2\pi k/5} + 1 \quad \text{for} \quad k = [0, 1, \ldots, 4].$$

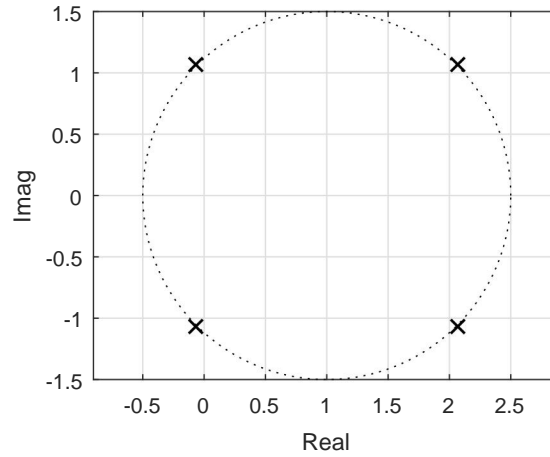Figure SB.1-14g graphs these solutions in the complex plane.
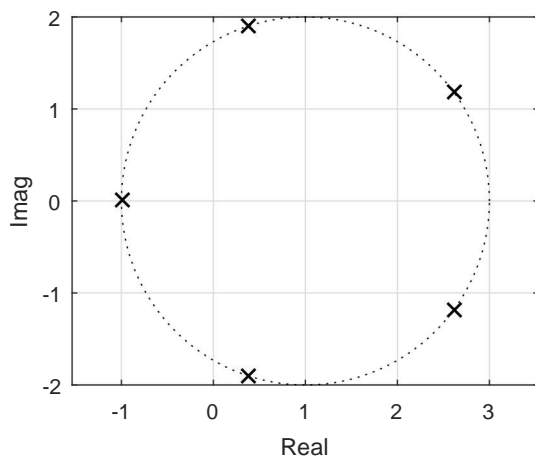
Figure SB.1-14g

Solution B.1-15

Expressing the righthand side in polar form, $w = \sqrt{j} = \left(e^{j(\pi/2+2\pi k)}\right)^{1/2} = e^{j(\pi/4+\pi k)}$. Thus, there are two distinct solutions

$$w = e^{j(\pi/4+\pi k)} \quad \text{for} \quad k = [0, 1].$$

That is, $w = \pm(1+j)/\sqrt{2}$.

Solution B.1-16

$\ln(-e) = \ln(e^{1+j(\pi+2\pi k)}) = 1 + j(\pi + 2\pi k)$. Since k can be any integer, there are an infinite number of solutions

$$\ln(-e) = 1 + j(\pi + 2\pi k) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k = 0$ solution $1 + j\pi$.

Solution B.1-17

$\log_{10}(-1) = \log_{10} e^{j(\pi+2\pi k)} = j(\pi + 2\pi k)\log_{10}(e)$. Since k can be any integer, there are an infinite number of solutions

$$\log_{10}(-1) = 0 + j(\pi + 2\pi k)\log_{10}(e) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k = 0$ solution $j\pi \log_{10}(e)$.

Solution B.1-18

(a) $w_{\text{a}} = \ln\left(\frac{1}{1+j}\right) = \ln\left(\frac{1}{\sqrt{2}e^{j\pi/4}}\right) = \ln\left((\sqrt{2})^{-1}e^{j(-\pi/4+2\pi k)}\right) = -\ln(\sqrt{2}) + j(-\pi/4 + 2\pi k)$. Since k can be any integer, there are an infinite number of solutions

$$w_{\text{a}} = \ln\left(\frac{1}{1+j}\right) = -\ln(\sqrt{2}) + j(-\pi/4 + 2\pi k) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k = 0$ solution.

(b) $w_{\text{b}} = \cos(1+j) = 0.5\left(e^{j(1+j)} + e^{-j(1+j)}\right) = 0.5\left(e^{-1}(\cos(1) + j\sin(1)) + e^{1}(\cos(1) - j\sin(1))\right) = \cos(1)\cosh(1) - j\sin(1)\sinh(1)$. That is

$$w_{\text{b}} = \cos(1+j) = \cos(1)\cosh(1) - j\sin(1)\sinh(1).$$

(c) $w_{\text{c}} = (1-j)^{j} = \left(\sqrt{2}e^{-j\pi/4}\right)^{j} = \left(e^{\ln(\sqrt{2})}e^{-j\pi/4}\right)^{j} = e^{j\ln(\sqrt{2})}e^{\pi/4} = e^{\pi/4}\left(\cos(\ln(\sqrt{2})) + j\sin(\ln(\sqrt{2}))\right)$. Thus,

$$w_{\text{c}} = (1-j)^{j} = e^{\pi/4}\cos(\ln(\sqrt{2})) + je^{\pi/4}\sin(\ln(\sqrt{2})).$$

Solution B.1-19

Letting $w = jy$, $\cos(w) = \cos(jy) = 0.5\left(e^{jjy} + e^{-jjy}\right) = 0.5\left(e^{-y} + e^{y}\right) = 2$. Multiplying both sides by $2e^{y}$ yields $1 + (e^{y})^{2} - 4e^{y} = (e^{y})^{2} - 4e^{y} + 1 = 0$. This is a quadratic equation in e^{y}. Applying the quadratic formula yields $e^{y} = \frac{4\pm\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. Solving for y gives $y = \ln\left(2 \pm \sqrt{3}\right)$. Thus,

$$w = jy = j\ln\left(2 \pm \sqrt{3}\right) = \pm j1.3170.$$

Solution B.1-20

(a) To express $e^{-x^{2}}$ as a Taylor series, recall that a Taylor series of e^{u} about zero is given by $e^{u} = \sum_{i=0}^{\infty}\frac{u^{i}}{i!}$. Substituting $-x^{2}$ for u yields

$$e^{-x^{2}} = \sum_{i=0}^{\infty}\frac{\left(-x^{2}\right)^{i}}{i!}.$$

(b) Integrating yields $\int e^{-x^{2}}dx = \int \sum_{i=0}^{\infty}\frac{(-1)^{i}x^{2i}}{i!}dx = \sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\int x^{2i}dx$ or

$$\int e^{-x^{2}}dx = \sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\frac{x^{2i+1}}{2i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus
$\int_0^1 e^{-x^2} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{2i+1}}{2i+1} \Big|_{x=1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = (-1).^i./(gamma(i+1).*(2*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 0.6667, 0.7667, 0.7429, 0.7475, 0.7467, 0.7468, 0.7468, 0.7468, 0.7468).$$

At a seven-term truncation, the result appears to converge to four digits.

Solution B.1-21

(a) To express e^{-x^3} as a Taylor series, recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $-x^3$ for u yields

$$e^{-x^3} = \sum_{i=0}^{\infty} \frac{(-x^3)^i}{i!}.$$

(b) Integrating yields $\int e^{-x^3} dx = \int \sum_{i=0}^{\infty} \frac{(-1)^i x^{3i}}{i!} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int x^{3i} dx$ or

$$\int e^{-x^3} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{3i+1}}{3i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus
$\int_0^1 e^{-x^3} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{3i+1}}{3i+1} \Big|_{x=1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(3i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = (-1).^i./(gamma(i+1).*(3*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 0.7500, 0.8214, 0.8048, 0.8080, 0.8074, 0.8075, 0.8075, 0.8075, 0.8075).$$

At a seven-term truncation, the result appears to converge to four digits.

Solution B.1-22

(a) To express $\cos(x^2) = 0.5 \left(e^{jx^2} + e^{-jx^2} \right)$ as a Taylor series, recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $\pm jx^2$ for u yields

$$\cos(x^2) = \sum_{i=0}^{\infty} 0.5 \left(\frac{(jx^2)^i}{i!} + \frac{(-jx^2)^i}{i!} \right).$$

(b) Integrating yields $\int \cos(x^2)dx = \int \sum_{i=0}^{\infty} 0.5 \left(\frac{(jx^2)^i}{i!} + \frac{(-jx^2)^i}{i!} \right) dx = \sum_{i=0}^{\infty} 0.5 \frac{(j)^i(1+(-1)^i)}{i!} \int \left(x^2 \right)^i dx$ or

$$\int \cos(x^2)dx = \sum_{i=0}^{\infty} 0.5 \frac{(j)^i(1+(-1)^i)}{i!} \frac{x^{2i+1}}{2i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 \cos(x^2)dx = \sum_{i=0}^{\infty} 0.5 \frac{(j)^i(1+(-1)^i)}{i!} \frac{x^{2i+1}}{2i+1} \Big|_{x=1} = \sum_{i=0}^{\infty} 0.5 \frac{(j)^i(1+(-1)^i)}{i!(2i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = 0.5*(j).^i.*(1+(-1).^i)./(gamma(i+1).*(2*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 1.0000, 0.9000, 0.9000, 0.9046, 0.9046, 0.9045, 0.9045, 0.9045, 0.9045).$$

At a seven-term truncation, the result appears to converge to four digits.

Solution B.1-23

(a) Using synthetic division, express $f_a(x) = \frac{1}{2-x^2} = \frac{1}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots$. Thus,

$$f_a(x) = \sum_{i=0}^{\infty} \left(\frac{1}{2} \right)^{i+1} x^{2i}.$$

(b) Rewrite as $f_b(x) = (0.5)^x = e^{-\ln(2)x}$. Recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $-\ln(2)x$ for u yields

$$f_b(x) = \sum_{i=0}^{\infty} \frac{(-\ln(2)x)^i}{i!}.$$

Solution B.1-24

(a) A Taylor series requires knowledge of a function and its derivatives as

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

In the present case,

$$f(x) = 1 + x + x^2 + x^3$$
$$\tfrac{d}{dx}f(x) = 1 + 2x + 3x^2$$
$$\tfrac{d^2}{dx^2}f(x) = 2 + 6x$$
$$\tfrac{d^3}{dx^3}f(x) = 6$$
$$\tfrac{d^4}{dx^4}f(x) = \frac{d^5}{dx^5}f(x) = \cdots = 0$$

Evaluating the non-zero terms using the expansion point $a = 1$ yields

$$f(x)|_{x=1} = 4$$
$$\tfrac{d}{dx}f(x)\big|_{x=1} = 6$$
$$\tfrac{d^2}{dx^2}f(x)\Big|_{x=1} = 8$$
$$\tfrac{d^3}{dx^3}f(x)\Big|_{x=1} = 6$$

Thus, the desired Taylor series is

$$f(x) = 4 + 6(x-1) + 4(x-1)^2 + (x-1)^3.$$

Not surprisingly, the Taylor series of a third-order polynomial is itself a third-order polynomial. If each term in the Taylor series were expanded, the simplified result would match the original expression $f(x) = 1 + x + x^2 + x^3$.

(b) It may seem a little odd to represent a polynomial expression with another polynomial expression of the same order. However, there can be good reason to do so. If the system described by the function $f(x)$ operates near the expansion point $x = a$, the Taylor series about this expansion point converges to a good result with fewer terms. For example, about $x = 1$, the truncated Taylor series $f(x) \approx 4 + 6(x-1)$ provides a good approximation to the cubic $f(x) = 1 + x + x^2 + x^3$. This linear approximation can be more efficiently computed on target hardware, such as an embedded processor.

Solution B.1-25

A Maclaurin series is a Taylor series with the expansion point a set to zero,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \bigg|_{a=0} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

(a) To begin, we notice that $f_a(x) = 2^x = e^{\ln(2)x}$. The kth derivative of $f_a(x)$ is

$$\frac{d^k}{dx^k}f_a(x) = [\ln(2)]^k \, e^{\ln(2)x}.$$

Evaluated at the $x = 0$ expansion point yields

$$\frac{d^k}{dx^k} f_{\mathrm{a}}(x)\bigg|_{x=0} = [\ln(2)]^k .$$

Substituting this result into the definition of the Maclaurin series yields

$$f_{\mathrm{a}}(x) = 2^x == \sum_{k=0}^{\infty} \frac{[\ln(2)]^k}{k!} x^k$$

(b) To begin, we notice that $f_{\mathrm{b}}(x) = \left(\frac{1}{3}\right)^x = e^{\ln\left(\frac{1}{3}\right)x}$. The kth derivative of $f_{\mathrm{b}}(x)$ is

$$\frac{d^k}{dx^k} f_{\mathrm{b}}(x) = \left[\ln\left(\frac{1}{3}\right)\right]^k e^{\ln\left(\frac{1}{3}\right)x}.$$

Evaluated at the $x = 0$ expansion point yields

$$\frac{d^k}{dx^k} f_{\mathrm{b}}(x)\bigg|_{x=0} = \left[\ln\left(\frac{1}{3}\right)\right]^k .$$

Substituting this result into the definition of the Maclaurin series yields

$$f_{\mathrm{b}}(x) = \left(\frac{1}{3}\right)^x == \sum_{k=0}^{\infty} \frac{\left[\ln\left(\frac{1}{3}\right)\right]^k}{k!} x^k$$

Solution B.2-1

(a) By inspection, signal $\cos(5\pi t + 3)$ has

$$\omega_0 = 5\pi, \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{5}{2}, \qquad T_0 = \frac{1}{f_0} = \frac{2}{5}.$$

(b) By inspection, signal $7 \sin\left(\frac{2t-\pi}{3}\right)$ has

$$\omega_0 = \frac{2}{3}, \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{3\pi}, \qquad T_0 = \frac{1}{f_0} = 3\pi.$$

Solution B.2-2

The expression of a generalized sinusoid is $x(t) = a\cos(2\pi f_0 t + b)$. To ensure $x(t)$ oscillates 15 times per second requires $f_0 = 15$. To ensure $x(t)$ has a peak amplitude of 3 requires $a = 3$. To ensure $x(0) = -1$ requires that $3\cos(b) = -1$ or $b = \cos^{-1}(-1/3) = 1.9106$. Thus, the desired signal $x(t)$ can be expressed as

$$x(t) = 3\cos\left(2\pi 15t + \cos^{-1}(-1/3)\right) = 3\cos\left(2\pi 15t + 1.9106\right).$$

Figure SB.2-2 graphs $x(t)$ over $0 \le t \le 1$.

```
>>   x = @(t) 3*cos(2*pi*15*t+acos(-1/3)); t = 0:.001:1;
>>   plot(t,x(t),'k-'); xlabel('t'); ylabel('x(t)'); axis([0 1 -3.1 3.1]);
```
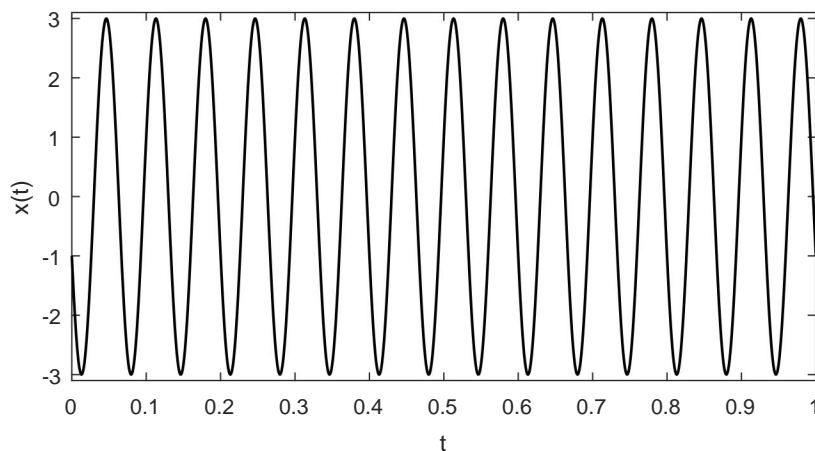
Figure SB.2-2

Solution B.2-3

For this problem, $x_1(t) = 2\cos(3t + 1)$ and $x_2(t) = -3\cos(3t - 2)$. Each part can be readily solved using Euler's formula and the real operator.

(a) In this part, we need to determine a_1 and b_1 so that $x_1(t) = a_1\cos(3t) + b_1\sin(3t)$.

$$
\begin{aligned}
x_1(t) &= \operatorname{Re}\left(2e^{j(3t+1)}\right) \\
&= \operatorname{Re}\left(2e^{j3t}\left[\cos(1) + j\sin(1)\right]\right) \\
&= 2\cos(1)\cos(3t) - 2\sin(1)\sin(3t).
\end{aligned}
$$

Thus, we see that

$$a_1 = 2\cos(1) = 1.0806 \qquad \text{and} \qquad b_1 = -2\sin(1) = -1.6829.$$

(b) In this part, we need to determine a_2 and b_2 so that $x_2(t) = a_2\cos(3t) + b_2\sin(3t)$.

$$
\begin{aligned}
x_2(t) &= \operatorname{Re}\left(-3e^{j(3t-2)}\right) \\
&= \operatorname{Re}\left(-3e^{j3t}\left[\cos(2) - j\sin(2)\right]\right) \\
&= -3\cos(2)\cos(3t) - 3\sin(2)\sin(3t).
\end{aligned}
$$

Thus, we see that

$$a_2 = -3\cos(2) = 1.2484 \qquad \text{and} \qquad b_2 = 3\sin(2) = 2.7279.$$

(c) In this part, we need to determine C and θ so that $x_1(t) + x_2(t) = C\cos(3t + \theta)$.

$$
\begin{aligned}
x_1(t) + x_2(t) &= \operatorname{Re}\left(2e^{j(3t+1)} - 3e^{j(3t-2)}\right) \\
&= \operatorname{Re}\left(e^{j3t}\left[2e^{j} - 3e^{-2j}\right]\right) \\
&= \operatorname{Re}\left(e^{j3t}4.988e^{j1.0850}\right) \\
&= 4.988\cos(3t + 1.0850).
\end{aligned}
$$

Thus, we see that

$$C = 4.988 \qquad \text{and} \qquad \theta = 1.0850.$$

Solution B.2-4

Solutions to this problem are based on Euler's identity.

(a)

$$\cosh(w) = \cosh(x + jy) = \frac{e^{x+jy} + e^{-x-jy}}{2}$$
$$= 0.5\left((\cos(y) + j\sin(y))e^x + (\cos(y) - j\sin(y))e^{-x}\right)$$
$$= 0.5\left(\cos(y)(e^x + e^{-x}) + j\sin(y)(e^x - e^{-x})\right)$$
$$= \cos(y)\cosh(x) + j\sin(y)\sinh(x)$$

Thus,
$$\cosh(w) = \cosh(x + jy) = \cosh(x)\cos(y) + j\sinh(x)\sin(y).$$

(b)

$$\sinh(w) = \sinh(x + jy) = \frac{e^{x+jy} - e^{-x-jy}}{2}$$
$$= 0.5\left((\cos(y) + j\sin(y))e^x - (\cos(y) - j\sin(y))e^{-x}\right)$$
$$= 0.5\left(\cos(y)(e^x - e^{-x}) + j\sin(y)(e^x + e^{-x})\right)$$
$$= \cos(y)\sinh(x) + j\sin(y)\cosh(x)$$

Thus,
$$\sinh(w) = \sinh(x + jy) = \sinh(x)\cos(y) + j\cosh(x)\sin(y).$$

Solution B.2-5

(a) Note, we can rewrite $2.5\cos(3t) - 1.5\sin(3t + \pi/3) = c\cos(3t + \phi)$ as $\mathrm{Re}\left(2.5e^{j3t} + j1.5e^{j(3t+\pi/3)}\right) = \mathrm{Re}\left(ce^{j(3t+\phi)}\right)$. Working with the left-hand side, $\mathrm{Re}\left(2.5e^{j3t} + j1.5e^{j(3t+\pi/3)}\right) = \mathrm{Re}\left(e^{j3t}(2.5 + 1.5e^{j(\pi/3+\pi/2)})\right)$. The unknown constants c and ϕ are determined by comparing the left- and right-hand sides.

$$c = |2.5 + 1.5e^{j(\pi/3+\pi/2)}| = \sqrt{(2.5 + 1.5\cos(5\pi/6))^2 + (1.5\sin(5\pi/6))^2} = 1.416$$

and
$$\phi = \angle\left(2.5 + 1.5e^{j(\pi/3+\pi/2)}\right) = \arctan\left(\frac{1.5\sin(5\pi/6)}{2.5 + 1.5\cos(5\pi/6)}\right) = 0.558.$$

(b) Note, $\cos(\theta \pm \phi) = \mathrm{Re}\left(e^{j(\theta\pm\phi)}\right) = \mathrm{Re}\left((\cos(\theta) + j\sin(\theta))(\cos(\phi) \pm j\sin(\phi))\right) = \mathrm{Re}\left((\cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)) + j(\sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi))\right) = (\cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi))$. Thus,
$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi).$$

(c) Noting that $\sin(\alpha x) = \frac{e^{j\alpha x} - e^{-j\alpha x}}{2j}$, first solve the indefinite integral $\int e^{wx}\sin(\alpha x)dx = \int e^{wx}\frac{e^{j\alpha x} - e^{-j\alpha x}}{2j}dx = \int \frac{e^{x(w+j\alpha)} - e^{x(w-j\alpha)}}{2j} = \frac{1}{2j(w+j\alpha)}e^{x(w+j\alpha)} - \frac{1}{2j(w-j\alpha)}e^{x(w-j\alpha)}$. Substituting the limits of integration yields

$$\int_a^b e^{wx}\sin(\alpha x)dx =$$
$$\frac{1}{2j(w + j\alpha)}\left(e^{b(w+j\alpha)} - e^{a(w+j\alpha)}\right) - \frac{1}{2j(w - j\alpha)}\left(e^{b(w-j\alpha)} - e^{a(w-j\alpha)}\right).$$

Solution B.2-6

A vehicle traveling at 70 mph travels 102.66 feet per second. To produce a sound with (fundamental) frequency of 1 kHz therefore requires grooving the highway shoulder at a rate of 1000 grooves per 102.66 feet, or 9.74 grooves per foot. To produces quarter second bursts of 1 kHz sounds every second therefore requires $\frac{1}{4}(102.66) = 25.67$ out of every 102.66 feet of shoulder with grooves spaced at 9.74 grooves per foot. Put another way, 250 equally-spaced grooves should occupy 25.67 out of every 102.66 feet of highway shoulder.

Solution B.3-1

In this problem, we sketch $x_a(t) = e^{-t}$, $x_b(t) = \sin(2\pi 5t)$, and $x_c(t) = e^{-t}\sin(2\pi 5t)$ over $(0 \le t \le 1)$. Signal $x_a(t)$ is just an exponentially decaying waveform with amplitude 1 at $t = 0$ that decays to $e^{-1} = 0.3679$ at $t = 1$. Signal $x_b(t)$ is a unit amplitude sine wave that oscillates 5 times per second. Lastly, $x_c(t)$ is just the product of $x_a(t)$ and $x_b(t)$. Figure SB.3-1 shows all three waveforms.
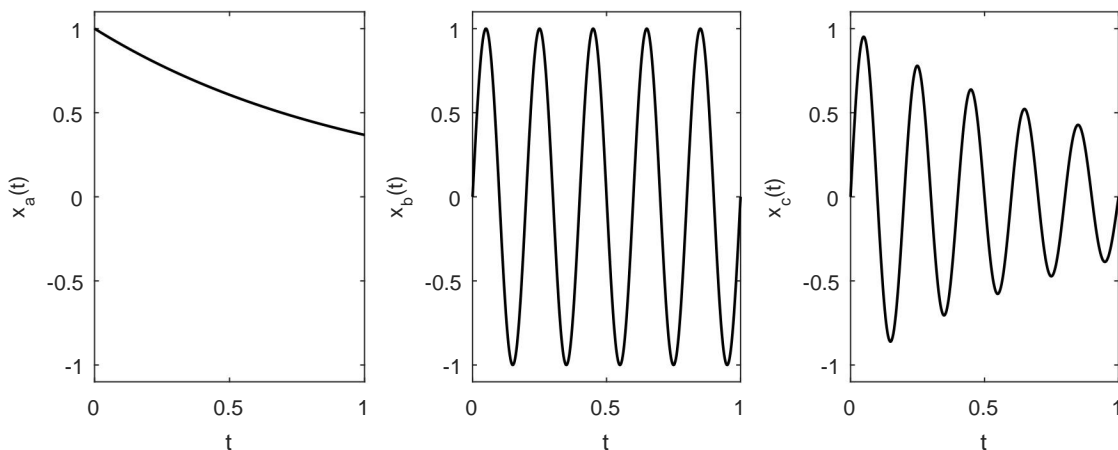

Figure SB.3-1

Solution B.3-2

In this problem, we model population as $p(t) = ae^{bt}$, where t indicates year. We can determine parameter b using the given 40-year doubling time. Since $p(0) = a$, $p(40) = 2a = ae^{40b}$. Solving for b yields

$$b = \ln(2)/40 = 0.0173.$$

To determine parameter a, we use the known 1950 population: $p(1950) = 2.5(10)^9 = ae^{1950\ln(2)/40}$. Solving for a yields

$$a = 2.5(10)^9 2^{-48.75} = 5.2811(10)^{-6}.$$

Given this model and its 40-year doubling time, we can predict the year t_{15} that the world population reaches 15 billion as

$$t_{15} = \ln\left(\frac{15(10)^9}{a}\right)/b = 2053.40.$$

Figure SB.3-2 shows $p(t)$ over $1950 \le t \le 2100$.

```
>>   a = 2.5*10^9*2^(-48.75); b = log(2)/40; p = @(t) a*exp(b*t);
>>   t = 1950:2100; t15 = log(15*10^9/a)/b;
>>   plot(t,p(t),'k-'); xlabel('year'); ylabel('population');
>>   line([1950 t15 t15],[15e9 15e9 0],'linestyle',':');
>>   set(gca,'xtick',[1950 2000 2053.4 2100]);
```
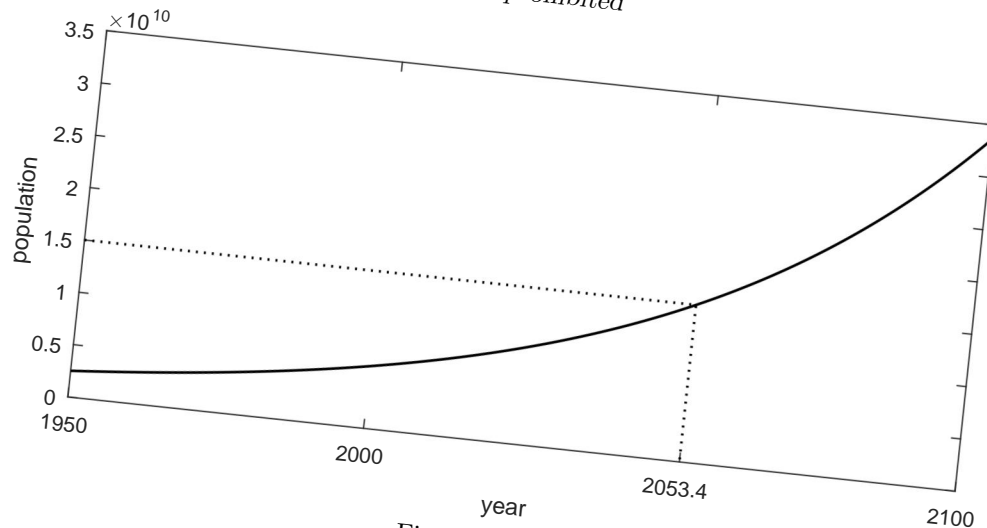
Figure SB.3-2

Like most models, this population model has strengths and limitations. The doubling time is based on twentieth century data, a time interval which includes the "initial" (1950) population of 2.5 billion. Not surprisingly, the model does pretty good over the interval $1950 \leq t \leq 2000$. As long as population trends do not significantly change, the 2053 estimate for a population of 15 billion is reasonable. Look, however, what happens when we move further away in time. This model, for example, estimates the population at year 0 as a, or 5 millionths of *a single person*! This number is clearly nonsensical, and far less than estimates based on historical data that place the world's human population in year 0 in the hundreds of millions.

Solution B.3-3

The general form is $x(t) = e^{-at}\cos(\omega t)$. At $t = 0$, $e^{-at} = 1$. Thus, a fifty percent decrease in two seconds requires $0.5 = e^{-a2}$, or $a = 0.5\ln(2) = 0.3466$. To oscillate three times per second requires $\omega = 6\pi$. Thus, one signal that meets design specifications is

$$x(t) = e^{-0.3466t}\cos(6\pi t).$$

```
>>   w = 3*2*pi; a = 0.5*log(2); x = @(t) exp(-a*t).*cos(w*t);
>>   t = [-2:.01:2]; plot(t,x(t),'k-'); xlabel('t'); ylabel('x(t)');
```

Figure SB.3-3 plots the signal over $-2 \leq t \leq 2$.
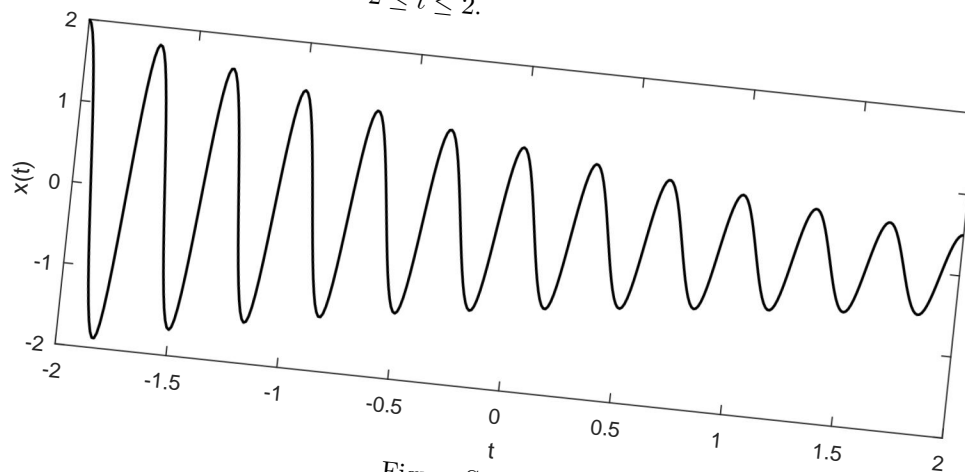
Figure SB.3-3

Solution B.3-4

(a) $x_a(t) = \text{Re}(2e^{(-1+j2\pi)t}) = 2e^{-t}\cos(2\pi t)$. This is 1 Hz cosine wave that exponentially decays by a factor of $1 - e^{-1} = 0.632$ every second. A signal peak is near $t = 0$, where the signal has an amplitude of 2. Figure SB.3-4a shows $x_a(t)$ over $0 \leq t \leq 3$.

Figure SB.3-4a

(b) $x_b(t) = \text{Im}(3 - e^{(1-j2\pi)t}) = e^t\sin(2\pi t)$. This is a 1 Hz sine wave that exponentially grows by a factor of $e^1 = 2.718$ every second. A signal peak is near $t = 1/4$, where the signal has an amplitude of 1.284. Figure SB.3-4b shows $x_b(t)$ over $0 \leq t \leq 3$.
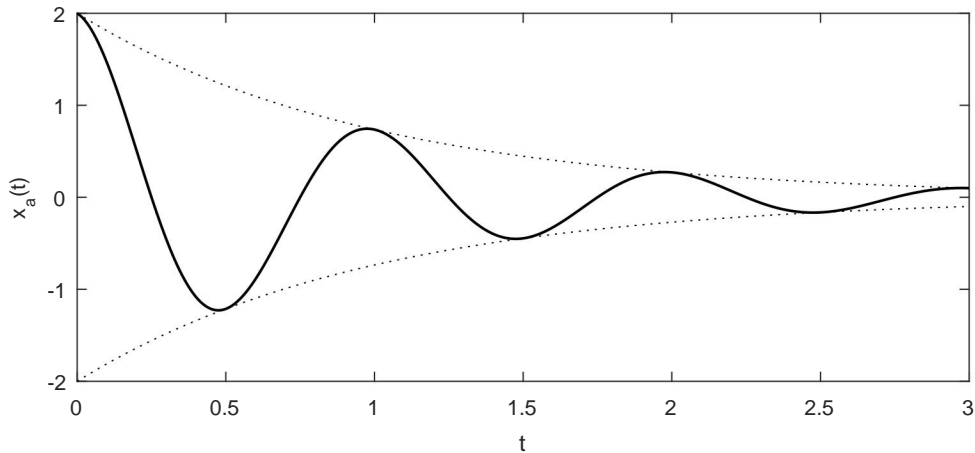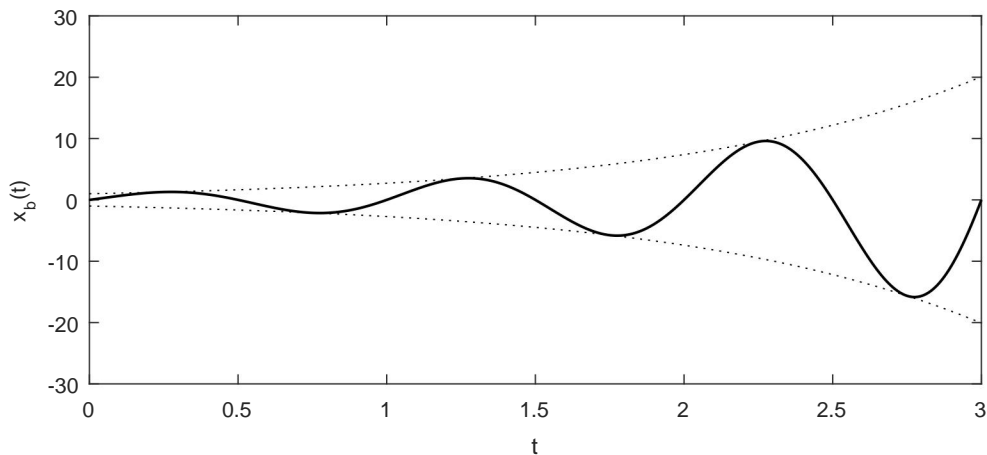

Figure SB.3-4b

(c) $x_c(t) = 3 - \text{Im}(e^{(1-j2\pi)t}) = 3 + e^t\sin(2\pi t)$. This is a 1Hz sine wave that exponentially grows by a factor of $e^1 = 2.718$ every second and has an offset of 3. A signal peak is near $t = 1/4$, where the signal has an amplitude of 4.284. Figure SB.3-4c shows $x_c(t)$ over $0 \leq t \leq 3$.

Figure SB.3-4c

Solution B.4-1

In this problem, we are interested in using hand calculations and Cramer's rule to solve the system of equations

$$\begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Solving for x_1, we obtain

$$x_1 = \frac{\begin{vmatrix} 3 & 2 \\ -1 & -4 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}} = \frac{3(-4) - 2(-1)}{-1(-4) - 3(2)} = \frac{-10}{-2} = 5$$

Solving for x_2, we obtain

$$x_2 = \frac{\begin{vmatrix} -1 & 3 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}} = \frac{-1(-1) - 3(3)}{-1(-4) - 3(2)} = \frac{-8}{-2} = 4$$

These hand-calculated results are readily confirmed using MATLAB.
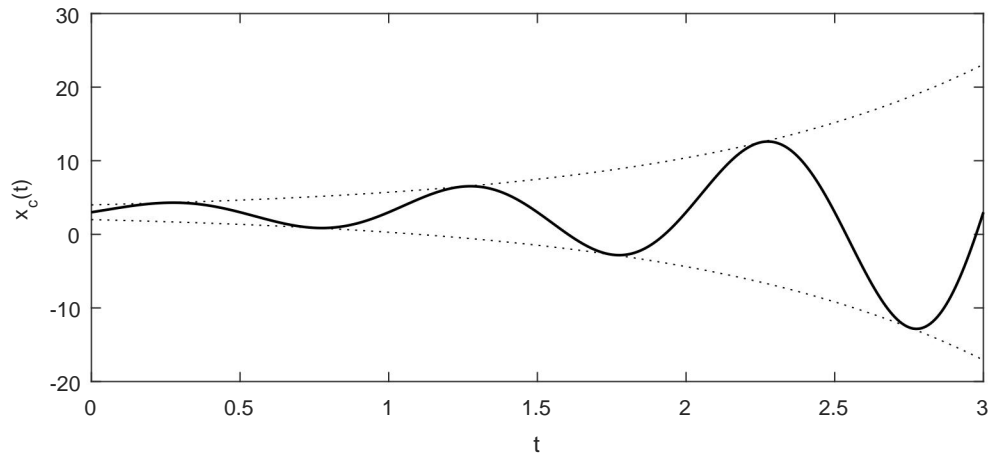
```
>>  x = inv([-1 2;3 -4])*[3;-1]
    x = 5.0000
        4.0000
```

Solution B.4-2

In this problem, we are interested in using hand calculations and Cramer's rule to solve the system of equations

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Solving for x_1, we obtain

$$x_1 = \frac{\begin{vmatrix} 7 & 2 & 0 \\ 8 & 3 & 4 \\ 9 & 0 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{vmatrix}} = \frac{7(18-0) - 2(48-36) + 0(0-27)}{1(18-0) - 2(0-20) + 0(0-15)} = \frac{126 - 24 + 0}{18 + 40 + 0} = \frac{51}{29}$$

Solving for x_2, we obtain

$$x_2 = \frac{\begin{vmatrix} 1 & 7 & 0 \\ 0 & 8 & 4 \\ 5 & 9 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{vmatrix}} = \frac{1(48-36) - 7(0-20) + 0(0-40)}{1(18-0) - 2(0-20) + 0(0-15)} = \frac{12 + 140 + 0}{18 + 40 + 0} = \frac{76}{29}$$

Solving for x_3, we obtain

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 7 \\ 0 & 3 & 8 \\ 5 & 0 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{vmatrix}} = \frac{1(27-0) - 2(0-40) + 7(0-15)}{1(18-0) - 2(0-20) + 0(0-15)} = \frac{27 + 80 - 105}{18 + 40 + 0} = \frac{1}{29}$$

These hand-calculated results are readily confirmed using MATLAB.

```
>>  format rat; x = inv([1 2 0;0 3 4;5 0 6])*[7;8;9]
    x = 51/29
         76/29
          1/29
```

Solution B.4-3

First, the system of equations is written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{Ax} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}.$$

$|\boldsymbol{A}| = 0 + 3 - 1 - (2 - 3 - 0) = 3.$

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & 0 \end{vmatrix} = 0 - 9 - 3 - (-6 - 3 - 0) = -3.$ Thus,

$$x_1 = \frac{-3}{|\boldsymbol{A}|} = \frac{-3}{3} = -1.$$

The same result is obtained in MATLAB by

```
>>  A = [1 1 1;1 2 3;1 -1 0];
>>  x_1 = det([[1;3;-3],A(:,2:3)])/det(A)
    x_1 = -1
```

(b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & -3 & 0 \end{vmatrix} = 0 + 3 - 3 - (3 - 9 - 0) = 6.$ Thus,

$$x_2 = \frac{6}{|A|} = \frac{6}{3} = 2.$$

The same result is obtained in MATLAB by

```
>>  A = [1 1 1;1 2 3;1 -1 0];
>>  x_2 = det([A(:,1),[1;3;-3],A(:,3)])/det(A)
    x_2 = 2
```

(c) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{vmatrix} = -6 - 1 + 3 - (2 - 3 - 3) = 0.$ Thus,

$$x_3 = \frac{0}{|A|} = \frac{0}{3} = 0.$$

The same result is obtained in MATLAB by

```
>>  A = [1 1 1;1 2 3;1 -1 0];
>>  x_3 = det([A(:,1:2),[1;3;-3]])/det(A)
    x_3 = 0
```

Solution B.5-1

In this problem, we determine the constants a_0, a_1, and a_2 of the partial fraction expansion

$$F(s) = \frac{s}{(s+1)^3}$$
$$= \frac{a_0}{(s+1)^3} + \frac{a_1}{(s+1)^2} + \frac{a_2}{(s+1)}$$

First, express both sides of the expression with a common denominator $F(s) = \frac{s}{(s+1)^3} = \frac{a_0}{(s+1)^3} + \frac{a_1}{(s+1)^2} + \frac{a_2}{(s+1)} = \frac{a_0 + a_1(s+1) + a_2(s+1)^2}{(s+1)^3} = \frac{a_2 s^2 + (a_1 + 2a_2)s + (a_0 + a_1 + a_2)}{(s+1)^3}$. Equating the coefficients of s^2 yields $a_2 = 0$. Thus $(a_1 + 2a_2) = a_1 = 1$. Finally $a_0 + a_1 + a_2 = a_0 + 1 + 0 = 0$ implies that $a_0 = -1$.

$$a_0 = -1, a_1 = 1, \text{ and } a_2 = 0.$$

Solution B.5-2

(a) $H_a(s) = \frac{s^2 + 5s + 6}{s^3 + s^2 + s + 1} = \frac{s^2 + 5s + 6}{(s-j)(s+j)(s+1)} = \frac{k_1}{s-j} + \frac{k_2}{s+j} + \frac{k_3}{s+1}$. Using the method of residues, $k_1 = \left.\frac{s^2 + 5s + 6}{(s+j)(s+1)}\right|_{s=j} = \frac{5(1+j)}{2j(1+j)} = -2.5j$. Since the system is real, $k_2 = k_1^* = 2.5j$. Lastly, $k_3 = \left.\frac{s^2 + 5s + 6}{s^2 + 1}\right|_{s=-1} = 1$. Thus,

$$H_a(s) = \frac{1}{s+1} + \frac{-2.5j}{s-j} + \frac{2.5j}{s+j} = \frac{1}{s+1} + \frac{5}{s^2+1}.$$

(b) $H_b(s) = \frac{1}{H_a(s)} = \frac{s^3 + s^2 + s + 1}{s^2 + 5s + 6} = s - 4 + \frac{15s + 25}{(s+2)(s+3)} = s - 4 + \frac{k_1}{s+2} + \frac{k_2}{s+3}$. Using the method of residues, $k_1 = \left.\frac{15s + 25}{s+3}\right|_{s=-2} = -5$ and $k_2 = \left.\frac{15s + 25}{s+2}\right|_{s=-3} = 20$. Thus,

$$H_b(s) = s - 4 + \frac{-5}{s+2} + \frac{20}{s+3}.$$

(c) $H_c(s) = \frac{1}{(s+1)^2(s^2+1)} = \frac{1}{(s+1)^2(s+j)(s-j)} = \frac{k_1}{s-j} + \frac{k_2}{s+j} + \frac{\tilde{a}_0}{(s+1)^2} + \frac{\tilde{a}_1}{(s+1)}$. Using the method

of residues, $k_1 = \frac{1}{(s+1)^2(s+j)}\Big|_{s=j} = \frac{1}{(1+j2-1)(j2)} = -0.25$. Since the corresponding roots are

complex conjugates, $k_2 = k_1^* = -0.25$. $\tilde{a}_0 = \frac{1}{s^2+1}\Big|_{s=-1} = 0.5$ and $\tilde{a}_1 = \frac{d}{ds}(s^2+1)^{-1}\Big|_{s=-1} =$

$-(s^2+1)^{-2}(2s)\big|_{s=-1} = \frac{-2}{-4} = 0.5$. Thus,

$$H_c(s) = \frac{-0.25}{s-j} + \frac{-0.25}{s+j} + \frac{0.5}{(s+1)^2} + \frac{0.5}{(s+1)}.$$

(d) $H_d(s) = \frac{s^2+5s+6}{3s^2+2s+1} = \frac{1}{3} + \frac{13s/3+17/3}{3s^2+2s+1} = \frac{13s/9+17/9}{s^2+2s/3+1/3}$. In some cases, this form is sufficient. A com-

plete partial fraction expansion, however, requires the denominator roots $s = \frac{-2/3\pm\sqrt{4/9-4/3}}{2} =$

$\frac{-1\pm j\sqrt{2}}{3} = -0.3333 \pm 0.4714j$. Thus, $H_d(s) = \frac{1}{3} + \frac{k_1}{s-(-1-j\sqrt{2})/3} + \frac{k_2}{s-(-1+j\sqrt{2})/3}$. Using the

method of residues, $k_1 = \frac{13s/9+17/9}{s-(-1+j\sqrt{2})/3}\Big|_{s=(-1-j\sqrt{2})/3} = 0.7222 + 1.4928j$. Since the system is

real, $k_2 = k_1^* = 0.7222 - 1.4928j$. Thus,

$$H_d(s) = \frac{1}{3} + \frac{0.7222 + 1.4928j}{s+0.3333+0.4714j} + \frac{0.7222 - 1.4928j}{s+0.3333-0.4714j}.$$

Solution B.5-3

(a) Since $F_a(x) = \frac{(x-1)(x-2)}{(x-3)^2}$ is not strictly proper, we long divide the numerator $x^2 - 3x + 2$ by
the denominator $x^2 - 6x + 9$ to yield

$$F_a(x) = 1 + \frac{3x-7}{(x-3)^2} = 1 + \frac{a_0}{(x-3)^2} + \frac{a_1}{x-3}.$$

To determine a_0 and a_1, we use Eq. (B.30).

$$a_0 = 3x - 7\big|_{x=3} = 2 \qquad \text{and} \qquad a_1 = \frac{d}{dx}(3x-7)\Big|_{x=3} = 3$$

Thus, the desired PFE is

$$F_a(x) = 1 + \frac{2}{(x-3)^2} + \frac{3}{x-3}.$$

(b) Since $F_b(x) = \frac{(x-1)^2}{(3x-1)(2x-1)}$ is not strictly proper, we long divide the numerator $x^2 - 2x + 1$ by
the denominator $6x^2 - 5x + 1$ to yield

$$F_b(x) = \frac{1}{6} + \frac{-\frac{7}{6}x+\frac{5}{6}}{(3x-1)(2x-1)} = \frac{1}{6} + \frac{k_1}{3x-1} + \frac{k_2}{2x-1}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \frac{-\frac{7}{6}x+\frac{5}{6}}{2x-1}\Big|_{x=\frac{1}{3}} = \frac{-\frac{7}{18}+\frac{15}{18}}{-\frac{1}{3}} = -\frac{4}{3}$$

and

$$k_2 = \frac{-\frac{7}{6}x+\frac{5}{6}}{3x-1}\Big|_{x=\frac{1}{2}} = \frac{-\frac{7}{12}+\frac{10}{12}}{\frac{1}{2}} = \frac{1}{2}.$$

Thus, the desired PFE is

$$F_b(x) = \frac{1}{6} + \frac{-\frac{4}{3}}{3x-1} + \frac{\frac{1}{2}}{2x-1} = \frac{1}{6} + \frac{-\frac{4}{9}}{x-\frac{1}{3}} + \frac{\frac{1}{4}}{x-\frac{1}{2}}.$$

(c)

$$F_c(x) = \frac{(x-1)^2}{(3x-1)^2(2x-1)} = \frac{a_0}{(3x-1)^2} + \frac{a_1}{3x-1} + \frac{k_1}{2x-1}.$$

Using the Heaviside cover-up method, we find that

$$a_0 = \frac{(x-1)^2}{2x-1}\bigg|_{x=\frac{1}{3}} = \frac{(-\frac{2}{3})^2}{-\frac{1}{3}} = -\frac{4}{3}$$

and

$$k_1 = \frac{(x-1)^2}{(3x-1)^2}\bigg|_{x=\frac{1}{2}} = \frac{(-\frac{1}{2})^2}{(\frac{1}{2})^2} = 1.$$

Combining the fraction to have a common denominator, the numerator is

$$a_0(2x-1) + a_1(3x-1)(2x-1) + k_1(3x-1)^2 = x^2 - 2x + 1.$$

Using the constant (non-x) terms in this expression, we see that

$$-a_0 + a_1 + k_1 = \frac{4}{3} + a_1 + 1 = 1 \quad \Rightarrow \quad a_1 = -\frac{4}{3}.$$

Thus, the desired PFE is

$$F_c(x) = \frac{-\frac{4}{3}}{(3x-1)^2} + \frac{-\frac{4}{3}}{3x-1} + \frac{1}{2x-1} = \frac{-\frac{4}{27}}{(x-\frac{1}{3})^2} + \frac{-\frac{4}{9}}{x-\frac{1}{3}} + \frac{\frac{1}{2}}{x-\frac{1}{2}}.$$

(d) By inspection (or using the quadratic equation, the denominator roots of $F_d(x)$ are -3 and -1. Since $F_d(x) = \frac{x^2-5x+6}{2x^2+8x+6}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_d(x) = \frac{1}{2} + \frac{-\frac{9}{2}x + \frac{3}{2}}{(x+3)(x+1)} = \frac{1}{2} + \frac{k_1}{x+3} + \frac{k_2}{x+1}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \frac{-\frac{9}{2}x + \frac{3}{2}}{x+1}\bigg|_{x=-3} = \frac{\frac{27}{2} + \frac{3}{2}}{-2} = -\frac{15}{2}$$

and

$$k_2 = \frac{-\frac{9}{2}x + \frac{3}{2}}{x+3}\bigg|_{x=-1} = \frac{\frac{9}{2} + \frac{3}{2}}{2} = 3.$$

Thus, the desired PFE is

$$F_d(x) = \frac{1}{2} + \frac{-\frac{15}{2}}{x+3} + \frac{3}{x+1}.$$

(e) By inspection (or using the quadratic equation, the denominator roots of $F_e(x)$ are 2 and -1. Since $F_e(x) = \frac{2x^2-3x-11}{x^2-x-2}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_e(x) = 2 + \frac{-x-7}{(x-2)(x+1)} = 2 + \frac{k_1}{x-2} + \frac{k_2}{x+1}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \frac{-x-7}{x+1}\bigg|_{x=2} = \frac{-9}{3} = -3$$

and

$$k_2 = \left.\frac{-x-7}{x-2}\right|_{x=-1} = \frac{-6}{-3} = 2.$$

Thus, the desired PFE is

$$F_e(x) = 2 + \frac{-3}{x-2} + \frac{2}{x+1}.$$

(f) By inspection (or using the quadratic equation, the denominator roots of $F_f(x)$ are -3 and 1. Since $F_f(x) = \frac{3+2x^2}{-3+2x+x^2}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_f(x) = 2 + \frac{-4x+9}{(x+3)}(x-1) = 2 + \frac{k_1}{x+3} + \frac{k_2}{x-1}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \left.\frac{-4x+9}{x-1}\right|_{x=-3} = \frac{12+9}{-4} = -\frac{21}{4}$$

and

$$k_2 = \left.\frac{-4x+9}{x+3}\right|_{x=1} = \frac{-4+9}{4} = \frac{5}{4}.$$

Thus, the desired PFE is

$$F_f(x) = 2 + \frac{-\frac{21}{4}}{x+3} + \frac{\frac{5}{4}}{x-1}.$$

(g) By inspection (or using the quadratic equation, the denominator roots of $F_g(x)$ are $\pm j$. Since $F_g(x) = \frac{x^3+2x^2+3x+4}{x^2+1}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_g(x) = x + 2 + \frac{2x+2}{(x+j)(x-j)} = x + 2 + \frac{k_1}{x+j} + \frac{k_2}{x-j}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \left.\frac{2x+2}{x-j}\right|_{x=-j} = \frac{-2j+2}{-2j} = 1 + j$$

and

$$k_2 = \left.\frac{2x+2}{x+j}\right|_{x=j} = \frac{2j+2}{2j} = 1 - j.$$

Thus, the desired PFE is

$$F_g(x) = x + 2 + \frac{1+j}{x+j} + \frac{1-j}{x-j}.$$

For those who prefer to combine conjugate roots, we can also express the PFE as

$$F_g(x) = x + 2 + \frac{2x+2}{x^2+1}.$$

(h) By inspection (or using the quadratic equation, the denominator roots of $F_h(x)$ are -3 and -2. Since $F_h(x) = \frac{1+2x+3x^2}{x^2+5x+6}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_h(x) = 3 + \frac{-13x-17}{(x+3)(x+2)} = 3 + \frac{k_1}{x+3} + \frac{k_2}{x+2}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \left.\frac{-13x - 17}{x + 2}\right|_{x=-3} = \frac{39 - 17}{-3 + 2} = -22$$

and

$$k_2 = \left.\frac{-13x - 17}{x + 3}\right|_{x=-2} = \frac{26 - 17}{-2 + 3} = 9.$$

Thus, the desired PFE is

$$F_{\text{h}}(x) = 3 + \frac{-22}{x + 3} + \frac{9}{x + 2}.$$

(i) By inspection (or using the quadratic equation, the denominator roots of $F_{\text{i}}(x)$ are $\pm 2j$. Since $F_{\text{i}}(x) = \frac{3x^3 - x^2 + 14x + 4}{x^2 + 4}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_{\text{i}}(x) = 3x - 1 + \frac{2x + 8}{(x + 2j)(x - 2j)} = 3x - 1 + \frac{k_1}{x + 2j} + \frac{k_2}{x - 2j}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \left.\frac{2x + 8}{x - 2j}\right|_{x=-2j} = \frac{-4j + 8}{-2j - 2j} = 1 + 2j.$$

Since $F_{\text{i}}(x)$ has real coefficients, the residues of conjugate roots are themselves conjugates,

$$k_2 = k_1^* = 1 - 2j.$$

Thus, the desired PFE is

$$F_{\text{i}}(x) = 3x - 1 + \frac{1 + 2j}{x + 2j} + \frac{1 - 2j}{x - 2j}.$$

For those who prefer to combine conjugate roots, we can also express the PFE as

$$F_{\text{i}}(x) = 3x - 1 + \frac{2x + 8}{x^2 + 4}.$$

(j) Converted to standard form, we see that

$$F_{\text{j}}(x) = \frac{2x^{-1} - 1 + 2x}{x - 5 + 6x^{-1}} = \frac{2x^2 - x + 2}{x^2 - 5x + 6}.$$

By inspection (or using the quadratic equation, the denominator roots of $F_{\text{h}}(x)$ are 3 and 2. Since $F_{\text{j}}(x) = \frac{2x^2 - x + 2}{x^2 - 5x + 6}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_{\text{j}}(x) = 2 + \frac{9x - 10}{(x - 3)(x - 2)} = 2 + \frac{k_1}{x - 3} + \frac{k_2}{x - 2}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \left.\frac{9x - 10}{x - 2}\right|_{x=3} = \frac{27 - 10}{3 - 2} = 17$$

and

$$k_2 = \left.\frac{9x - 10}{x - 3}\right|_{x=2} = \frac{18 - 10}{2 - 3} = -8.$$

Thus, the desired PFE is

$$F_{\text{j}}(x) = 2 + \frac{17}{x - 3} + \frac{-8}{x - 2}.$$

(k) By inspection (or using the quadratic equation, the denominator roots of $F_k(x)$ are -2 and 1. Since $F_k(x) = 3\frac{-5x^2-9x+23}{x^2+x-2}$ is not strictly proper, we long divide the numerator by the denominator to yield

$$F_k(x) = -15 + \frac{-12x+39}{(x+2)(x-1)} = -15 + \frac{k_1}{x+2} + \frac{k_2}{x-1}.$$

Using the Heaviside cover-up method, we find that

$$k_1 = \left.\frac{-12x+39}{x-1}\right|_{x=-2} = \frac{24+39}{-2-1} = -21$$

and

$$k_2 = \left.\frac{-12x+39}{x+2}\right|_{x=1} = \frac{-12+39}{1+2} = 9.$$

Thus, the desired PFE is

$$F_k(x) = -15 + \frac{-21}{x+2} + \frac{9}{x-1}.$$

Solution B.6-1

(a) A matrix representation is

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{Ax} = \boldsymbol{y} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

(b) By inspection, $x_1 = 3$ and $x_2 = -2$ can be obtained by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Thus, $a = 1, b = 0, c = 3, d = 0, e = 1$, and $f = -2$ is one possible set of constants. These constants are not unique. Any linear combination of the rows yields the same solution set. For example, $a = 2, b = 0, c = 6, d = 1, e = 1$, and $f = 1$ also works.

To ensure unique values of x_1 and x_2, the matrix $\boldsymbol{A}$ must be full rank.

(c) For no solutions to exist, the matrix $\boldsymbol{A}$ must be rank deficient, and $[\boldsymbol{A}, \boldsymbol{y}]$ must increase the rank of $\boldsymbol{A}$ by one. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The rank of $\boldsymbol{A}$ is one and the rank of $[\boldsymbol{A}, \boldsymbol{y}]$ is two. Thus, there are no solutions. MATLAB verifies the desired ranks are obtained.

```
>>  A = [1 1;2 2]; y = [1;1]; [rank(A), rank([A,y])]
    ans = 1    2
```

(d) For an infinite number of solutions to exist, the matrix $\boldsymbol{A}$ must be rank deficient, and $[\boldsymbol{A}, \boldsymbol{y}]$ must not increase the rank of $\boldsymbol{A}$. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The rank of $\boldsymbol{A}$ is one and the rank of $[\boldsymbol{A}, \boldsymbol{y}]$ is also one. Thus, there are an infinite number of solutions. MATLAB verifies the desired ranks are obtained.

```
>>  A = [1 1;2 2]; y = [1;2]; [rank(A), rank([A,y])]
    ans = 1    1
```

Solution B.6-2

The system of equations is first written in matrix form.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \boldsymbol{Ax} =
\begin{bmatrix}
4 \\
2 \\
0 \\
-2
\end{bmatrix}.
$$

Next, the result is obtained using MATLAB.

```
>>  A = [1 1 1 1;1 1 1 -1;1 1 -1 -1;1 -1 -1 -1]; x = A\[4;2;0;-2]
    x =  1
         1
         1
         1
```

That is, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 1$.

Solution B.6-3

First, the system of equations is written in matrix form.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -2 & 3 & 0 \\
1 & 0 & -1 & 7 \\
0 & -2 & 3 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \boldsymbol{Ax} =
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}.
$$

The result is obtained using MATLAB.

```
>>  A = [1 1 1 1;1 -2 3 0;1 0 -1 7;0 -2 3 -4]; x = A\[1;2;3;4]
    x = -30.0000
          8.0000
         16.0000
          7.0000
```

That is, $x_1 = -30$, $x_2 = 8$, $x_3 = 16$, and $x_4 = 7$.

Solution B.6-4

We are given that a signal $f(t) = a\cos(3t) + b\sin(3t)$ reaches a peak amplitude of 5 at $t = 1.8799$ and has a zero crossing at $t = 0.3091$. These facts can be coded matrix style as

$$
\begin{bmatrix}
\cos(5.6397) & \sin(5.6397) \\
\cos(0.9273) & \sin(0.9273)
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
=
\begin{bmatrix}
5 \\
0
\end{bmatrix}.
$$

Next, we use MATLAB to solve for the unknowns a and b.

```
>>  inv([cos(5.6397) sin(5.6397);cos(0.9273) sin(0.9273)])*[5;0]
    ans = 4.0000
         -3.0000
```

Thus, we see that $a = 4$ and $b = -3$.

Solution B.6-5

Define

$$
\mathbf{x} =
\begin{bmatrix}
1 & 3 \\
-2 & 4
\end{bmatrix}, \quad
\mathbf{y} =
\begin{bmatrix}
-5 \\
2
\end{bmatrix},
$$

$$\text{and} \quad \mathbf{z} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(a)

$$\mathbf{f}_\mathrm{a} = \mathbf{y}^T\mathbf{y} = \begin{bmatrix} -5 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix} = -5(-5) + 2(2) = 25 + 4 = 29.$$

This result is readily confirmed using MATLAB.

```
>>  x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fa = y.'*y
    fa = 29
```

(b)

$$\mathbf{f}_\mathrm{b} = \mathbf{y}\mathbf{y}^T = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \begin{bmatrix} -5 & 2 \end{bmatrix} = \begin{bmatrix} -5(-5) & -5(2) \\ 2(-5) & 2(2) \end{bmatrix} = \begin{bmatrix} 25 & -10 \\ -10 & 4 \end{bmatrix}.$$

This result is readily confirmed using MATLAB.

```
>>  x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fb = y*y.'
    fb =  25   -10
         -10     4
```

(c)

$$\mathbf{f}_\mathrm{c} = \mathbf{x}\mathbf{y} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(-5) + 3(2) \\ -2(-5) + 4(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix}.$$

This result is readily confirmed using MATLAB.

```
>>  x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fc = x*y
    fc =  1
         18
```

(d)

$$\mathbf{f}_\mathrm{d} = \mathbf{x}^T\mathbf{y} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(-5) - 2(2) \\ 3(-5) + 4(2) \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}.$$

This result is readily confirmed using MATLAB.

```
>>  x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fd = x.'*y
    fd = -9
         -7
```

(e)

$$\mathbf{f}_\mathrm{e} = \mathbf{y}^T\mathbf{x} = \begin{bmatrix} -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -5(1) + 2(-2) & -5(3) + 2(4) \end{bmatrix} = \begin{bmatrix} -9 & -7 \end{bmatrix}$$

This result is readily confirmed using MATLAB.

```
>>  x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fe = y.'*x
    fe = -9    -7
```

(f)

$$\mathbf{f}_\mathrm{f} = \mathbf{x}\mathbf{z} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1(0) + 3(-1) & 1(1) + 3(0) \\ -2(0) + 4(-1) & -2(1) + 4(0) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -4 & -2 \end{bmatrix}.$$

This result is readily confirmed using MATLAB.

```
>>   x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; ff = x*z
     ff = -3      1
          -4     -2
```

(g)

$$\mathbf{f_g} = \mathbf{zxz} = \mathbf{zf_f} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0(-3)+1(-4) & 0(1)+1(-2) \\ -1(-3)+0(-4) & -1(1)+0(-2) \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 3 & -1 \end{bmatrix}.$$

This result is readily confirmed using MATLAB.

```
>>   x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fg = z*x*z
     fg = -4     -2
           3     -1
```

(h)

$$\mathbf{f_h} = \mathbf{x}^T - \mathbf{z} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-0 & -2-1 \\ 3-(-1) & 4-0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 4 \end{bmatrix}.$$

This result is readily confirmed using MATLAB.

```
>>   x = [1 3;-2 4]; y = [-5;2]; z = [0 1;-1 0]; fh = x.'-z
     fh = 1     -3
          4      4
```

Solution B.7-1

(a) Figure SB.7-1a shows $x_a(t) = \text{Re}\big(2e^{(-1+j2\pi)t}\big)$ over $0 \le t \le 3$.

```
>>   t = 0:.001:3; xa = @(t) real(2*exp((-1+1j*2*pi)*t));
>>   plot(t,xa(t),'k-',t,2*exp(-t),'k:',t,-2*exp(-t),'k:');
>>   xlabel('t'); ylabel('x_a(t)');
```
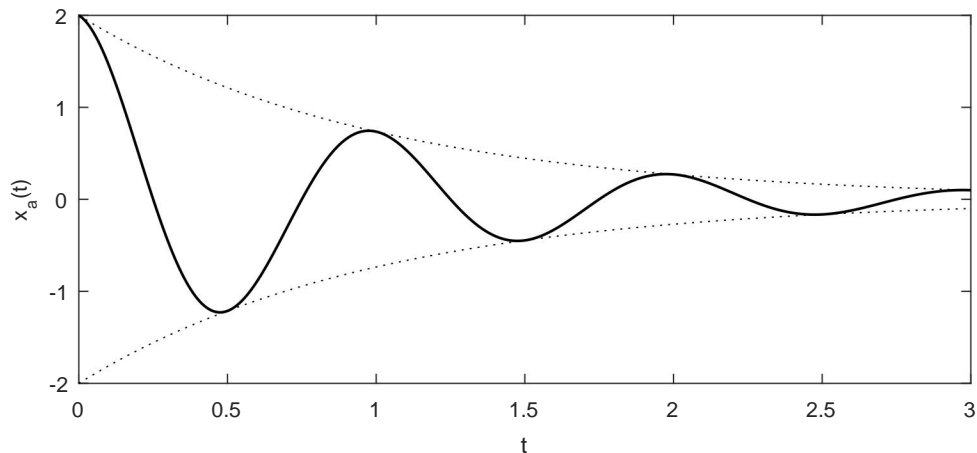

Figure SB.7-1a

(b) Figure SB.7-1b shows $x_b(t) = \text{Im}\big(3 - e^{(1-j2\pi)t}\big)$ over $0 \le t \le 3$.

```
>>   t = 0:.001:3; xb = @(t) imag(3-exp((1-1j*2*pi)*t));
>>   plot(t,xb(t),'k-',t,exp(t),'k:',t,-exp(t),'k:');
>>   xlabel('t'); ylabel('x_b(t)');
```
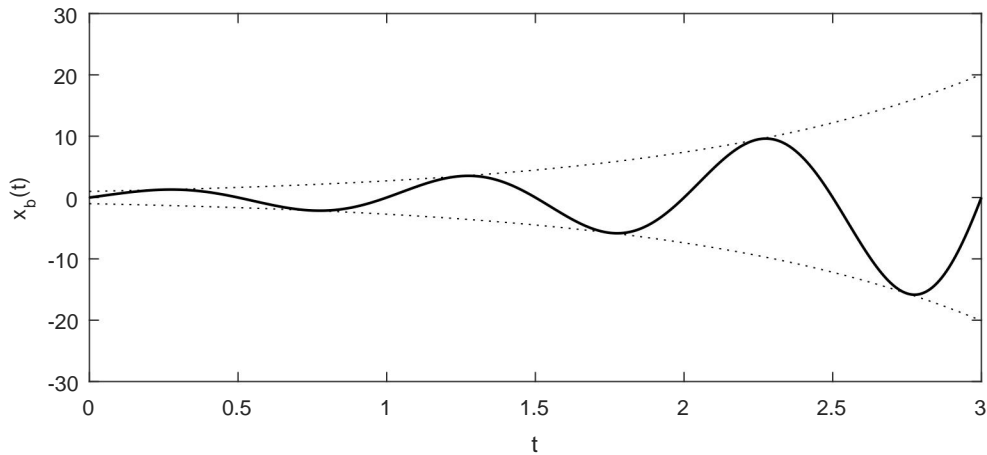
Figure SB.7-1b

(c) Figure SB.7-1c shows $x_c(t) = 3 - \text{Im}\big(e^{(1-j2\pi)t}\big)$ over $0 \le t \le 3$.

```
>>  t = 0:.001:3; xc = @(t) 3-imag(exp((1-1j*2*pi)*t));
>>  plot(t,xc(t),'k-',t,3+exp(t),'k:',t,3-exp(t),'k:');
>>  xlabel('t'); ylabel('x_c(t)');
```
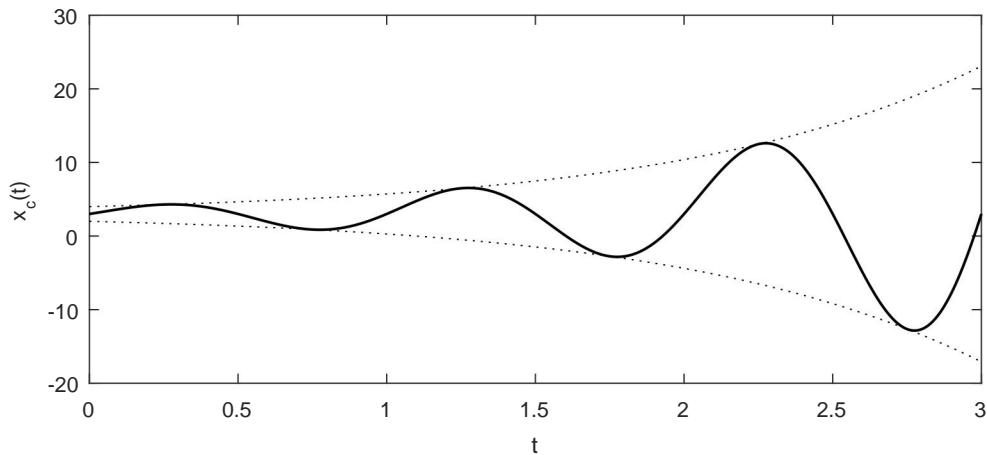

Figure SB.7-1c

Solution B.7-2

```
>>  x = @(t) t.*sin(2*pi*t); t = linspace(0,10,501);
>>  plot(t,x(t),'k-'); xlabel('t'); ylabel('x(t)');
>>  m = max(x(t))
    m = 9.2417
>>  set(gca,'ytick',[-10,-5 0 5 m]); grid on
```

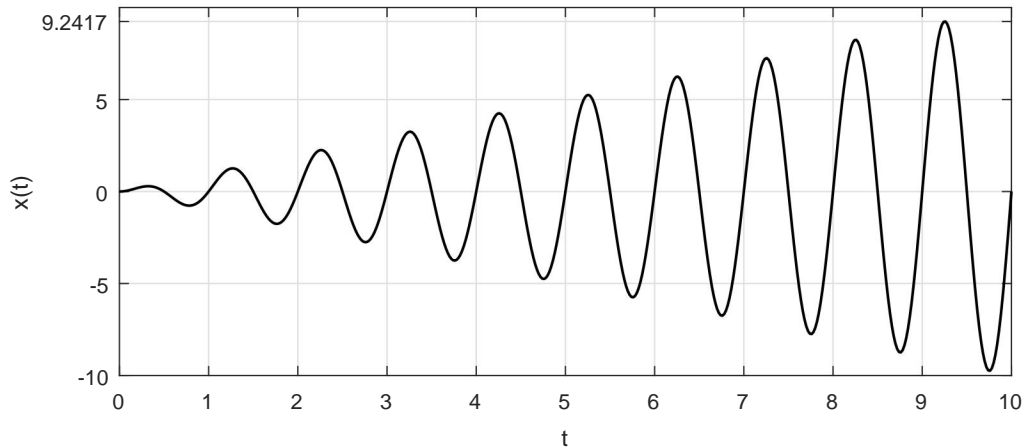Figure SB.7-2 shows $x(t)$ over $0 \le t \le 10$. Over this time interval, $x(t)$ has a maximum value of 9.2417.

Figure SB.7-2

Solution B.7-3

Since $\cos(t)$ oscillates at $\frac{1}{2\pi}$ Hz, t should cover at least 2π seconds to span one period. Since $\sin(20t)$ has a period of $\frac{2\pi}{20} = 0.314$ seconds, the step size of t should be less than 0.0314 to ensure at least ten samples per period of this fastest component.

```
>>  t = [0:.01:8]; x = cos(t).*sin(20*t);
>>  plot(t,x,'k-'); xlabel('t'); ylabel('x(t)');
```

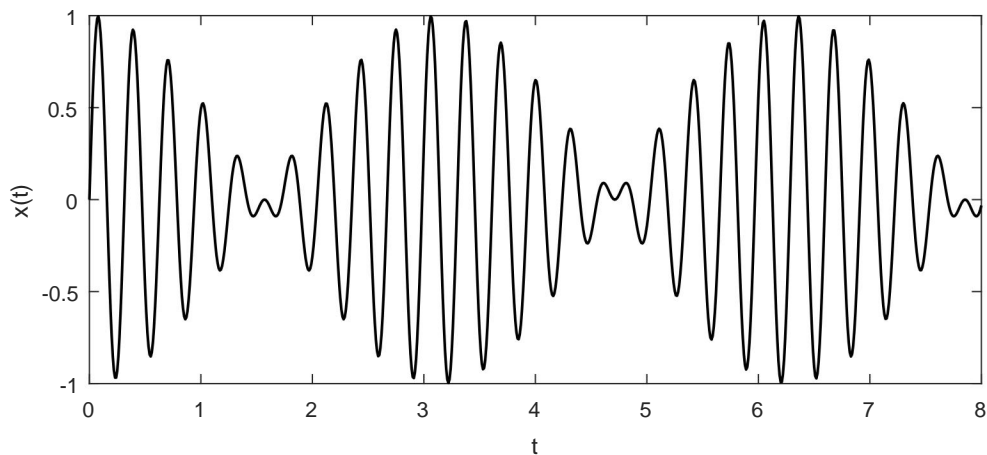Figure SB.7-3 shows $x(t) = \cos{(t)}\sin{(20t)}$ over $0 \leq t \leq 8$.

Figure SB.7-3

Solution B.7-4

The highest frequency is 10Hz, so the step size of t should be 0.01 or less to provide ten samples per period of the fastest component. The lowest frequency is 1Hz, so t should span at least one second to cover one period of the slowest component.

```
>>  t = [0:.005:2]; kt = (1:10)'*t; x = sum(cos(2*pi*kt));
>>  plot(t,x,'k'); xlabel('t'); ylabel('x(t)');
```

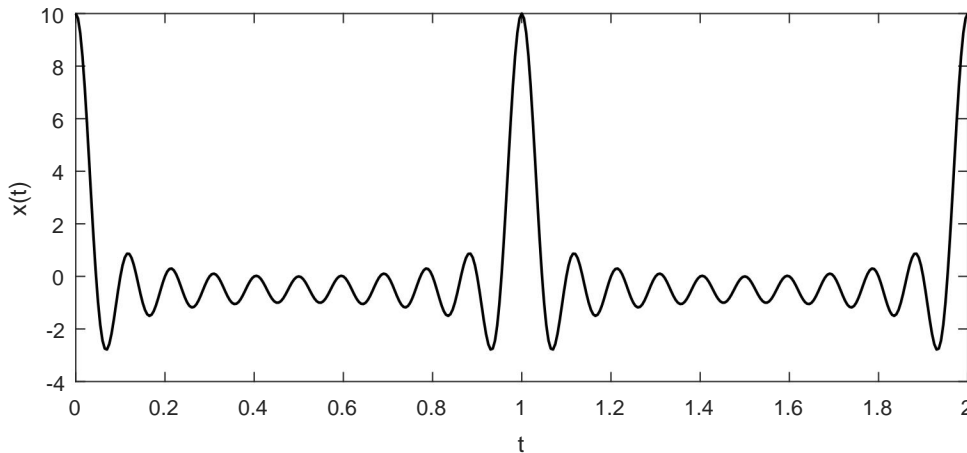Figure SB.7-4 shows $x(t) = \sum_{k=1}^{10} \cos(2\pi kt)$ over $0 \leq t \leq 2$.

Figure SB.7-4

Solution B.7-5

There are many approximations possible for the sound of a bell. In the most simple case, we can model a bell as a decaying exponential. A small, light bell will have a high pitch and not sustain a sound for long. Thus, we might choose a base oscillation of 1kHz. A reasonably quick decay rate is obtained if the envelop decreases by 90% every second, or $e^{\ln(0.1)t}$. Thus, our bell model is $x(t) = e^{\ln(0.1)t}\cos(2\pi 1000t)$. The result, however, is somewhat "flat". Adding harmonics, such as $\cos(2\pi 2000t)$, adds richness to the sound. Furthermore, some low frequency modulation, perhaps as a result of a hand initially ringing the bell, improves the sound. For example, $y(t) = e^{\ln(0.1)t}\cos(2\pi 3t)\left(\cos(2\pi 1000t) + 0.1\cos(2\pi 2000t)\right)$ sounds more natural than $x(t)$. The possibilities are endless.

```
>>  t = [0:1/8000:3.5]; a = log(0.1); x = exp(a*t).*(cos(2*pi*1000*t));
>>  y = exp(a*t).*(cos(2*pi*3*t).*(cos(2*pi*1000*t)+0.1*cos(2*pi*2000*t)));
>>  sound([x,y],8000);
```

If a large, heavy bell is desired, the frequency and decay rates need to be reduced. For example, $z(t) = e^{\ln(0.5)t}\cos(2\pi 3t)\left(\cos(2\pi 100t) + 0.1\cos(2\pi 200t)\right)$.

```
>>  t = [0:1/8000:5]; a = log(0.5);
>>  z = exp(a*t).*(cos(2*pi*3*t).*(cos(2*pi*200*t)+0.1*cos(2*pi*400*t)));
>>  sound(z,8000);
```

Solution B.7-6

(a) Begin by choosing a point on the unit circle, $w = e^{j\Omega}$. Multiplying w by itself yields $ww = w^2 = e^{j2\Omega}$. Taking this result and again multiplying by w yields $ww^2 = w^3 = e^{j3\Omega}$. At step n, the result is $w^n = e^{jn\Omega}$. From Euler's identity, we know $w^n = e^{jn\Omega} = \cos(n\Omega) + j\sin(n\Omega)$. The process does indeed provide the desired quadrature sinusoids: the real part provides the cosine term and the imaginary part yields the sine term.

(b) To produce a periodic signal, Ω needs to be a rational multiple of 2π. Most simply, choose $\Omega = 2\pi/N$, where N is the number of points computed per oscillation of each sinusoid. For reasonable quality sinusoids, N should be some moderately large integer, say 10 or 20. Although the quality of the sinusoids increases as N is increased, the required processing speed also increases with N. Thus, N represents a compromise between signal quality and processor speed. Taking $N = 20$, for example, yields $w = e^{j\pi/10}$. In this case, only $\frac{1}{100000N} = 500e-9$ seconds (500ns) are available to process each sample. This is feasible with current processor technologies.

(c) Although not required by the procedure, a vector $x[n]$ is maintained so that the signal outputs can be plotted.

```
>>  N = 20; w = exp(j*2*pi/N); w_n = w;
>>  I = 40; x = zeros(1,I); x(1) = w_n;
>>  for i = 1:I; x(i) = w_n; w_n = w_n*w; end
>>  plot([1:I],real(x),'k-',[1:I],imag(x),'k--');
>>  xlabel('n'); ylabel('Amplitude');
```

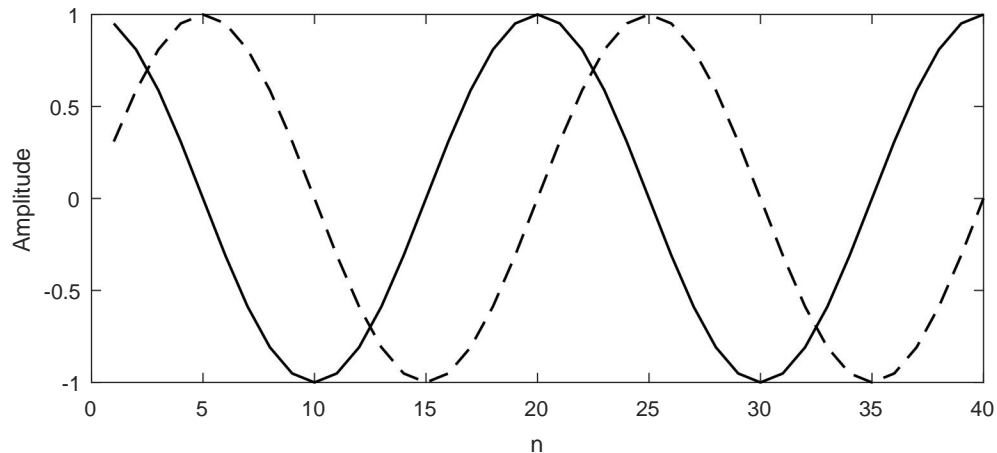Figure SB.7-6 shows the resulting quadrature sinusoids.

Figure SB.7-6

(d) To work, this procedure requires several assumptions. First for periodicity, the frequency Ω should be of the form $\Omega = 2\pi/N$, where N is an integer. Second, we assume N is chosen large enough to provide good-quality sinusoids yet provide ample time during each step to compute the next value. Each of these assumptions can generally be met. However, there are at least two limitations that may affect the suitability of this procedure:

- Since digital processors represent numbers with a finite number of bits, there is often an error associated with representing w. Instead of w, the computer stores $w + \Delta$. Due to the iterative nature of the procedure, the error grows with time. Generally, if $|w+\Delta| > 1$ then the signals will exponentially grow and if $|w + \delta| < 1$ the signals will exponentially decay. This limitation can prevent the procedure from working correctly over an indefinite time period.

- For the output signals to be truly periodic, the processor must take exactly the same amount of time between steps. This is impossible; timing errors are always present. Additionally, if the desired output frequency is not divisible by the processor clock speed, the resulting signals with either not be truly periodic, have slight frequency errors, or both.

Solution B.7-7

The MATLAB `residue` command computes the partial fraction expansion of a rational function by providing three quantities: the residues, the poles, and the direct terms.

(a)
```
>>  [r,p,k] = residue([1 5 6],[1 1 1 1])
    r =   1.0000
          0.0000 - 2.5000i
          0.0000 + 2.5000i
```

```
      p = -1.0000
          -0.0000 + 1.0000i
          -0.0000 - 1.0000i
      k = []
```

Thus,

$$H_1(s) = \frac{1}{s+1} + \frac{-2.5j}{s-j} + \frac{2.5j}{s+j} = \frac{1}{s+1} + \frac{5}{s^2+1}.$$

(b) >> `[r,p,k] = residue([1 1 1 1],[1 5 6])`
```
      r = 20.0000
          -5.0000
      p = -3.0000
          -2.0000
      k =  1    -4
```

Thus,

$$H_2(s) = \frac{20}{s+3} + \frac{-5}{s+2} + s - 4.$$

(c) >> `[r,p,k] = residue(1,poly([-1,-1,j,-j]))`
```
      r =  0.5000
           0.5000
          -0.2500 - 0.0000i
          -0.2500 + 0.0000i
      p = -1.0000
          -1.0000
           0.0000 + 1.0000i
           0.0000 - 1.0000i
      k =  []
```

Thus,

$$H_3(s) = \frac{0.5}{(s+1)} + \frac{0.5}{(s+1)^2} + \frac{-0.25}{s-j} + \frac{-0.25}{s+j}.$$

(d) >> `[r,p,k] = residue([1 5 6],[3 2 1])`
```
      r =  0.7222 - 1.4928i
           0.7222 + 1.4928i
      p = -0.3333 + 0.4714i
          -0.3333 - 0.4714i
      k =  0.3333
```

Thus,

$$H_4(s) = \frac{1}{3} + \frac{0.7222 - 1.4928j}{s + 0.3333 - 0.4714j} + \frac{0.7222 + 1.4928j}{s + 0.3333 + 0.4714j}.$$

Solution B.7-8

(a) >> `format rat; [r,p,k] = residue([1 -3 2],[1 -6 9])`
```
      r =  3
           2
      p =  3
           3
      k =  1
```

Thus,

$$F_a(x) = 1 + \frac{2}{(x-3)^2} + \frac{3}{x-3}.$$

(b) `>> format rat; [r,p,k] = residue([1 -2 1],[6 -5 1])`
```
r =    1/4
       -4/9
p =    1/2
       1/3
k =    1/6
```
Thus,

$$F_b(x) = \frac{1}{6} + \frac{-\frac{4}{9}}{x - \frac{1}{3}} + \frac{\frac{1}{4}}{x - \frac{1}{2}}.$$

(c) `>> format rat; [r,p,k] = residue([1 -2 1],[18 -21 8 -1])`
```
r =    1/2
       -4/9
       -4/27
p =    1/2
       1/3
       1/3
k =    []
```
Thus,

$$F_c(x) = \frac{-\frac{4}{27}}{(x - \frac{1}{3})^2} + \frac{-\frac{4}{9}}{x - \frac{1}{3}} + \frac{\frac{1}{2}}{x - \frac{1}{2}}.$$

(d) `>> format rat; [r,p,k] = residue([1 -5 6],[2 8 6])`
```
r    -15/2
       3
p =   -3
      -1
k =    1/2
```
Thus,

$$F_d(x) = \frac{1}{2} + \frac{-\frac{15}{2}}{x + 3} + \frac{3}{x + 1}.$$

(e) `>> format rat; [r,p,k] = residue([2 -3 -11],[1 -1 -2])`
```
r =   -3
       2
p =    2
      -1
k =    2
```
Thus,

$$F_e(x) = 2 + \frac{-3}{x - 2} + \frac{2}{x + 1}.$$

(f) `>> format rat; [r,p,k] = residue([2 0 3],[1 2 -3])`
```
r = -21/4
      5/4
p =   -3
       1
k =    2
```
Thus,

$$F_f(x) = 2 + \frac{-\frac{21}{4}}{x + 3} + \frac{\frac{5}{4}}{x - 1}.$$

(g) `>> format rat; [r,p,k] = residue([1 2 3 4],[1 0 1])`
```
      r =    1-1i
             1+1i
      p =    0+1i
             0-1i
      k =    1     2
```
Thus,
$$F_{\mathrm{g}}(x) = x + 2 + \frac{1+j}{x+j} + \frac{1-j}{x-j}.$$

(h) `>> format rat; [r,p,k] = residue([3 2 1],[1 5 6])`
```
      r = -22
             9
      p =   -3
            -2
      k =    3
```
Thus,
$$F_{\mathrm{h}}(x) = 3 + \frac{-22}{x+3} + \frac{9}{x+2}.$$

(i) `>> format rat; [r,p,k] = residue([3 -1 14 4],[1 0 4])`
```
      r =    1-2i
             1+2i
      p =    0+2i
             0-2i
      k =    3     -1
```
Thus,
$$F_{\mathrm{i}}(x) = 3x - 1 + \frac{1+2j}{x+2j} + \frac{1-2j}{x-2j}.$$

(j) `>> format rat; [r,p,k] = residue([2 -1 2],[1 -5 6])`
```
      r =   17
            -8
      p =    3
             2
      k =    2
```
Thus,
$$F_{\mathrm{j}}(x) = 2 + \frac{17}{x-3} + \frac{-8}{x-2}.$$

(k) `>> format rat; [r,p,k] = residue([-15 -27 69],[1 1 -2])`
```
      r = -21
             9
      p =   -2
             1
      k = -15
```
Thus,
$$F_{\mathrm{k}}(x) = -15 + \frac{-21}{x+2} + \frac{9}{x-1}.$$

Solution B.7-9

The MATLAB `residue` output for some rational function $F(x) = \frac{B(x)}{A(x)}$ is

```
>>  [r,p,k] = residue(b,a)
   r =   0 + 2.0000i
         0 - 2.0000i
   p =   3
        -3
   k =   0 + 1.0000i
```

Thus,

$$F(x) = j + \frac{2j}{x-3} + \frac{-2j}{x+3}.$$

Next, we recombine the terms.

$$F(x) = j\frac{(x+3)(x-3)}{(x+3)(x-3)} + \frac{2j(x+3)}{(x-3)(x+3)} + \frac{-2j(x-3)}{(x-3)(x+3)}.$$

Simplifying, we obtain

$$F(x) = \frac{jx^2 + 3j}{x^2 - 9}.$$

Thus, the length-3 vectors `a` and `b` are

$$a = [1\ 0\ -9]\ \text{and}\ b = [1j\ 0\ 3j].$$

It is a simple matter to check the solution.

```
>>  a = [1 0 -9]; b = [1j 0 3j];
>>  [r,p,k] = residue(b,a)
   r =   0 + 2.0000i
         0 - 2.0000i
   p =   3
        -3
   k =   0 + 1.0000i
```

Solution B.7-10

Many solutions are possible to this problem, but the procedure is the same in each case. Consider a fictitious phone number 555-5555. Then, $H_N(s) = \frac{5s^2+5s+5+5s^{-1}}{5s^2+5s+5}\frac{s}{s} = \frac{5s^3+5s^2+5s+5}{5s^3+5s^2+5s+0}$. The partial fraction expansion of $H_N(s)$ is obtained using the MATLAB residue command.

```
>>  [r,p,k] = residue([5 5 5 5],[5 5 5 0])
   r = -0.5000 + 0.2887i
       -0.5000 - 0.2887i
        1.0000
   p = -0.5000 + 0.8660i
       -0.5000 - 0.8660i
        0
   k =   1
```

Thus,

$$H_N(s) = 1 + \frac{-0.5000 + 0.2887j}{s + 0.5000 - 0.8660j} + \frac{-0.5000 - 0.2887j}{s + 0.5000 + 0.8660j} + \frac{1}{s}.$$

Solution B.7-11

(a)
```
>>  omega = linspace(-pi,pi,201);
>>  fr = cos(omega); fi = 0.1*sin(2*omega);
>>  plot(fr,fi,'k-'); xlabel('Re(f)'); ylabel('Im(f)');
>>  axis([-1.1 1.1 -1.1 1.1]); axis equal;
```
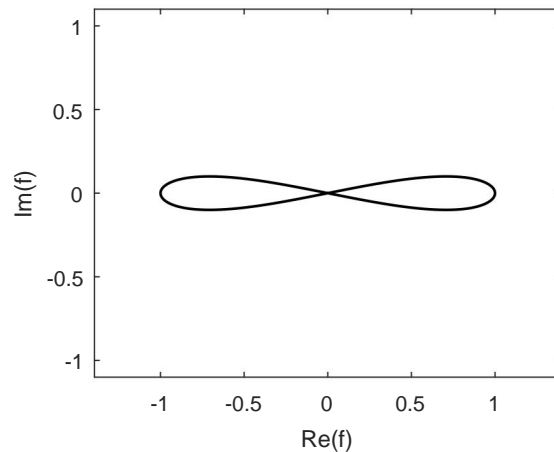Figure SB.7-11a shows the resulting Lissajous figure which resembles a horizontal propeller.

Figure SB.7-11a

(b) Multiplying w by $e^{j\theta}$ adds θ to the angle of w and thereby rotates w by θ. Also, $we^{j\theta} = (x+jy)(\cos(\theta)+j\sin(\theta)) = (x\cos(\theta)-y\sin(\theta))+j(x\sin(\theta)+y\cos(\theta))$. Furthermore, $\mathbf{Rw} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta)-y\sin(\theta) \\ x\sin(\theta)+y\cos(\theta) \end{bmatrix}$. Thus, $\mathbf{Rw}$ and $we^{j\theta}$ are equivalent, and $\mathbf{Rw}$ rotates w by θ.

(c)
```
>>  theta = 10*pi/180; R = [cos(theta) -sin(theta);sin(theta) cos(theta)];
>>  f = [fr;fi]; f = R*f;
>>  plot(f(1,:),f(2,:),'k-'); xlabel('Re(Rf)'); ylabel('Im(Rf)');
>>  axis([-1.1 1.1 -1.1 1.1]); axis equal;
```
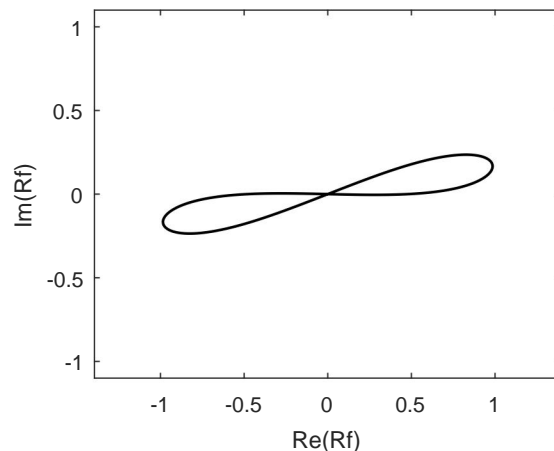Figure SB.7-11b shows the resulting Lissajous figure that has rotated 10 degrees CCW.

Figure SB.7-11b

(d) If Rf rotates f by θ, then $RRf = R(Rf)$ rotates f by 2θ. Similarly, $RRRf$ rotates f by 3θ. In general, $(R^N)f$ rotates f by $N\theta$.

(e) As suggested in part (b), multiplying $f(\omega)$ by the function $e^{j\theta}$ simply rotates f by θ. For example, the previous plot is also obtained by

```
>>  f = fr + j*fi; f = f*exp(j*theta);
>>  plot(real(f),imag(f),'k-');
>>  xlabel('Re(fe^{j\theta})'); ylabel('Im(fe^{j\theta})');
>>  axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

Figure SB.7-11c shows the resulting Lissajous figure that has rotated 10 degrees CCW by this alternate method.
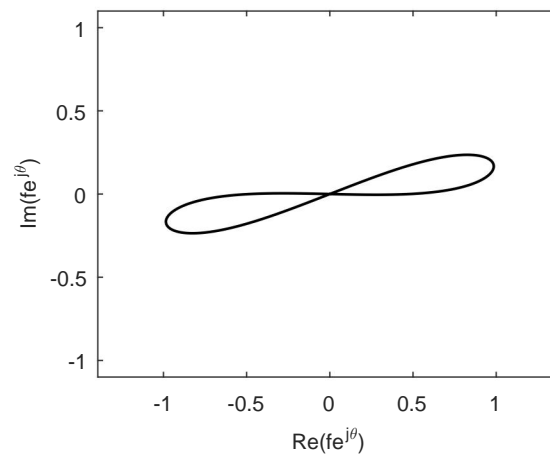

Figure SB.7-11c